CHAPTER-1

Chapter-1

Convolution and Correlation

(1.1) Convolution integration:

If $f(t)$ and $g(t)$ are two time functions. Then convolution of $f(t)$ and $g(t)$ is a function $h(t)$ defined by

$$h(t) = (f * g)(t) = \int_{-\infty}^{\infty} f(t-y)\, g(y)dy \qquad \text{-------}(1.1.1)$$

Where $f(t-y)$ is the image of $f(y)$ in ordinate axis shifted by the quantity t. The multiplication of $f(t-y)$ & $g(y)$ is integrated for each value of t from $-\infty$ to ∞.

A general rule for determining the limits of integration can be started as :

Given two functions with lower nonzero values L_1 & L_2 and upper nonzero values U_1 & U_2, choose the lower limit of integration as

$$\text{Max}\left\{L_1, L_2\right\}$$

And the upper limit of integration as

$$\text{Min}\left\{U_1, U_2\right\}$$

These lower and upper non-zero values remains unchanged for the fixed functions, however the values of the sliding function $f(t-y)$ change as t changes.

Convolution integral can be alternately defined as

$$h(t) = (f * g)(t) = \int_{-\infty}^{\infty} f(t)\ g(t-y)dy \qquad \text{-------}(1.1.2)$$

That is, either f(t) or g(t) can be folded and shifted.

<u>Example</u> Let

$$f(t) = e^{-t}\ ,\ t\geq 0$$
$$= 0\ ,\quad t<0$$

and $g(t) = \sin t,\ 0\leq t\leq \pi/2$

$$= 0\ ,\quad \text{otherwise}$$

Find $f(t)*g(t)$.

Solution :- From equation (1.1.1)

$$h(t) = (f * g)(t) = \int_{-\infty}^{\infty} f(t-y)\ g(y)\ dy$$

The integral limits are obtained by using the above procedure. The lower and upper limit of the function $f(t-y) = e^{-(t-y)}$ is $-\infty$ and t resp. For the function g(t) the lower value is 0 and the upper non zero value is $\pi/2$. Therefore for $0\leq t\leq \pi/2$

Upper limit = Min $[t, \pi/2] = t$

for $t \geq \pi/2$, Upper limit $= \pi/2$

and Lower limit = Max $[-\infty, o] = o$

$$\therefore\ h(t)=\begin{cases}\int_0^t e^{-(t-y)}\sin y\,dy, & 0\le t\le \pi/2\\ \int_0^{\pi/2} e^{-(t-y)}\sin y\,dy, & t\ge \pi/2\\ 0, & t\le 0\end{cases}$$

Evaluating we obtain

$$\begin{aligned} h(t) &= 0, \quad t\le 0\\ &= 1/2\ (\sin t - \cos t + e-t),\ 0\le t\le \pi/2\\ &= (e^{-t}/2)\ (1+e^{\pi/2}),\quad t\ge \pi/2\end{aligned}$$

- **Convolution Theorem** :- The relationship between the convolution integral and its F.T. is known as convolution theorem. It is stated as if F(w) & G(w) are the F.T. of f(t) & g(t) resp., then the F.T. of f(t)* g(t) is F(w).G(w).

Proof :- We have

$$h(t) = \int_{-\infty}^{\infty} f(t-y)\ g(y)\ dy$$

Form F.T. of both sides.

$$\int_{-\infty}^{\infty} h(t)e^{-iwt}\,dt = \int_{-\infty}^{\infty} e^{-iwt}\left[\int_{-\infty}^{\infty} f(t-y)g(y)dy\right]dt$$

$$\therefore\ H(w) = \int_{-\infty}^{\infty} g(y)\ \left[\int_{-\infty}^{\infty} e^{-iwt} f(t-y)\ dt\right]\ dy \qquad \text{------}(1.1.3)$$

Putting x = t-y the term in bracket becomes

$$\int_{-\infty}^{\infty} f(x) e^{-iw(x+y)}\, dx = e^{-iwy} \int_{-\infty}^{\infty} f(x)\, e^{-iwx}\, dx$$

$$= e^{-iwy}\, F(w)$$

Equation (1.1.3) can be written as

$$H(w) = \int_{-\infty}^{\infty} g(y)\, e^{-iwy}\, F(w)\, dy$$

$$= F(w) \int_{-\infty}^{\infty} g(y)\, e^{-iwy}\, dy$$

$$= F(w) \, . \, G(w)$$

Thus,

$$F[f(t) * g(t)] = F[f(t)] \, . \, F[g(t)]$$

- **Some properties :-**

i) Convolution is commutative ;

$$f(t) * g(t) = g(t) * f(t), \quad t \in (-\infty, \infty)$$

ii) Convolution is associative ;

$$h(t) * [f(t) * g(t)] = [h(t) * f(t)] * g(t)$$

iii) Convolution is distributive over addition;

$$h(t) * [f(t) + g(t)] = [h(t) * f(t)] + [h(t) * g(t)]$$

Example :- Determine the F.T. of $e^{-\alpha t^2} * e^{-\beta t^2}$

$$f(t) = e^{-\alpha t^2}, \quad g(t) = e^{-\beta t^2}$$

$$F[f(t)] = \int_{-\infty}^{\infty} e^{-iwt}\, e^{-\alpha t^2}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-(\alpha t^2 + iwt)}\, dt$$

