CHAPTER - II

ABSTRACT

In this second chapter study of p-valent functions, holomorphic and meromorphic is taken into consideration. We generalise the research work carried out by Kulkarni - Thakare [2] and Kulkarni - Joshi [3].

SECTION I

I) INTRODUCTION

In this section of chapter II we study the properties of the family of multivalent functions denoted by $Sp(\alpha,\xi)$ in the unit disc. The family $Sp(\alpha,\xi)$ satisfies the condition

$$\left| \frac{\dfrac{zf'}{f} - p}{\left[2\xi\left(\dfrac{zf'}{f} - \alpha\right) - \left(\dfrac{zf'}{f} - p\right)\right]} \right| < 1,$$

with appropriate restriction on α & ξ.

Members of $Sp(\alpha,\xi)$ have been characterised, the holomorphic functions with negative coefficient that are in $Sp(\alpha,\xi)$ are also characterised and several properties are obtained. We also determine the span of the index parameter in the integrals of the form

$$\int_0^z \left(\frac{f(t)}{t}\right)^{\delta} (p(t))^{\rho/n} \, dt$$

where $f \in Sp(\alpha,\xi)$, p is a polynomial of degree n whose all the zeros lie outside or on the unit circle and ρ is a fixed non-negative number.

In section II a sub family denoted by $Dp(\alpha,\xi)$ of the class of holomorphic multivalent functions in the unit disc in the complex plane satisfying the condition.

$$\left| \frac{f'(z) - p}{2\xi\,(f'(z) - \alpha) - (f'(z) - p)} \right| < 1$$

with appropriate restriction on α & ξ is introduced. A characterisation in terms of integral representation of members of Dp (α,ξ) is studied, also sharp bounds on the sizes of $| f |$, $| f' |$ and Re (f) where $f \in$ Dp (α,ξ) are obtained.

The characterisation have also been specialised to those members of Dp (α,ξ) which have negative coefficients. We conclude the section III with the study of p-valent meromorphic functions and obtain the results exactly on the same lines of Kulkarni - Joshi [3]. We claim that all the results obtained in section - I, section - II, and section -III are entirely new and not found in the literature.

II] A Characterisation and Elementary Properties

In this section we obtain elementary properties of the members of Sp (α,ξ) We present a lemma that leads to a characterisation of members of Sp (α,ξ)

Lemma 2.1

Suppose that G(z) is holomorphic in U with G(0) = p Then G satisfies the condition

$$\left| \frac{p - G(z)}{2\xi\,(G(z) - \alpha) - (G(z) - p)} \right| < 1 \text{ in } U$$

if and only if

$$\left| \frac{p - G(z)}{2\xi\,(G(z) - \alpha) - (G(z) - p)} \right| = z\,\theta(z)$$

with appropriate restriction on α & ξ.

where $\theta(z)$ is holomorphic in U with $|\theta(z)| \leq 1$

Proof Let us consider

$$g(z) = \frac{p - G(z)}{2\xi\,(G(z) - \alpha) - (G(z) - p)}$$

We observe that $g(0) = 0$ and $|g(z)| < 1$

$\therefore$ By Schwarz's Lemma we can write down $g(z) = z\phi(z)$ where $\phi(z)$

is holomorphic and $|\phi(z)| < 1$ in U. This is equivalent to $g(z) = z\theta(z)$ with $\theta(z)$ holomorphic in U and $|\theta(z)| \le 1$ in U. From this we get

$$\left| \frac{p - G(z)}{2\xi(G(z) - \alpha) - (G(z) - p)} \right| = z\,\theta(z)$$

$$p - G(z) = 2\xi\, G(z)\, z\,\theta(z) - 2\alpha\xi z\,\theta(z) - G(z) z\,\theta(z) + pz\,\theta(z)$$

This yields that

$$G(z) = \frac{p - z\,\theta(z)(p - 2\alpha\xi)}{1 + z\,\theta(z)(2\xi - 1)}$$

Conversely we assume that G(z) having the above representation is holomorphic in U. One easily sees that G(z) satisfies.

$$\left| \frac{p - G(z)}{2\xi(G(z) - \alpha) - (G(z) - p)} \right| < 1$$

and the proof is complete.

Let us have the following characterisation of members of $Sp(\alpha, \xi)$

Lemma 2.2

Suppose that $f \in Sp(\alpha, \xi)$ with appropriate restriction on α & ξ

Then f has the following integral representation

$$f(z) = z^p \exp\left[2\xi(p - \alpha) \int_X \log(1 + xz(1 - 2\xi)\, d\mu(x)\right]$$

where $x = \{x : |x| = 1\}$ and $\int_X d\mu(x) = 1$

Proof : From the definition of $f \in Sp(\alpha, \xi)$

We have

$$\frac{zf'}{f} - p = \quad \left[2\xi\left(\frac{zf'}{f} - \alpha \right) - \left(\frac{zf'}{f} - p \right) \right]$$

$$= - \quad xz \left[2\xi\left(\frac{zf'}{f} - \alpha \right) - \left(\frac{zf'}{f} - p \right) \right]$$

where $|x| = 1$, $|z| = 1$

$$\frac{zf'}{f} = p - xz \left[\frac{zf'}{f}(2\xi - 1) + (p - 2\xi\alpha) \right]$$

$$\frac{zf'}{f} + xz \left[\frac{zf'}{f}(2\xi - 1) \right] = p - xz(p - 2\xi\alpha)$$

$$\frac{zf'}{f} \left[1 + xz(2\xi - 1) \right] = p - xz(p - 2\xi\alpha)$$

$$\frac{zf'}{f} = \frac{p - xz(p - 2\xi\alpha)}{1 + xz(2\xi - 1)}$$

$$\frac{f'}{f} = \frac{p - xz(p - 2\xi\alpha)}{\left[1 + xz(2\xi - 1)\right]z}$$

This can be further transcribed as

$$\log f = p \log z - 2\xi(p - \alpha) \int_{X} \frac{x}{1 + xz(2\xi - 1)} \, d\mu(x)$$

This is equivalent to

$$f(z) = z^{p} \operatorname{Exp}\left[\frac{-2\xi(p - \alpha)}{2\xi - 1} \int_{X} \log(1 + xz(2\xi - 1)) \, d\mu(x) \right]$$

Theorem 2.3

Suppose f is holomorphic in U with f(0)=0 Then $f \in Sp(\alpha,\xi)$ if and only if

$$f(z) = \operatorname{Exp}\left[\int_{o}^{z} \frac{p - t\theta(t)(p - 2\xi\alpha)}{[1 + t\theta(t)(2\xi - 1)]t} \right] dt$$

Proof : Let f belong to $Sp(\alpha,\xi)$ then zf'/f satisfies the condition of lemma 2.1 and zf'/f has the representation

We have

$$G(z) = \frac{zf'}{f} = \frac{p - z\theta(z)(p - 2\xi\alpha)}{1 + z\theta(z)(2\xi - 1)z}$$

$$\log f = \int_{o}^{z} \frac{p - t\theta(t)(p - 2\xi\alpha)}{\left[1 + t\theta(t)(2\xi - 1)\right]t} \, dt$$

$$f = \text{Exp}\left[\int_0^z \frac{p - t\theta(t)\,(p-2\xi\alpha)}{\left[1 + t\theta(t)\,(2\xi-1)\right]t}\,dt\right]$$

Conversely

Let us suppose that f has the above integral representation. Simple computation gives that $\frac{zf'}{f}$ has the representation

$$\frac{zf'}{f} = \frac{p - t\theta(t)\,(1-2\xi\alpha)}{\left[1 + t\theta(t)\,(2\xi-1)\right]t}$$

But then it leads to the conditions of lemma 2.1 which f must satisfy and completes the proof.