But $\alpha t^2 + iwt = \alpha[(t+iw/2\alpha)^2 + (w^2/4\alpha^2)]$

$$= \alpha(t + iw/2\alpha)^2 + w^2/4\alpha$$

Put $Z = \sqrt{\alpha}\ (t + iw/2\alpha)$

$dz = \sqrt{\alpha}\ dt$ or $dt = dz/\sqrt{\alpha}$

Therefore $\alpha t^2 + iwt = z^2 + (w^2/4\alpha)$ and

$$\int_{-\infty}^{\infty} e^{-(\alpha t^2 + iwt)}\, dt = \int_{-\infty}^{\infty} e^{-(z^2 + w^2/4\alpha)}\, dz/\sqrt{\alpha}$$

$$= \frac{e^{-w^2/4\alpha}}{\sqrt{\alpha}} \int_{-\infty}^{\infty} e^{-z^2} dz$$

$$= \sqrt{(\pi/\alpha)}\, e^{-w^2/4\alpha} \qquad \text{---------(1.1.4)}$$

Similarly

$$F[g(t)] = \sqrt{(\pi/\beta)}\, e^{-w^2/4\beta} \qquad \text{-------(1.1.5)}$$

Using convolution theorem,

$$F\ [f(t) * g\ (t)] = F[f(t)].\ \ F[g(t)]$$

$$= \sqrt{(\pi/\alpha)}\, e^{-w^2/4\alpha} \cdot \sqrt{(\pi/\beta)}\, e^{-w^2/4\beta}$$

$$= (\pi/\sqrt{\alpha\beta})\, e^{-(w^2/4)\ (1/\alpha + 1/\beta)}$$

Example : Determine $h(t) * g(t)$ where

$$f(t) = e^{-at},\ t > 0$$

$$= 0,\quad t < 0$$

and

$$g(t) = e^{-bt}\ ,\quad t > 0$$
$$= 0,\quad t < 0$$

Solution : By Definition,

$$(f*g)(t) = \int_{-\infty}^{\infty} f(t-y)\, g(y)\, dy$$

The lower and upper values of t for f(t-y) are 0 & t which are the same for the function g(y). Thus

$$(f*g)(t) = \begin{cases} \int_0^t e^{-a(t-y)}\, e^{-by}\, dy & , \quad t>0 \\ 0 & , \quad t<0 \end{cases}$$

$$= \begin{cases} \dfrac{e^{-at}}{(a-b)} [e^{(a-b)t} - 1] & , \quad t>0 \\ 0 & , \quad t<0 \end{cases}$$

$$= \begin{cases} \dfrac{1}{(a-b)} [e^{-bt} - e^{-at}] & , \quad t>0 \\ 0 & , \quad t<0 \end{cases}$$

If either f(t) or g(t) is an impulse function then convolution integral is of simplest type. Let us consider,

$$g(t) = \delta(t-T) + \delta(t + T)$$

then convolution integral gives

$$h(t) = (f*g)(t) = \int_{-\infty}^{\infty} f(t-y)\ g\ (y)\ dy$$

$$= \int_{-\infty}^{\infty} f(t-y)[\delta(y-T) + \delta(y+T)]dy$$

But we know,

$$\int_{-\infty}^{\infty} \delta(y-T)\ f(y)\ dy = f(T)$$

$$\therefore h(t) = \int_{-\infty}^{\infty} f(t-y)\ \delta(y-T)dy + \int_{-\infty}^{\infty} f(t-y)\ \delta(y+T)dy$$

$$= f(t-T) + f(t + T)$$

Thus the convolution of the function f(t) with an impulse function is evaluated by simply reconstructing f(t) with the position of the impulse function replacing the ordinate of f(t).

If g(t) is a series of impulse functions then convolution of f(t) with g(t) is simply obtained by replacing each impulse by the function f(t).

- **Frequency convolution theorem** :- It is the relation between the Fourier transform of the product of the functions in time domain and the convolution of their F.T. in frequency domain.

  It is stated as if F(w) & G(w) are the F. T. of f(t)and g(t) respectively. then.

$$F[f(t)\ g(t)] = F(w) * G(w) \quad \text{---------------}(1.1.6)$$

Proof :- By definition,

$$H(w) = F(w) * G(w) = \int_{-\infty}^{\infty} F(w-y)\ G(y)\ dy$$

Taking the inverse F.T. on both sides we get,

$$\int_{-\infty}^{\infty} e^{iwt}\ H(w)\,dw = \int_{-\infty}^{\infty} e^{iwt}\left[\int_{-\infty}^{\infty} F(w-y)\ G(y)\,dy\right]dw$$

$$= \int_{-\infty}^{\infty} G(y)\left[\int_{-\infty}^{\infty} e^{iwt}\ F(w-y)\,dw\right]\ dy$$

$$= \int_{-\infty}^{\infty} e^{ity}\ G(y)\ f(t)\,dy$$

$$= f(t)\int_{-\infty}^{\infty} e^{ity}\ G(y)\,dy$$

$$h(t) = f(t).\ g(t)$$

Thus inverse F.T. of convolution in frequency domain is equal to the product of the functions in time domain, or F.T. of the product of the functions is equal to the convolution of their Fourier transforms.

* **Parseval's Theorem :-** If F(w) is the F.T. of f(t) then

$$\int_{-\infty}^{\infty} [f(t)]^2\ dt = \int_{-\infty}^{\infty} |F(w)|^2\ dw \quad \text{-------}(1.1.7)$$

Proof :- Consider the function $h(t) = f(t).f(t)$

By convolution theorem

$$F[h(t)] = F(w) * F(w)$$

$$\therefore \quad \int_{-\infty}^{\infty} e^{-iwt}\, h(t)\,dt = \int_{-\infty}^{\infty} F(w-y)\; F(y)\,dy$$

$$\therefore \quad \int_{-\infty}^{\infty} e^{-iwt}\, [f(t)]^2 dt = \int_{-\infty}^{\infty} F(w-y)\; F(y)\,dy$$

Putting $w = 0$ in the above expression, we get

$$= \quad \int_{-\infty}^{\infty} [f(t)]^2 dt = \int_{-\infty}^{\infty} F(w)\; F(-w)\,dw$$

(Replacing y by w)

Since $F(w) = R(w) + iI(w)$,

Where $R(w)$ = Real part & $I(w)$ = Imaginary part.

$\therefore \quad F(-w) = R(-w) + iI(-w)$

and it can be proved that $R(w)$ is even function and $I(w)$ is an odd function ie.

$$R(-w) = R(w) \;\&\; I(-w) = -I(w)$$

$\therefore \quad F(-w) = R(w) - iI(w)$

this gives

$$F(w).\; F(-w) = R^2(w) + I^2(w) = |F(w)|^2$$

$$\int_{-\infty}^{\infty} [f(t)]^2\, dt = \int_{-\infty}^{\infty} |F(w)|^2\, dw \quad \text{proved,}$$

Thus parseval's theorem states that the energy in waveform f(t) is equal to the energy in F(w).

(1.2) Correlation :- The integral defined by

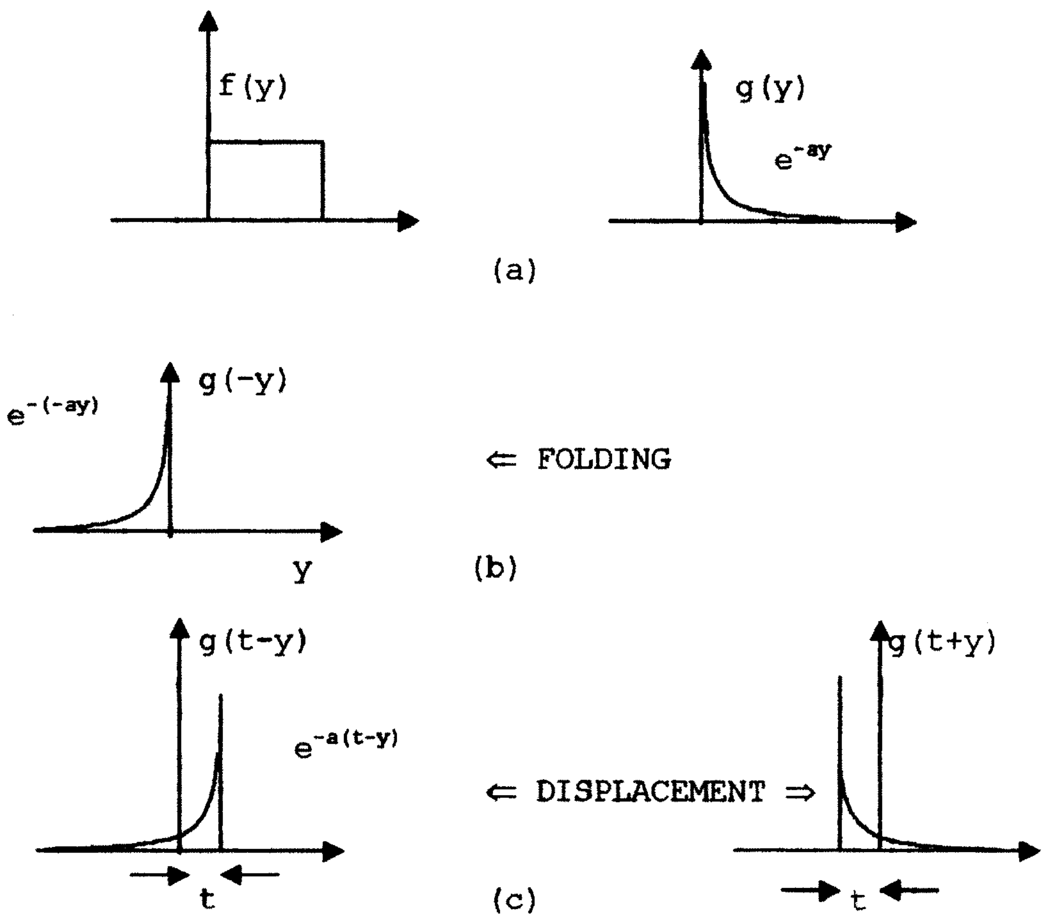
$$y(t) = \int_{-\infty}^{\infty} f(t+y)\, g(y)\, dy \qquad \text{-------(1.2.1)}$$