We state the following distortion theorem:

Theorem 2.4 Let f be in $Sp(\alpha,\xi)$ then for z in U

$$\frac{p - (p-2\xi\alpha)|z|}{1+(2\xi-1)|z|} \leqslant \text{Re}\left(\frac{zf'}{f}\right) \leqslant \frac{p+(p-2\xi\alpha)|z|}{1-(2\xi-1)|z|}$$

<u>Proof</u> As f belongs to $Sp(\alpha,\xi)$ we have from lemma 2.1

$$G(z) = \frac{p - z\theta(z)\,(p-2\xi\alpha)}{1 + z\theta(z)\,(2\xi-1)}$$

$$G(z) = \frac{zf'}{f} = \frac{p - z\theta(z)\,(p-2\xi\alpha)}{1 + z\theta(z)\,(2\xi-1)}$$

we use $-|z| \leqslant \operatorname{Re}(z) \leqslant |z|$

$$\therefore \quad \operatorname{Re}\left(\frac{zf'}{f}\right) = \operatorname{Re}\left[\frac{p - z\theta(z)(p - 2\xi\alpha)}{1 + z\theta(z)(2\xi - 1)}\right]$$

$$\frac{p - (p - 2\xi\alpha)|z|}{1 + (2\xi - 1)|z|} \leq \operatorname{Re}\left(\frac{zf'}{f}\right) \leq \frac{p + (p - 2\xi\alpha)|z|}{1 - (2\xi - 1)|z|}$$

III] Functions with Negative coefficients

We specialise our consideration for those members of $Sp(\alpha,\xi)$ that have negative coefficients. The motivation to carry out such study arises from investigation carried out by Kulkarni - Thakare[2]

Let T be the subclass of holomorphic function in U having the power series representation

$$f(z) = z^p - \sum_{n=p+1}^{\infty} |a_n|\, z^n$$

For those holomorphic function which lie in both $Sp(\alpha,\xi)$ and T we obtain several refined results.

Let $S^*_p(\alpha,\xi) = Sp(\alpha,\xi) \cap T$

First we state a coefficient theorem that completely characterises the members of $Sp(\alpha,\xi)$

Theorem 3.1

A function $f(z) = z^p - \sum_{n=p+1}^{\infty} |a_n| z^n$ is in $S^*_p(\alpha,\xi)$

if and only if

$$\sum_{n=p+1}^{\infty} |a_n| \{(n-p) - (2\xi\alpha - 2n\xi - n - p)\} \leq 2\xi(p-\alpha)$$

Proof Suppose $f(z) \in S^*_p(\alpha,\xi)$. To prove the coefficient inequalities

$$f(z) = z^p - \sum_{n=p+1}^{\infty} |a_n| z^n$$

for $|z| = 1$ we have

$$= |zf' - pf| - |2\xi(zf' - \alpha f) - (zf' - pf)|$$

$$= \left| - \sum_{n=p+1}^{\infty} |a_n| z^n (n-p) \right| - \left| 2\xi(p-\alpha) - \sum_{n=p+1}^{\infty} n|a_n| + \alpha \sum_{n=p+1}^{\infty} |a_n| + \sum_{n=p+1}^{\infty} |a_n| - \sum_{n=p+1}^{\infty} |a_n| p \right|$$

$$\leq \sum_{n=p+1}^{\infty} |a_n| ((n-p) + (2n\xi - 2\xi\alpha - n + p) - 2\xi(p-\alpha) \leq 0$$

by hypothesis)

Thus by maximum modules theorem function f is in $S^{*}_{p}(\alpha,\xi)$. For the converse we assume that the coefficient inequality is satisfied to prove f belongs to $S^{*}p(\alpha, \xi)$

$$\left| \frac{zf' - pf}{2\xi\,(zf' - \alpha f) - (zf' - pf)} \right|$$

$$= \left| \frac{-\sum_{n=p+1}^{\infty} |a_n|\, z^{n}\,(n-p)}{2\xi z(p-\alpha) + \sum_{n=p+1}^{\infty} |a_n|\, z^{n}\,(n-p+2\xi\alpha-2n\xi)} \right| < 1$$

since $|\,\mathrm{Re}(z)\,| \leq |z|$ for all z we have

$$\mathrm{Re}\left\{ \frac{-\sum |a_n|\, z^{n}\,(n-p)}{2\xi z(p-\alpha) + \sum_{n=p+1}^{\infty} |a_n|\, z^{n}\,(n-p+2\xi\alpha-2n\xi)} \right\} < 1$$

We select the values of z on the real axis so that $f'(z)$ is real. Simplification of the denominator in the above expression and letting

$z \to 1$ through real values we obtain

$$\sum_{n=p+1}^{\infty} |a_n| \left\{(n-p) + (p-n-2\xi\alpha+2n\xi)\right\} \leqslant 2\xi\,(p-\alpha)$$

and it results in the required condition.

The Result is sharp for the function

$$f(z) = z^p - \left[\frac{2\xi(p-\alpha)}{(n-p) + (p-n-2\xi n+2\xi\alpha)}\right] z^n$$

The following two results show that the family $S^*p(\alpha,\xi)$ is closed under taking arithmetic mean and convex linear combination.

Theorem 3.2

If $f(z) = z^p - \sum_{n=p+1}^{\infty} |a_n| z^n$ and

$g(z) = z^p - \sum_{n=p+1}^{\infty} |b_n| z^n$ are in $S^*p(\alpha,\xi)$

then

$$h(z) = z^p - \tfrac{1}{2} \sum_{n=p+1}^{\infty} |a_n + b_n| z^n \in S^*p(\alpha,\xi)$$

__Proof__ As f and g both being members of $S^*p(\alpha,\xi)$ we have in accordance with theorem 3.1

$$\sum_{n=p+1}^{\infty} [(n-p) - (n-p+2\xi\alpha-2n\xi)] \; |a_n| \leq 2\xi(p-\alpha) \quad \text{--------- (A)}$$

and

$$\sum_{n=p+1}^{\infty} [(n-p) - (n-p+2\xi\alpha-2n\xi)] \; |b_n| \leq 2\xi(p-\alpha) \quad \text{--------- (B)}$$

To show that h is a member of $S^*p(\alpha,\xi)$ it is enough to show that

$$\tfrac{1}{2} \sum_{n=p+1}^{\infty} [(n-p) - (n-p+2\xi\alpha-2n\xi)] \; |a_n+b_n| \leq 2\xi(p-\alpha)$$

This is exactly an immediate consequence of (A) and (B)

Theorem 3.3

Let

$$f_n(z) = \left[\frac{2\xi\ (p-\alpha)}{(n-p)\ +\ (p-n\ -2\xi\alpha - 2n\xi)}\right] z^n$$

for n= 2,3,........... then $f \in S^*p(\alpha,\xi)$ if and only if it can be expressed in the form

$$f(z) = z^p - \sum_{n=p+1}^{\infty} \lambda_n\ f_n(z)$$

where $\lambda_n \geq 0$ (n = 1,2,3...........) and $\sum_{n=1}^{\infty} \lambda_n = 1$

Proof Let us suppose that

$$f(z) = z^p - \sum_{n=p+1}^{\infty} \lambda_n\ f_n(z)$$

$$= z^p - \sum_{n=p+1}^{\infty} \frac{2\xi\ (p-\alpha)}{(n-p)\ +\ (p-n\ -2\xi\alpha + 2n\xi)} \lambda_n\ z^n$$

Now

$$\sum_{n=p+1}^{\infty} \left\{ \frac{(n-p) + (p-n - 2\xi\alpha + 2n\xi)}{2\xi\ (p-\alpha)} \lambda_n \frac{2\xi\ (p-\alpha)}{(n-p) + (p-n - 2\xi\alpha + 2n\xi)} \right] \leq 1$$

so that by theorem 3.1 $f \in S^*p(\alpha,\xi)$

Conversely suppose that $f \in S^*p(\alpha,\xi)$ therefore by theorem 3.1 we have

$$|a_n| \leq \frac{2\xi\ (p-\alpha)}{(n-p) - (n-p + 2\alpha\xi - 2n\xi)} \qquad (n = 2,3,........)$$

$$\text{let } \lambda_n = \frac{(n-p) + (2n\xi - 2\alpha\xi - n + p)}{2\xi\ (p-\alpha)} a_n$$

Then we have

$$\sum_{n=p+1}^{\infty} \lambda_n \leq 1 \qquad -\lambda_n \geqslant 0 \qquad \therefore \lambda_1 = 1 - \sum_{n=p+1}^{\infty} \lambda_n$$

We have

$$f(z) = z^p - \sum_{n=p+1}^{\infty} \lambda_n f_n(z)$$

and the proof is complete

[S. R. Kulkarni Ph. D. Thesis, 1981, Shivaji University, Kolhapur (Unpublished)]

Corollary - 1 :

Let f be in $S(\alpha,\beta,\xi)$. Then we have

$$\frac{1-\beta(1-2\xi\alpha)}{1+\beta(2\xi-1)} \leq \operatorname{Re}(zf'/f) \leq \frac{1+\beta(1-2\xi\alpha)}{1-\beta(2\xi-1)}$$

Let us also list the following known pariticular cases of Theorem 2.3 that give distortion properties for various sub families of S.

Corollary - 2 :

If $z = re^{i\theta}$, $z_1 = Re^{i\phi}$, where $0 \leq r < 1$ and $R > 1$, then

$$\frac{-1}{R-1} \leq \operatorname{Re} \frac{z}{z-z_1} \leq \frac{1}{R+1}.$$

We shall also recall in brief some partinent concepts that would be needed. The family $K(\delta)$, $(0 \leq \delta < 1)$, consists of those univalent holomorphic functions f in E, which are convex of order δ, i. e. $\operatorname{Re}\{1 + zf''/f'\} > \delta$. Clearly $K(0) = K$, the usual class of convex functions. A holomorphic function $f \in S$ is said to be close-to-convex of order δ, $(0 \leq \delta < 1)$ if $|\operatorname{Arg} zf'/f| < \delta\pi/2$ for all $z \in E$ and some $\eta F \in S^*$, with $|\eta| = 1$. We then write $f \in C(\delta)$. For $\delta = 1$, one obtains the usual class C of close-to-convex functions. Patil and Thakare [8] have given the following characterization of close-to-convex functions of order δ which is a generalisation of similar characterization obtained by Kaplan [6] for close-to-convex functions.

IV] SPAN OF THE INDEX PARAMETER

Here we are concerned with the integrals of the form

$$H(z) = \int_{o}^{z} \left(\frac{f(t)}{t} \right)^{\delta} (p(t))^{\rho/n} \, dt$$

with appropriate restrication on α & ξ where $f \in Sp(\alpha, \xi)$, ρ is a fixed non negative number, $p \in \wp$ $R > 1$. Our interest lies in the determination of the span of index δ so that the integral given by above is either convex or close-to-convex in the unit disc U for $\rho = 0$ and $p = \xi = 1$ we get the results for the starlike functions of order α

Theorem 4.1

Let $p \in \wp(n,R)$, ρ a fixed non-negative number $R > 1$ and $f \in Sp(\alpha, \xi)$ then the integral H given by

$$H(z) = \int_{o}^{z} \left(\frac{f(t)}{t} \right)^{\delta} (p(t))^{\rho/n} \, dt \text{ is in } K(\lambda)$$

$(o \leq \lambda < p)$ with $|z| < 1$ for

$$a \leq \delta \leq \frac{(1-\lambda)(1+(1-2\xi))}{-2+2\xi(1+\alpha)} + a$$

where

$$a = \frac{-\rho(1+(1-2\xi))}{-2+2\xi(\alpha+1)(R-1)}$$

__Proof__ By routine calculations we get

$$1 + \frac{zH''(z)}{H'(z)} = (1-\delta) + \delta\left(\frac{zf'}{f}\right) + \rho/n \sum_{k=1}^{n} \frac{z}{z-zk}$$

By using distortion property and some corollary (1) of theorem 2.3 of $Sp(\alpha, \xi)$

$$\mathrm{Re}\left(1+\frac{zH''(z)}{H'(z)}\right) \geq (1-\delta) - \mathrm{Re}\left\{\delta\left[\frac{p-(2\xi\alpha-p)}{1+(1-2\xi)}\right]-\left[\frac{\rho}{R-1}\right]\right\}$$

$$\mathrm{Re}\left(1+\frac{zH''(z)}{H'(z)}\right) \geq (1-\delta) + \left\{\delta\left[\frac{p-(2\xi\alpha-p)}{1+(1-2\xi)}\right]-\left[\frac{\rho}{R-1}\right]\right\}$$

Now $H(z) \in K(\lambda)$, convex function of order λ therefore by definition of convex function

$$\mathrm{Re}\left\{1+\frac{zH''(z)}{H'(z)}\right\} > \lambda \quad \therefore 0 \leq \lambda \leq p$$

i.e. $$a \leq \delta \leq \frac{(1-\lambda)(1+(1-2\xi))}{-2+2\xi(1+\alpha)} + a$$

where

$$a = \frac{-\rho(1+(1-2\xi))}{-2+2\xi(\alpha+1)(R-1)}$$

We need the following results due to Kulkarni [3] for our further study state the corollaries.

Theorem 4.2

Suppose that $f \in Sp(\alpha,\xi)$, $p \in \wp(n,R)$ and ρ is any fixed non-negative numbers $R > 1$ then $H \in \mathfrak{C}(\lambda)$ in U $(0 \leq \lambda < p)$ where $H(z)$ is given by

$$H(z) = \int_o^z \left(\frac{f(t)}{t}\right)^{\delta} (p(t))^{\rho/n}\, dt \text{ is in } K(\lambda) \qquad \text{for}$$

$$\frac{\rho(1+(1-2\xi)}{(-2+2\xi\alpha+2\xi)} - \frac{\lambda(1+(1-2\xi)}{-2-2\xi\alpha+2\xi} \leq \delta \leq \frac{(2+\lambda)(1+(1-2\xi)}{4(-1+\xi(1+\alpha)} - \frac{\rho(1+(1-2\xi)}{(R-1)(-2+2\xi\alpha+2\xi)}$$

Proof : By usual computation we get

$$\mathrm{Re}\left[1+\frac{zH''}{H'}\right] = (1-\delta)+\delta\,\mathrm{Re}\left[\frac{zf'}{f}\right]+\rho/n\sum_{k=1}^{n}\mathrm{Re}\left(\frac{z}{z-zk}\right)$$

where $\delta>0$ we have in view of corollary 1 of page 41 (S.R.K.) and corollary on page 51 which have been stated further the following

$$\int_{\theta 1}^{\theta 2}\mathrm{Re}\left(1+\frac{zH''}{H'}\right)d\theta > \left\{(1-\delta)+\delta\,\frac{p-(p-2\xi\alpha)}{1+(1-2\xi)}-\frac{\rho}{R-1}\right\}(\theta 2-\theta 1)$$

$$\left\{(1-\delta)+\delta\left\{\frac{p-(p-2\xi\alpha)}{1+(1-2\xi)}\right\}-\frac{\rho}{R-1}\right\}(\theta 2-\theta 1)>\lambda$$