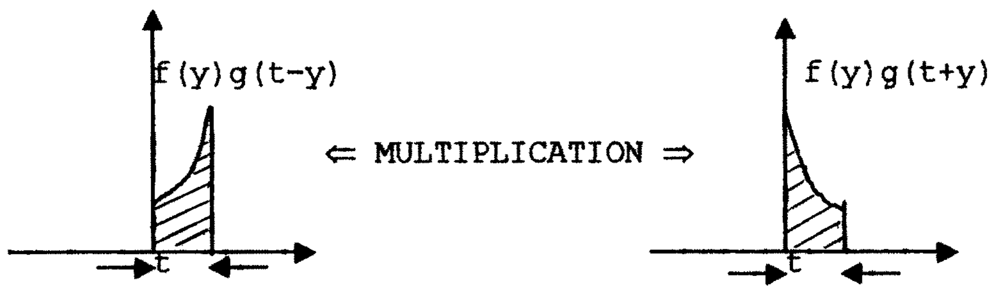
is called the correlation integral and the functions f(t), g(t) are called correlated.

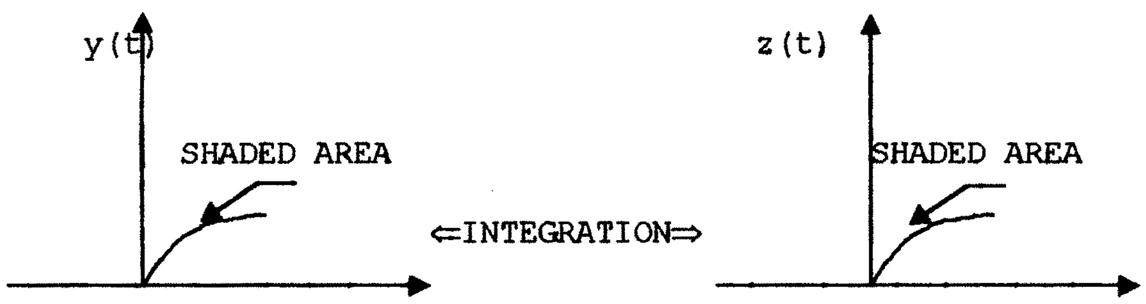
Also the autocorrelation function of square integrable functions is defined by

$$y(t) = \int_{-\infty}^{\infty} f(t+y)\, \bar{f}(y)\, dy \qquad \text{-------(1.2.2)}$$

* **The relation between convolution and correlation:**







The convolution & correlation integrals are closely related. The convolution is the sequence of folding, displacement, multiplication, and integration. But the correlation does not contain the folding. Thus when the function is even function the convolution and correlation are equivalent since an even function and its image is identical.

Ex. Determine the autocorrelation function of the waveform

$$f(t) = e^{-at}, \quad t>0$$
$$= 0 \quad , \quad t<0$$

Solution :- By definition,

$$y(t) = \int_{-\infty}^{\infty} f(t+y)\ \overline{f}(y)\ dy$$

$$\therefore\ y(t) = \int_{-\infty}^{0} f(t+y)\ \overline{f}(y)\ dy + \int_{0}^{\infty} f(t+y)\ \overline{f}(y)\ dy$$