Now by Kaplans theorem $H(z)\in \mathcal{C}(\lambda)$

$$-\lambda\pi\le\int_{\theta 1}^{\theta 2}\mathrm{Re}\left\{1+\frac{zH''(z)}{H'(z)}\right\}d\theta\le\lambda\pi+2\pi$$

continuing the theorem we get

$$\int_{\theta 1}^{\theta 2}\mathrm{Re}\left[1+\frac{zH''(z)}{H'(z)}\right]d\theta \geq \int_{\theta 1}^{\theta 2}(1-\delta)+\delta\left\{\frac{p-(p-2\xi\alpha)}{1+(1-2\xi)}-\frac{\rho}{R-1}\right\}d\theta$$

$$\int_{\theta 1}^{\theta 2}\mathrm{Re}\left[1+\frac{zH''(z)}{H'(z)}\right]d\theta > -\lambda\pi$$

for

$$0\le\delta\le\frac{(2+\lambda)(1+(1-2\xi)}{4(-1+\xi(1+\alpha))}-\frac{\rho(1+(1-2\xi)}{(R-1)(-2+2\xi\alpha+2\xi)}$$

we have the right hand side of the above inequality never less than $-\lambda\pi$ for

$$\delta \le \frac{(2+\lambda)(1+(1-2\xi)}{4(-1+\xi(1+\alpha))} - \frac{\rho(1+(1-2\xi)}{(R-1)(-2+2\xi\alpha+2\xi)}$$

On the other hand for $\delta < 0$ we have on account of corollary 1 of page 41 Kulkarni S.R. [2] and corollary of page 51

$$\int_{\theta 1}^{\theta 2} \operatorname{Re}\left\{ 1+\frac{zH''}{H'} \right\} d\theta < \left[(1-\delta)+\delta\left\{\frac{p-(p-2\xi\alpha)}{1+(1-2\xi)}\right\} - \frac{\rho}{R-1})\right](\theta 2-\theta 1)$$

The right hand side of the above inequality is always less than $\lambda\pi + 2\pi$ for

$$\frac{\rho(1+(1-2\xi)}{(-2+2\xi\alpha+2\xi)(R+1)} - \frac{\lambda(1+(1-2\xi)}{(-2+2\xi\alpha+2\xi)} \le \delta < 0$$

Theorem 4.3

Let M(z) and N(z) be the polynomials belonging to $\wp$ (m,R1) and $\wp$(n,R2) with m $\ge$1 n $\ge$0 and R1, R2 > 1, ρ is a fixed non-negative number and Let f $\in$ Sp(α,ξ) with usual restrinction on α,ξ and then the integral G(z) given by

$$G(z) = \int_{o}^{z} \left(\frac{f(t)}{t}\right)^{\delta} \left(\frac{m(t)}{n(t)}\right)^{\rho} dt \text{ is in } K(\lambda)$$

$0 \le \lambda < p$ when $|z| < 1$ for

$$\frac{-\rho(1+(1-2\xi)}{-2+2\xi\alpha+2\xi)}\left[\frac{M}{R1-1}+\frac{N}{R2+1}\right] \le \delta \le \frac{(1-\lambda)(1+(1-2\xi)}{-2+2\xi\alpha+2\xi}$$

$$\frac{-\rho(1+(1-2\xi)}{-2+2\xi\alpha+2\xi)}\left[\frac{M}{R1-1}+\frac{N}{R2+1}\right]$$

Proof We have

$$G(z) = \int_0^z \left[\frac{f(t)}{t}\right]^{\delta} \left[\frac{M(t)}{N(t)}\right]^{\rho} dt$$

$$\frac{zG''(z)}{G'(z)} = \delta\left[\frac{zf'(z)}{f(z)} - 1\right] + \rho\left[\frac{zM'(z)}{M(z)} - \frac{zN'(z)}{N(z)}\right]$$

$$= (1-\delta) + \frac{\delta\, zf'(z)}{f(z)} + \rho \sum_{1}^{m} \frac{z}{z-zk} - \rho \sum_{m+1}^{m+n} \frac{z}{z-zk}$$

Now by distortion property of Sp (α,ξ)

$$1 + \frac{zG''(z)}{G'(z)} \geq (1-\delta) + \delta\left[\frac{p-(p-2\xi\alpha)}{1+(1-2\xi)}\right] - \frac{\rho M}{R1-1} - \frac{\rho N}{R2+1}$$

Now if $G(z) \in K(\lambda)$ then by definition of convex function

$$\text{Re}\left\{ 1 + \frac{zG''(z)}{G'(z)} \right\} > \lambda$$

for

$$\delta \leq \frac{(1-\lambda)(1+(1-2\xi)}{-2+2\xi\alpha+2\xi} - \rho\left\{\frac{(1+(1-2\xi)}{-2+2\xi\alpha+2\xi)}\right\}\left[\frac{M}{R1-1} + \frac{N}{R2+1}\right]$$

$$\text{where } a = -\rho\left\{\frac{(1+(1-2\xi)}{-2+2\xi\alpha+2\xi)}\right\}\left\{\frac{M}{R1-1} + \frac{N}{R2+1}\right\}$$

$$\therefore\ a \leq \delta \leq a \frac{(1-\lambda)(1+(1-2\xi)}{-2+2\xi\alpha+2\xi}$$

Theorem 4.4

Suppose f∈ Sp(α,ξ) M(z) and N(z) polynomials belonging to ρ(m, R1) and ρ(n, R2) with M>,1 , N>,0 and R1, R2 >1 ρ is a fixed non-negative number and let f∈ Sp(α,ξ) with usual restriction on α,ξ then the integral

$$G(z) = \int_{o}^{z} \left[\frac{f(t)}{t}\right]^{\delta} \left[\frac{M(t)}{N(t)}\right]^{\rho} dt \text{ is in } \mathbb{C}(\lambda)$$

$0 \leq \lambda < P$, when $|z| < 1$ for

$$\frac{-\rho(1+(1-2\xi))}{-2+2\xi\alpha+2\xi}\left[\frac{M}{R1-1} + \frac{N}{R2+1}\right] - \frac{\lambda(1+(1-2\xi))}{4(-1+\xi(\alpha+1)} \leq \delta \leq$$

$$\frac{(2+\lambda)(1+(1-2\xi))}{4(-1+\xi(\alpha+1)} - \frac{\rho\left[\frac{M}{R1-1} - \frac{N}{R2-1}\right](1+(1-2\xi))}{-2+2\xi\alpha+2\xi}$$

Proof We have

$$G(z)= \int_{o}^{z} \left[\frac{f(t)}{t}\right]^{\delta} \left[\frac{M(t)}{N(t)}\right]^{\rho} dt$$

$$\therefore \left\{1+\frac{zG''(z)}{G'(z)}\right\} = (1-\delta) + \delta \frac{zf'(z)}{f(z)} + \rho \sum_{k=1}^{m} \frac{z}{z-zk} - \rho \sum_{k=m+1}^{n} \frac{z}{z-zk}$$

Now By distortion property of Sp (α,ξ) We have

$$\int_{\theta 1}^{\theta 2} \operatorname{Re}\left\{1+\frac{zG''(z)}{G'(z)}\right\} d\theta \geq \left\{(1-\delta)+\delta\left\{\frac{p-(p-2\xi\alpha)}{1+(2\xi-1)}\right\}-\frac{\rho M}{R1-1}-\frac{\rho N}{R2+1}\right\}(\theta 2-\theta 1)$$

By Kaplans theorem if $G(z) \in \mathbf{C}(\lambda)$ Then

$$\int_{\theta 1}^{\theta 2} \operatorname{Re}\left\{1+\frac{zG''(z)}{G'(z)}\right\} d\theta > -\lambda\pi$$

for

$$\delta \leq \frac{(2+\lambda)(1+(1-2\xi)}{4(-1+\xi(1+\alpha))} - \frac{\rho\left[\frac{M}{R1-1}+\frac{N}{R2-1}\right]}{-2-2\xi\alpha+2\xi}(1+(1-2\xi))$$

and

$$\frac{-\rho(1+(1-2\xi))}{-2+2\xi\alpha+2\xi}\left[\frac{M}{R1-1}+\frac{N}{R2+1}\right]\frac{-\lambda(1+(1-2\xi))}{4(-1+\xi(1+\alpha)}\leq\delta$$

In the definition of Sp(α,ξ) we replace $\frac{zf'}{f}$ by $\left(1+\frac{zf''}{f'}\right)$ with the same restriction on α and ξ We note that this new class denoted by Kp (α,ξ) is a related to the class Sp(α,ξ) in the same manner as the class of convex univalent function to the class of starlike functions.

Taking into consideration the relationship between starlike and convex functions we write that $f \in$ Kp (α,ξ) if

$$\left|\frac{\frac{zf''}{f'}}{2\xi\left(p-\alpha+\frac{zf''}{f'}\right)-\frac{zf''}{f'}}\right| < 1$$

$z \in U$ we are interested in determining the span of the index parameter δ so that the following integral is p-valent or is in some subclass of S

$$G(z) = \int_0^z [f'(t)]^{\delta} [p(t)]^{\rho/n} dt$$

Now we state only the results

Theorem 5.1'

Let $P \in \mathcal{P}(n,R)$, ρ a fixed non-negative number $R > 1$ and $f \in Kp(\alpha,\xi)$ then the integral G given by

$$G(z) = \int_0^z (f'(t))^{\delta} (p(t))^{\rho/n} dt$$

$$\text{Replacing } p(t) \text{ by } \left(\frac{M(t)}{N(t)}\right) \text{ is in } Kp(\lambda)$$

$0 \le \lambda < p$ with $|z| < 1$ for

$$a \le \delta \le \frac{(1-\lambda)(1+(2\xi-1))}{2\xi(1-\alpha)} + a$$

with

$$a = \frac{-\rho(1+(2\xi-1))}{(R-1)(2\xi(1-\alpha))}$$

Theorem 5.2'

Suppose that $f \in Kp(\alpha,\xi)$, $p \in \mathcal{P}(n,R)$ and ρ is any fixed non-negative number $R>1$ then G defined by

$$G(z) = \int_0^z (f'(t))^{\delta} (p(t))^{\rho/n} dt \quad \text{replacing } p(t) \text{ by } (M/(t) / N(t))$$

is $C(\lambda)$ $0 \le \lambda \le p$ for

$$\frac{\rho(1+(2\xi-1)}{(R+1)(2\xi(1-\alpha)} - \frac{\lambda(1+(2\xi-1)}{4\xi(1-\alpha)} \le \delta \le \frac{(\lambda-2)(1+(2\xi-1)}{4\xi(1-\alpha)} - \frac{\rho(1+(2\xi-1)}{2\xi(1-\alpha)(R-1)}$$

Theorem 5.3'

Let M(z) and N(z) be the polynomials belonging to $\wp$ (m, R1) and $\wp$ (n, R2) with $n \ge 0$ and R1, R2 > 1 and let $f \in K p(\alpha,\xi)$ then the integral

$$G(z) = \int_0^z (f'(t))^{\delta} (p(t))^{\rho/n} dt$$

Replacing p(t) by $\left(\frac{M(t)}{N(t)}\right)$ is in $K(\lambda)$, $\lambda > p$

when | z | < | for

$$\frac{-\rho(1+(2\xi-1)}{2\xi(1-\alpha)} \left[\frac{M}{R1-1} + \frac{N}{R2+1}\right] \le \delta \le \frac{(1-\lambda)(1+(2\xi-1)}{2\xi(1-\alpha)} -$$

$$\frac{-\rho(1+(2\xi-1)}{2\xi(1-\alpha)} \left(\frac{M}{R1-1} + \frac{N}{R2+1}\right)$$

Theorem 5.4'

Suppose $f \in Kp(\alpha,\xi)$, M(z) and N(z) polynomials belonging to $\wp$ (m,R1) and $\wp$ (n,R2) with $m \le 1$, $n \ge 0$ and R1, R2 > 1 Then the integral

$$G(z) = \int_{0}^{z} (f'(t))^{\delta} (p(t))^{\rho/n} dt$$

Replacing p(t) by $\left(\frac{M(t)}{N(t)}\right)$ is in $C(\lambda)$, $\lambda < 1$

for

$$\frac{-\lambda(1+(2\xi-1)}{4\xi(1-\alpha)} - \rho \frac{\left(\frac{M}{R1-1} + \frac{N}{R2+1}\right)(1+(2\xi-1)}{2\xi(1-\alpha)} \leq \delta \leq$$

$$\frac{(2+\lambda)(1+2\xi-1)}{4\xi(1-\alpha)} - \rho \frac{\left(\frac{M}{R1-1} + \frac{N}{R2+1}\right)}{2\xi(1-\alpha)} [1+(2\xi-1)]$$

SECTION - II

I] INTRODUCTION

A new subclass of p-valent function in the unit disc has been introduced in this section. The motivation starts from the desire to generalise the class studied by Kulkarni - Thakare. The family that we intend to introduce is deeply connected with the family $Sp(\alpha,\xi)$ as in section I. We denote this new class by $Dp(\alpha,\xi)$ This class can be obtained by replacing $\frac{zf'}{f}$ by f'

$Dp(\alpha,\xi)$ is a subfamily of p-valent functions f that are holomorphic in the unit disc U and satisfy the condition

$$\left| \frac{f'(z) - p}{[2\xi(f'(z)-\alpha) - (f'(z)-p)} \right| < 1$$

with appropriate restriction on α, ξ.

In this section we obtain characterisation of members of Dp (α,ξ) in terms of integral representation. We also derive sharp distortion theorems on the sizes of $|f|$, $|f'|$ and $Re(f')$ where $f\in$ Dp (α,ξ) Finally we specialise our consideration to those holomorphic functions which are in Dp (α,ξ) and whose series expansion have negative coefficients. All our results are shown to be sharp.

II] Characterisation and Related Properties of Dp (α,ξ)

We derive a lemma that gives us a representation formula for functions in Dp (α,ξ)

Lemma 2.1

Suppose that G(z) is holomorphic in U, G(0)=1

with appropriate restrictions on α & ξ

$$\left| \frac{p - G(z)}{2\xi\ (G(z) - \alpha) - (G(z) - p)} \right| < 1$$

for z in U then

$$G(z) = \frac{p - z\theta(z)\ [p + 2\xi\alpha]}{1 + z\theta(z)\ (1 - 2\xi)}\ z^{p-1}$$

where $\theta(z)$ is holomorphic and $|\theta(z)| \leq 1$ for z in U

<u>**Proof**</u> consider

$$g(z) = \frac{p - G(z)}{2\xi\ (G(z) - \alpha) - (G(z) - p)}$$

We observe that $g(0) = 0$ and $|g(z)| < 1$ Therefore by applying Schwarz lemma we have $g(z) = z\,\phi(z)$ where $\phi(z)$ is holomorphic and $|\phi(z)| \leq 1$ in U. This is equivalent to $g(z) = z\,\theta(z)$ with $\theta(z)$ holomorphic and $|\theta(z)| \leq 1$ in U

$$\left| \frac{p - G(z)}{2\xi\,(G(z) - \alpha) - (G(z) - p)} \right| = z\theta(z)$$

is given by

$$G(z) = \frac{p - z\,\theta(z)\,(p - 2\alpha\xi)}{1 + z\,\theta(z)\,(2\xi - 1)}\, z^{p-1}$$

Conversely we assume that G(z) having the above representation

$$G(z) = \frac{p - z\,\theta(z)\,(p - 2\xi\alpha)}{1 + z\,\theta(z)\,(2\xi - 1)}\, z^{p-1}$$

is holomorphic in U It is quite clear that

$$G(z) \text{ satisfies } \left| \frac{p - G(z)}{2\xi\,(G(z) - \alpha) - (G(z) - p)} \right| < 1$$

and completes the proof. This formula hence formulates the characterisation of members of $Dp(\alpha,\xi)$.