$$= \int_{o}^{\infty} e^{-a(t+y)}\ e^{-ay}\ dy,\ t > 0$$

$$= e^{-at}\int_{o}^{\infty} e^{-2ay}\ dy,\ t > 0$$

$$= e^{-at} \left(\frac{e^{-2ay}}{-2a}\right)_{0}^{\infty}$$

$$= \frac{e^{-at}}{2a},\ t > 0$$

$$= \frac{e^{at}}{2a},\ t < 0$$

$$\therefore\ y(t) = \frac{e^{-a|t|}}{2a},\quad -\infty < t < \infty$$

Ex : Determine the correlation integral for the following functions

$$f(t) = 1,\quad 0 < t < a$$

$$= 0,\quad t \geq a,\ t \leq 0$$

and

$$g(t) = (b/a)t,\ o \leq t < a$$

$$= 0\quad ,\ t \geq a$$

Sol. By definition, correlation of f(t) & g(t) is given by

$$y(t) = \int_{-\infty}^{\infty} f(t+y)\, g(y)\, dy$$

Waveforms of the given functions are as below

The lower and upper values of y for f(t+y) are -t & a-t and those for g(y) are 0 & a.

Therefore for $0 \leq t \leq a$

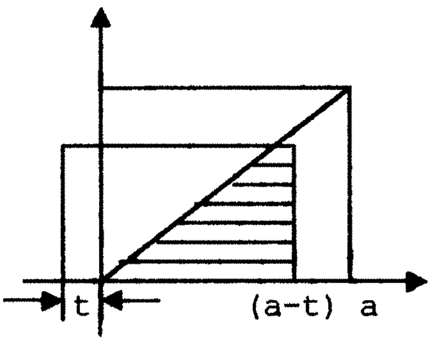
Upper limit of integration = Min [a-t, a] = a-t

Lower limit of integration = Max [-t, 0]= 0

$$\therefore \quad y(t) = \int_{0}^{a-t} 1.(by/a)\, dy, \qquad 0 \leq t \leq a$$

$$= \frac{b}{a}\left[\frac{y^2}{2}\right]_0^{a-t} = \frac{b}{2a}\left[(a-t)^2\right], \quad 0 \leq t \leq a$$

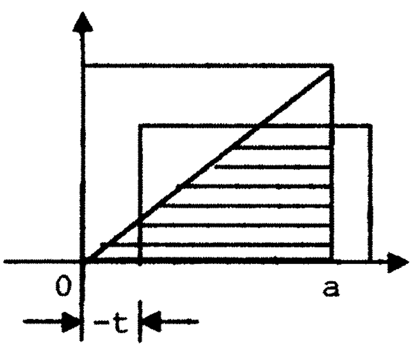
It is shown as below.



Now, for $-a \leq t \leq 0$, the lower and upper values of y for $f(t+y)$ are $-t$ & $a-t$ and therefore,

$$\text{Upper limit of int.} = \text{Min.}\left\{a-t,\ a\right\} = a$$

$$\text{Lower limit of int.} = \text{Max.}\left\{-t,\ o\right\} = -t$$



$$\therefore \quad y(t) = \int_{-t}^{a} 1.\left[\frac{b}{a}\,y\right] dy, \quad -a \leq t \leq 0$$

$$\therefore \quad y(t) = \frac{b}{2a}\,(a^2-t^2), \ -a \leq t \leq 0$$

$$\therefore \ y(t) = \begin{cases} \dfrac{b}{2a}\,(a-t)^2, & 0 \leq t \leq a \\ \dfrac{b}{2a}\,(a^2-t^2), & -a \leq t \leq 0 \\ 0 \quad , & \text{otherwise} \end{cases}$$

Correlation theorem: It gives the relation between the correlation integral and its F.T.

Statement : If F(w) and G(w) are the F.T. of f(t) and g(t) respectively and

$$y(t)=\int_{-\infty}^{\infty} f(t+y)\,g(y)\,dy,\ \text{then}$$

F.T. of y(t) is F(w).G*(w), where G*(w) is the complex conjugate of G(w).

Proof :- We have

$$y(t) = \int_{-\infty}^{\infty} f(t+y)\ g(y)\ dy$$

find F.T. on both sides.

$$\therefore \int_{-\infty}^{\infty} \bar{e}^{iwt}\, y(t) = [\int_{-\infty}^{\infty} \bar{e}^{iwt}\ (\int_{-\infty}^{\infty} f(t+y)g(y)dy)]\ dt$$