Theorem 2.2

Let f be holomorphic in U with $f(0) = 0$. Then $f \in Dp(\alpha,\xi)$.if and only if.

$$f(z) = \int_0^z \left[\frac{p - t\,\theta(t)\,(p - 2\xi\alpha)}{1 + (1 - 2\xi)\,t\,\theta(t)}\, t^{p-1} \right] dt$$

where $\theta(z)$ is holomorphic and satisfies $|\theta(z)| \leq 1$ in U and with appropriate restriction on α,ξ.

Proof Let $f \in Dp(\alpha,\xi)$ then f' satisfies the condition of lemma 2.1 Hence f' must have a representation of the type

$$\frac{f'}{z^{p-1}} = \frac{p - z\,\theta(z)\,(p-2\xi\alpha)}{1+z\,\theta(z)\,(2\xi-1)}$$

integrating we get

$$f = \int_o^z \left[\frac{p - t\,\theta(t)(p-2\xi\alpha)}{1+t\,\theta(t)\,(2\xi-1)}\, t^{p-1}\right] dt$$

conversely

Let f be given by

$$f(z) = \int_o^z \left[\frac{p - t\,\theta(t)(p-2\xi\alpha)}{1+t\,\theta(t)\,(2\xi-1)}\, t^{p-1}\right] dt$$

Then simple computation yields that f' has the representation.

$$G(z) = \frac{p - z\theta(z)(p-2\xi\alpha)}{1+z\theta(z)\,(2\xi-1)}\, z^{p-1}$$

Thus as a consequences of lemma 2.1 f' equivalently satisfies the condition

$$\left| \frac{p - G(z)}{2\xi(G(z)-\alpha) - (G(z)-p)} \right| < 1$$

We shall explicitly apply the contents of the above lemma to obtain sharp Distortion theorem on the sizes of $|f|$, $|f'|$ and Re (f') where f is in $Dp(\alpha,\xi)$

Theorem 2.3

If $f \in Dp(\alpha, \xi)$ then for $|z| = r$, $0 < r < 1$ we have

(I) $$|f'(z)| \leq \frac{p + (p-2\xi\alpha)r}{1-(2\xi-1)r} r^{p-1}$$

(II) $$\frac{p - r(p-2\alpha\xi)}{1+r(2\xi-1)} r^{p-1} \leq \operatorname{Re}(f') \leq \frac{p + r(p-2\xi\alpha)}{1-r(2\xi-1)} r^{p-1}$$

(III) $$|f(z)| \leq \int_o^{|z|} |f'(te^{i\theta})| \, dt \leq \int_o^{|z|} \frac{p + t(p-2\xi\alpha)}{1-t(2\xi-1)} t^{p-1} \, dt$$

$$= |z|^p - \frac{(2\xi-1)^{-p} \, 2\xi(p-\alpha)}{(2\xi-1)} \log(1-|z|(2\xi-1))$$

(IV) $$|f(z)| \geq \int_o^{|z|} \operatorname{Re}(f'(t)) \, dt \geq \int_o^{|z|} \frac{p - t(p-2\xi\alpha)}{1+t(2\xi-1)} t^{p-1} \, dt$$

$$= |z|^p - \frac{(2\xi-1)^{-p} \, (2\xi(p-\alpha))}{(2\xi-1)} \log(1+|z|(2\xi-1))$$

Proof (I) By lemma 2.3 we have $|\theta(z)| \leq 1$

we have

$$f' = \frac{p - z\theta(z)(p-2\xi\alpha)}{1+z\theta(z)(2\xi-1)} z^{p-1}$$

$$|f'(z)| = \left| \frac{p - z\theta(z)(p-2\xi\alpha)}{1+z\theta(z)(2\xi-1)} \right| z^{p-1}$$

for $|\theta(z)| \leq 1$ we get

$$\frac{p-|z|(p-2\xi\alpha)}{1+|z|(2\xi-1)} \; \frac{|z|^{p}}{|z|}$$

also for $|z| = r$ we get

$$\frac{p-(p-2\xi\alpha)r}{1+r(2\xi-1)} \times \frac{r^{p}}{r}$$

$$|f'(z)| \leqslant \frac{p-(p-2\xi\alpha)r}{1+r(2\xi-1)} \, r^{p-1}$$

(II) $$\frac{p-r(p-2\xi\alpha)}{1+r(2\xi-1)} \, r^{p-1} \leq \operatorname{Re} f' \leq \frac{p+r(p-2\xi\alpha)}{1-r(2\xi-1)} \, r^{p-1}$$

Proof

By lemma 2.1 we know that f lies in a disc whose centre is

$$\frac{p+r^{2}(2p\xi-p-4\xi^{2}\alpha)}{1-(2\xi-1)^{2}r^{2}}$$

and has radius equal to

$$\frac{2\xi r(\alpha(1+r)-p}{1-(2\xi-1)^{2}r^{2}}$$

The real axis intersects the disc having end points

$$\frac{p-(p-2\xi r)r}{1-(2\xi-1)r} \, r^{p-1} \quad \text{and} \quad \frac{p+r(p-2\xi\alpha)}{1-(2\xi-1)} \, r^{p-1}$$

This makes possible for us to write down

$$\frac{p-r(p+2\xi\alpha)}{1+r(1-2\xi)}\, r^{p-1} \leq \operatorname{Re} f' \leq \frac{p+r(p-2\xi\alpha)}{1-r(1-2\xi)}\, r^{p-1}$$

(III) We obtain this result from (1) by integrating

$$|f(z)| \leq \int_0^{|z|} |f'(te^{i\theta})|\, dt \leq \int_0^{|z|} \frac{p+t(p-2\xi\alpha)}{1-t(2\xi-1)}\, t^{p-1}\, dt$$

Proof Using (1) and integrating we obtain

$$|f(z)| \leq \int_0^{|z|} |f'(te^{i\theta})|\, dt \leq \int_0^{|z|} \frac{p+t(p-2\xi\alpha)}{1-t(2\xi-1)}\, t^{p-1}\, dt$$

$$\int_0^{|z|} \frac{p+t(p-2\xi\alpha)}{1-t(2\xi-1)}\, t^{p-1}\, dt = \int_0^{|z|} \frac{p+t(p-2\xi\alpha)}{(1-t(2\xi-1)t^{1-p}}\, dt$$

We get

$$\frac{p+t(p-2\xi\alpha)}{1-t(2\xi-1)\, t^{1-p}} = \frac{p}{t^{1-p}} + \frac{(2\xi-1)^{-p}(2\xi(p-\alpha))}{1-t(2\xi-1)}$$

integrating

$$\int_0^{|z|} \frac{p+t(p-2\xi\alpha)}{1-t(2\xi-1)t^{1-p}}\, dt = \int_0^{|z|} \frac{p}{t^{1-p}}\, dt + \int_0^{|z|} \frac{(2\xi-1)^{-p}(2\xi(p-\alpha))}{1-t(2\xi-1)}\, dt$$

$$\int_0^{|z|} \frac{p+t(p-2\xi\alpha)}{1-t(2\xi-1)t^{1-p}}\, dt \leq |z| - \frac{p\,(2\xi-1)^{-p}(2\xi(p-\alpha))}{(2\xi-1)} \log(1-|z|(2\xi-1))$$