Put t + y = x and rewrite the term in bracket as

$$\int_{-\infty}^{\infty} \bar{e}^{iw(x-y)}\ f(x)\ dx = e^{iwy} \int_{-\infty}^{\infty} \bar{e}^{iwx}\ f(x)\ dx$$

$$= \bar{e}^{iwy}\ F(w)$$

$$\therefore y(w) = \int_{-\infty}^{\infty} g(y)\ e^{iwy}\ F(w)\ dy$$

$$\therefore Y(w) = F(W)[\int_{-\infty}^{\infty} g(y)\ \cos wy\ dy + i \int_{-\infty}^{\infty} g(y)\ \sin wy\ dy]$$

$$= \quad F(w)\ [\ R(w) + iI\ (w)] \qquad \text{-------}(1.2.3)$$

But the F.T. of g(y) is given by

$$G(w) = \int_{-\infty}^{\infty} e^{-iwy}\, g(y)\, dy$$

$$= \int_{-\infty}^{\infty} g(y) \cos wy\, dy - i \int_{-\infty}^{\infty} g(y) \sin wy\, dy$$

$$= \quad R(w) - iI(w) \qquad \text{-------}(1.2.4)$$

The bracketed part in (1.2.3) is complex conjugate of (1.2.4), equation (1.2.3) can be written as

$$Y(w) = F(w).G^*(w) \quad \text{proved.}$$

If g(t) is even function then G(w) is purely real and hence $G(w) = G^*(w)$ i.e.

$$Y(w) = F(w).\ G(w)$$

Thus, in this case the correlation is equivalent with the convolution.

(1.3) Fourier Series

If f(t) is periodic function of period T_0 then it is expressed as a Fourier Series given by

$$f(t) = (a_0/2) + \sum_{n=1}^{\infty}[a_n \cos(n\pi t/T_0) + b_n \sin(n\pi t/T_0)]$$

$$= (a_o/2) + \sum_{n=1}^{\infty}[a_n \cos(n\pi w_o t) + b_n \sin(n\pi w_o t)]$$

$$\text{-------}(1.3.1)$$

Where w_o is the fundamental frequency given as $w_o=1/T_0$. The coefficient a_n, b_n are given by

$$a_n = (1/T_0) \int_{-To/2}^{To/2} f(t)\cos(n\pi w_o t)\,dt, \quad n=0,1,2,\ldots \qquad \text{-------}(1.3.2)$$

$$b_n = (1/T_0) \int_{-To/2}^{To/2} f(t)\,\sin(n\pi w_o t)\,dt, \quad n=0,1,2,\ldots \qquad \text{-------}(1.3.3)$$

We can write expression (1.3.1) as

$$f(t) = (a_o/2) + \sum_{n=1}^{\infty}\left[a_n \left(\frac{e^{in\pi wot} + e^{-in\pi wot}}{2}\right) + b_n \left(\frac{e^{in\pi wot} - e^{-in\pi wot}}{2i}\right)\right]$$

$$= (a_o/2) + (1/2)\sum_{n=1}^{\infty}[a_n - ib_n)e^{in\pi wot} +$$

$$+(1/2)\sum_{n=1}^{\infty}[a_n + ib_n)e^{-in\pi wot} \qquad \text{------}(1.3.4)$$

Now, using expression (1.3.2)

$$a_{-n} = (1/T_o) \int_{-To/2}^{To/2} f(t)\,\cos(n\pi w_o t)\,dt = a_n, \quad n=1,2,$$

and

$$b_{-n} = (1/T_o) \int_{-To/2}^{To/2} f(t)\ \sin(-n\pi w_o t)dt,$$

$$= (-1/T_o) \int_{-To/2}^{To/2} f(t)\ \sin(-n\pi w_o t)dt = -b_n \quad n = 1,2,3\ \ldots$$

Thus $a_{-n} = a_n$ and $b_{-n} = -b_n$ -------(1.3.5)

This gives

$$\sum_{n=1}^{\infty} a_n\ e^{-in\pi w_o t} = \sum_{n=-1}^{-\infty} a_n\ e^{in\pi w_o t} \quad \text{-------(1.3.6)}$$

and

$$\sum_{n=-1}^{-\infty} ib_n\ e^{-in\pi w_o t} = -\sum_{n=-1}^{-\infty} ib_n\ e^{in\pi w_o t} \quad \text{-------(1.3.7)}$$

putting (1.3.6) & (1.3.7) in (1.3.4), we get

$$f(t) = (a_o/2) + (1/2)\sum_{n=1}^{\infty}(a_n - ib_n)\ e^{in\pi w_o t} +$$

$$+ (1/2)\sum_{n=-1}^{-\infty}(a_n - ib_n)\ e^{in\pi wot}$$

$$= (a_o/2) + \sum_{n=-\infty}^{\infty}(a_n - ib_n)\ e^{in\pi wot}$$

We can write it as

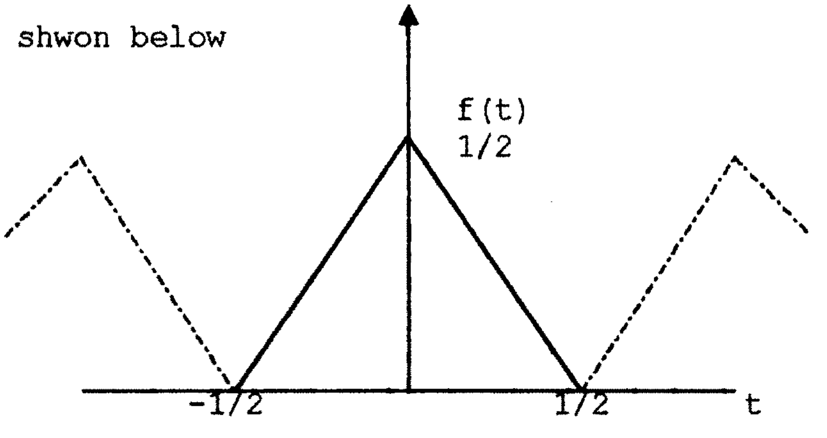
$$f(t) = \sum_{n=-\infty}^{\infty} A_n\ e^{in\pi wot} \quad \text{-------(1.3.8)}$$

Where $A_n = a_n - ib_n$, $n=0,\pm1,\pm2,\pm3,\pm\ldots$.

The equations (1.3.2), (1.3.3) & (1.3.5) together gives

$$A_n = (1/T_o)\int_{-To/2}^{To/2} f(t)\, e^{-in\pi w_o t}\, dt, \qquad \text{-------}(1.3.9)$$

Example :- Determine the Fourier series of the periodic function as shwon below



Sol|ⁿ : Given waveform is symmetric in ordinate axis hence given function is an even function defined as

$$f(t) = t+1/2, \quad -1/2 \le t \le 0$$

$$= -t+1/2, \quad 0 \le t \le 1/2$$

$$A_n = \int_{-1/2}^{1/2} f(t)\cos(n\pi w_o t)\, dt,$$

$$= \int_{-1/2}^{1/2} f(t)\cos(n\pi t)dt \quad (\text{since } T_o = 1/w_o = 1)$$

$$= \int_{-1/2}^{0} (t+1/2)\ (\cos n\pi t)\ dt + \int_{0}^{1/2} (-t+1/2)\cos(n\pi t)\ dt$$

$$= \frac{2}{n^2 \pi^2} \quad [1- \cos n\pi/2]$$

$$A_n = \begin{cases} \dfrac{2}{n^2 \pi^2} & \text{if } n= 1,3,5,\ldots \\ 0 & \text{if } n= 2,4,6,\ldots \\ \dfrac{4}{\pi^4} & \text{if } n= 0 \end{cases}$$

Hence

$$.\grave{}. \quad f(t) = (2/\pi^4) + \sum_{n=-\infty}^{\infty} A_n \cos(n\pi w_o t)$$

$$.\grave{}. \quad f(t) = \frac{2}{\pi^4} + \frac{2}{\pi^2} \cos(\pi w_o t) + \frac{2}{9\pi^2} \cos(3\pi w_o t) + \ldots$$

Consider the periodic triangular function

$$\left.\begin{aligned} f(t) &= \frac{2}{T_o} + \frac{4t}{T_o^2}, \quad \frac{T_o}{2} \leq t \leq 0 \\ &= \frac{2}{T_o} - \frac{4t}{T_o^2}, \quad 0 \leq t \leq \frac{T_o}{2} \end{aligned}\right\} \text{------}(1.3.10)$$

As above its Fourier Series is an infinite set of sinusoids. The same relationship can be obtained by using the Fourier Integral.

By convolution theorem the periodic triangular waveform (period To) is nothing but the convolution of the single triangle and infinite equidistant impulses given by

$$g(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT_o) \qquad \text{-------(1.3.11)}$$

Therefore periodic waveform

$$h(t)= f(t) * g(t) \qquad \text{-------(1.3.12)}$$

Finding F.T. on both sides and using convolution theorem,

$$H(w) = F(w) . G(w)$$

$$= F(w)(1/T_o)\sum_{n=-\infty}^{\infty} \delta(w-n/T_o)$$

$$= (1/T_o) \sum_{n=-\infty}^{\infty} F(n/T_o)\ \delta(w-n/T_o) \qquad \text{-------(1.3.13)}$$

(Because $h(t)\ \delta(t-t_o) = h(t_o)\ \delta(t-t_o)$)

Expression (1.3.13) shows that the F.T. of the periodic waveform is the infinite set of sinusoids with amplitude of $F(n/T_o)$.

In the definition of A_n the limits of integration are from $-T_o/2$ To $T_o/2$ and

$$f(t) = h(t), \quad -T_o/2 < t < T_o/2 \qquad \text{-------(1.3.14)}$$

Therefore, using f(t) in (1.3.9) we get

$$A_n = (1/T_o) \int_{-To/2}^{To/2} f(t) e^{-inw_o\pi t}\ dt$$

$$= (1/T_o)\ F(nw_o) = (1/T_o)F(n/T_o) \quad \text{-------}(1.3.15)$$

Thus the coefficients obtained by Fourier integral and the Fourier series are identical for periodic function.

(1.4) Waveform Sampling

Similar to the transform theory of continuous and impulse functions of time we can develop the same about the sampled waveforms.