(IV) Using (II) and integrating we obtain

$$|f(z)| \geq \int_{o}^{|z|} \operatorname{Re}(f'(t))\,dt \geq \int_{o}^{|z|} \frac{p-t(p-2\xi\alpha)}{1+t(2\xi-1)\,t^{1-p}}\,dt$$

$$\therefore \int_{o}^{|z|} \frac{p-t(p-2\xi\alpha)}{1+t(2\xi-1)\,t^{1-p}}\,dt = |z|^{p} - \frac{(2\xi-1)^{-p}\,(2\xi(p-\alpha))}{2\xi-1}\log\left(1+|z|(2\xi-1)\right)$$

$$\therefore f(z) \geq \int_{o}^{|z|} \operatorname{Re}(f'(t))\,dt \geq \int_{o}^{|z|} \frac{p-t(p-2\xi\alpha)}{1+t(2\xi-1)\,t^{1-p}}\,dt$$

$$= |z|^{p} - \frac{(2\xi-1)^{-p}\,(2\xi(p-\alpha))}{(2\xi-1)}\log\left(1+|z|(2\xi-1)\right)$$

III) Functions with Negative coefficients

We carry out the investigation that are very similar to our consideration of section - I. We recall the definition of T mentioned therein. The family $P^*p(\alpha,\xi) = T \cap Dp(\alpha,\xi)$ gives us the beautiful results not found in the literature. We begin with a characterisation of members of $P^*p(\alpha,\xi)$

Theorem 3.1

A holomorphic function

$$f(z) = z^p - \sum_{n=p+1}^{\infty} |an|\, z^n \text{ is in } P^*p(\alpha,\xi) \text{ if and only if}$$

$$\sum_{n=p+1}^{\infty} n\,|an|\, 2\xi \leq 2\xi\,(p-\alpha)$$ this result is sharp.

Proof

Let $|z| = 1$ then

$$|f' - pz^{p-1}| - |2\xi(f' - \alpha z^{p-1}) - (f' - pz^{p-1})|$$

$$= \left| -\sum_{n=p+1}^{\infty} n|\,an\,|z^{n-1} \right| - \left| 2\xi\,(p - \sum_{n=p+1}^{\infty} n\,|\,an\,| - \alpha) + \sum_{n=p+1}^{\infty} n|an| \right|$$

$$\leq \sum_{n=p+1}^{\infty} n|an|\,(1+2\xi-1) - (2\xi p + 2\xi\,\alpha)$$

≤ 0 by hypothesis .

Thus by maximum modules theorem we have $f \in P^*p(\alpha\,\xi)$. For the converse let us assume that

$$\left| \frac{f' - pz^{p-1}}{2\xi(f' - \alpha z^{p-1}) - (f' - pz^{p-1})} \right|$$

$$\left| \frac{-\sum_{n=p+1}^{\infty} n|a_n| z^{n-1}}{2\xi z^{p-1}(p-\alpha) - \sum_{n=p+1}^{\infty} n|a_n| z^{n-1}(2\xi-1)} \right| < 1$$

for $|z|<1$ since $|Re(z)| \leq |z|$ for all z we have

$$Re\left\{ \frac{\sum_{n=p+1}^{\infty} n|a_n| z^{n-1}}{2\xi z^{p-1}(p-\alpha) - \sum_{n=p+1}^{\infty} n|a_n| z^{n-1}(2\xi-1)} \right\} \leq 1$$

for $|z|<1$ we choose the values of z on the real axis so that f' (z) is real simplifying denominater in the above expression and letting $z \to 1$ through real values we obtain

$$\sum_{n=p-1}^{\infty} n|a_n| \leq 2\xi(p-\alpha) - \sum_{n=p+1}^{\infty} n|a_n| (2\xi-1)$$

and we get the required condition. We obtain bounds on | f | and | f' | in the following theorem

Theorem 3.2

If $f \in P^*p(\alpha,\xi)$ then for $|z| = r$ $(0 < r < 1)$ we have

(I) $r^p - r^2 \left(\frac{p-\alpha}{n}\right) \leq |f(z)| \leq r^p + r^2 \left(\frac{p-\alpha}{n}\right)$

(II) $1 - r(p-\alpha) \leq |f'(z)| \leq 1+r(p-\alpha)$

Proof From theorem 3 1 we have

$$\sum_{n=p+1}^{\infty} |a_n| \leq \frac{2\xi(p-\alpha)}{2n\xi}$$

$\therefore$ equivalently $$\sum_{n=p+1}^{\infty} |a_n| \le \frac{(p-\alpha)}{n}$$

Hence $$|f(z)| \le r^p + r^2 \left(\frac{(p-\alpha)}{n}\right)$$

similarly
$$|f(z)| \ge r^p - r^2 \left(\frac{p-\alpha}{n}\right)$$
$$\therefore r^p - r^2 \left(\frac{p-\alpha}{n}\right) \le |f(z)| \le r^p + r^2 \left(\frac{p-\alpha}{n}\right)$$

(ii) we have

$$\sum_{n=p+1}^{\infty} n\,|a_n| \le \frac{2\xi(p-\alpha)}{2\xi}$$

$$\sum_{n=p+1}^{\infty} |a_n| \le \frac{p-\alpha}{n}$$

Now $$f(z) = z^p - \sum_{n=p+1}^{\infty} |a_n|\, z^n$$

$$f'(z) = pz^{p-1} - \sum_{n=p+1}^{\infty} n\,|a_n|\, z^{n-1}$$

$$|f'(z)| = \left| pz^{p-1} - \sum_{n=p+1}^{\infty} n\,|a_n|\, z^{n-1} \right|$$

$$= p\, r^{p-1} + r^{n-1} (p-\alpha)$$

n=2 , p=1

Also

$$|f'(z)| \geq 1 - r^{n-1} \sum_{n=p+1}^{\infty} n\,|a_n|$$

$$1 - r\,(p-\alpha) \leq |f'(z)| \leq 1 + r\,(p-\alpha)$$

Our next concern lies with the problem of determining the radius of convexity for the members of $P^*p(\alpha,\xi)$.

Theorem 3.3

If $f \in P^*p(\alpha,\xi)$ then f is convex in the disc $|z| < r = r(\alpha,\xi)$ where

$$r(\alpha,\xi) = \operatorname{Inf}_n \left[\frac{1}{n(p-\alpha)} \right]^{1/n-p} \qquad n = 2,3\ldots\ldots\ldots\ldots$$

The result is sharp and the external function is

$$f(z) = z^p - \left[\frac{p-\alpha}{n} \right] z^n$$

Proof It is enough to show that $|zf''/f'| < 1$ for $z| < 1$
first we note that

$$|zf''/f'| \leq \frac{\{ p(p-1) - \sum_{n=p+1}^{\infty} n(n-1)\,|a_n|\, z^{n-p} \}}{\{ p - \sum_{n=p+1}^{\infty} |a_n|\, n\, z^{n-p} \}}$$

The conclusion follows provided that

$$\left| p^2 - p - \sum_{n=p+1}^{\infty} n^2 |a_n| z^{n-p} + \sum_{n=p+1}^{\infty} n |a_n| z^{n-p} \right| \leq \left| p - \sum_{n=p+1}^{\infty} n |a_n| z^{n-p} \right|$$

This reduces after simplification to

$$\sum_{n=p+1}^{\infty} n^2 \ |a_n| \ z^{n-p} \leq p^2$$

By theorem 3.1 we have

$$\sum_{n=p+1}^{\infty} n \, |a_n| \, 2\xi \leq 2\xi \, (p-\alpha)$$

Hence f is convex if

$$n^2 \ |z|^{n-p} \leq \frac{2n\xi}{2\xi(p-\alpha)} \qquad n = 2,3,\ldots\ldots\ldots\ldots\ldots$$

$$|z| \leq \left\{ \frac{2\xi}{2n\xi(p-\alpha)} \right\}^{1/n-p} \qquad n = 2,3,\ldots\ldots\ldots\ldots\ldots$$

$$|z| \leq \left\{ \frac{1}{n(p-\alpha)} \right\}^{1/n-p} \qquad n = 2,3,\ldots\ldots\ldots\ldots$$

This complete the proof.