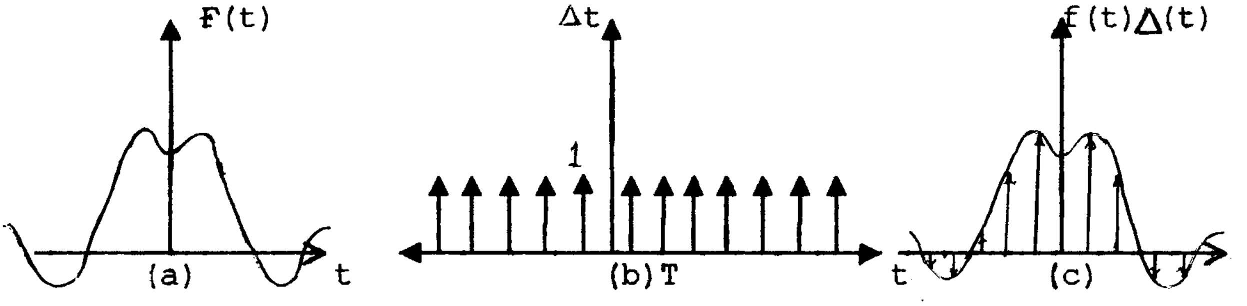
It the function f(t) is continuous at t=T then a sample of f(t) at time equal to T is expressed as

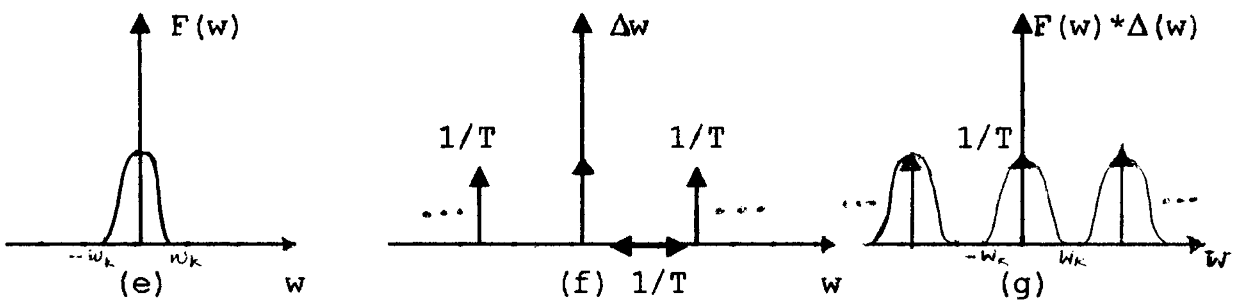
$$F^{\wedge}(t) = f(t)\ \delta(t-T) = f(T)\ \delta(t-T) \quad \text{-------}(1.4.1)$$

The amplitude of the resultant waveform at t=T is equal to the function value at t = T, If f(t) is continuous at t = nT, n = o, ±1, ±2. . . then

$$f^{\wedge}(t) = \sum_{n=-\infty}^{\infty} f(nT)\ \delta(t-nT) \quad \text{-------}(1.4.2)$$

is termed the sampled waveform f(t)with sample interval T. Thus $\hat{f}(t)$ is the infinite sequence of equidistant impulses and amplitude of each impulse is the function value at that point. Eq.(1.4.2) is the product of a continuous function f(t) and the sequence of impulses we can use the convolution theorem to find the F.T. of the sampled waveform. Thus F.T. of the sampled waveform is the periodic function where wave in one period is equal to the F.T. of the function f(t). It is illustrated below.





If T is very large in Δt, the equidistant impulses in Δw will be very close to each other. Because of the decreased spacing of the frequency impulses, their convolution with the frequency function F(w) results in overlapping waveform. This distortion of the desired F.T. of a sampled function is known as aliasing. This error can be removed by sampling the time function at a sufficiently high rate. Generally this overlapping occurs until the separation of impulses of $\Delta(w)$ is increased to $1/T = 2W_k$, where w_k is the highest frequency component of F.T. of the continuous function f(t). Thus to avoid the overlapping $T=1/2w_k$.

Furthermore, these samples can be combined to reconstruct identically the continuous waveform.

- **Sampling theorem :-**

It states that if the F.T. of a function f(t) is Zero for all frequencies greater than a certain frequency w_c, then the continuos function f(t) can be uniquely represented using the sampled values,

$$\hat{f}(t) = f(nT)\sum_{n=-\infty}^{\infty} \delta(t-nT)$$

Where $T = 1/2w_c$

Thus the necessary conditions are

i) The F.T. of f(t) should be band limited.

ii) The sample spacing ie T should be $1/2w_c$ ie impulse functions are required to be separated by $1/T = 2W_c$. If $T< 1/2w_c$, then aliasing will result and if $T>1/2W_c$, the theorem holds.