We explicitly show that the family $P^*p\,(\alpha,\xi)$ is closed under the formation of Arithmetic means.

Theorem 3.4

If $f(z) = z^p - \sum_{n=p+1}^{\infty} |a_n| z^n$ and

$g(z) = z^p - \sum_{n=p+1}^{\infty} |b_n| z^n$ are in $P^*p(\alpha, \xi)$

Then $h(z) = z^p - 1/2 \sum_{n=p+1}^{\infty} |a_n + b_n| z^n$

is also in $P^*p(\alpha, \xi)$

Proof

Since f and g are in $P^*p(\alpha, \xi$ we have

$$\sum_{n=p+1}^{\infty} n |a_n| 2\xi \leq 2\xi \ (p-\alpha)$$

and $\sum_{n=p+1}^{\infty} n |b_n \ 2\xi \leq 2\xi \ (p-\alpha)$

for h to be a member of $P^*p(\alpha, \xi)$ it is enough to show that

$$\frac{1}{2} \sum_{n=p+1}^{\infty} \{ n \, 2\xi \ |a_n + b_n| \} \leq 2\xi \ (p-\alpha)$$

which follows immediately by the use of above two inequalities therefore

$$h(z) = z^p - \sum_{n=p+1}^{\infty} \left| \frac{a_n + b_n}{2} \right| z^n$$

is also in $P^*p(\alpha,\xi)$

Finally we show that the convex linear combination of members of $P^*p(\alpha,\xi)$ is again a member of $P^*p(\alpha,\xi)$. We show that the family $P^*p(\alpha,\xi)$ is closed under the formation of convex linear combination.

Theorem 3.5

Let

$$fn(z) = \left(\frac{1-\alpha}{n}\right) z^n \qquad n = 2,3,\ldots\ldots\ldots\ldots$$

Then $f \in P^*p(\alpha,\xi)$ if and only if it can be expressed in the form

$$f(z) = z^p - \sum_{n=p+1}^{\infty} \lambda n \; fn(z)$$

where $\lambda n \geq 0$ ($n=1,2,\ldots\ldots$)

and $\sum_{n=p+1}^{\infty} \lambda n = 1$

__**Proof**__

Let us suppose that

$$f(z) = z^p - \sum_{n=p+1}^{\infty} \lambda n \; fn(z)$$

$$= z^p - \sum_{n=p+1}^{\infty} \left(\frac{1-\alpha}{n}\right) \lambda n \, z^n$$

Then $\sum_{n=p+1}^{\infty} \left\{ \frac{n}{1-\alpha} \lambda n \frac{1-\alpha}{n} \right\} \leq 1$

and by theorem 3.1 $f \in P^*p(\alpha,\xi)$

Conversely we suppose that $f \in P^*p(\alpha.\xi)$

By theorem - 3.1 we have $|a_n| \leq \dfrac{p-\alpha}{n}$ $\quad n = 2,3,\ldots\ldots$

setting $\lambda_n = \dfrac{n}{p-\alpha} |a_n|$ $\quad n = 2,3,\ldots\ldots$

then we have $\sum_{n=p+1}^{\infty} \lambda_n \leq 1 \qquad \lambda_n \geq 0$

$\therefore \lambda_1 = p z^{p-1} - \sum_{n=p+1}^{\infty} \lambda_n$ so that

we have $f(z) = z^p - \sum_{n=p+1}^{\infty} \lambda_n f_n(z)$

and the proof is complete.

SECTION - III

In this last section we introduce a class $\sigma p^*(\alpha,\beta,\xi) = \sigma p(\alpha,\beta,\xi) \cap T$ of meromorphic p-valent functions of the type

$$f(z) = \frac{1}{z^p} - \sum_{n=p}^{\infty} |a_n| z^n \quad \text{in punctured disc}$$

$$U^* = \{ z : 0 < |z| < 1 \}$$

and investigate same mapping properties of F(z) when F(z) is in $\sigma p^*(\alpha,\beta,\xi)$

$$\text{where} \quad F(z) = c \int_0^1 u^c f(uz)\, du \qquad c > 1$$

we also consider the converse problem.

INTRODUCTION

Let σp denote the class of functions $f(z) = \frac{1}{z^p} - \sum_{n=p}^{\infty} |a_n| z^n$

which are meromorphic and p-valent in the $U^* = \{z : 0 < |z| < 1\}$. We have introduced a subclass $\sigma p(\alpha,\beta,\xi)$ of σp that satisfies the following condition

$$\left| \frac{\frac{zf'(z)}{f(z)} + p}{2\xi\left(\frac{zf'(z)}{f(z)} + \alpha\right) - \left(\frac{zf'(z)}{f(z)} + p\right)} \right| < \beta,$$

$0 < \beta \leq p$ with appropriate restriction on α and ξ .

We have specialised our considerations for those members of $\sigma p(\alpha,\beta,\xi)$ that have negative coefficients.

Let T be the subclass of σp of meromorphic function in U* that have the power series representation of the form

$$f(z) = \frac{1}{z^p} - \sum_{n=p}^{\infty} |a_n| z^n$$

We have investigated some results for those meromorphic functions which are in both family $\sigma p(\alpha,\beta,\xi)$ and T Let $\sigma^* p(\alpha,\beta,\xi) = \sigma p(\alpha,\beta,\xi) \cap T$. The motivation to carry out such study arises from Kulkarni - Joshi [3] in particular we have studied mapping property of F(z) when f(z) is in $\sigma^* p(\alpha,\beta,\xi)$ where

$$F(z) = c \int_0^1 u^c f(uz)\, du \qquad c > 1$$

Theorem (1)

A function $f(z) = \frac{1}{z^p} - \sum_{n=p}^{\infty} |a_n| z^n$ is in $\sigma p^*(\alpha,\beta,\xi)$

if and only if $\sum_{n=p}^{\infty} [\, n+p+\beta\,(2\xi n + 2\xi\alpha - n - p)\,] \le 2\beta\xi(p-\alpha)$

<u>Proof</u> Let us

$$\sum_{n=p}^{\infty} [\, n+p+\beta\,(2\xi n + 2\xi\alpha - n - p)\,]\, |a_n| \le 2\beta\xi(p-\alpha)$$

since $|zf' + pf| - \beta\, |2\xi(zf' - \alpha f') - (zf' + pf)| < 0$

$$\left| \sum_{n=p}^{\infty} (n+p)\,|a_n|\, z^n \right| - \beta\, \left| -2\xi(p-\alpha)\, z^{-p} - \sum_{n=p}^{\infty} (n+\alpha)\,|a_n|\, z^n + \sum_{n=p}^{\infty} (n+p)\,|a_n| z^n \right| < 0$$

for $|z| = 0 < r < 1$ above expression is bounded above by

$$\sum_{n=p}^{\infty} (n+p)\,|a_n|\,r^n - 2\beta\xi(p-\alpha) + \beta\sum_{n=p}^{\infty} (2\xi\alpha+2n\xi-n-1)\,|a_n|\,r^n$$

$$= \sum_{n=p}^{\infty} [\,n+p+\beta(2\xi\alpha+2n\xi-n-1)]\,|a_n| - 2\beta\xi(p-\alpha) \leq 0$$

therefore $f(z) \in 6^* (\alpha,\beta,\xi)$.

Now we prove the result conversely

Let

$$\left| \frac{\dfrac{zf'}{f} + p}{2\xi\left(\dfrac{zf'}{f} + \alpha\right) - \left(\dfrac{zf}{f} + p\right)} \right| < \beta$$

$$\left| \frac{\sum_{p=1}^{\infty} (n+p)\,|a_n|\,z^n}{2\xi(p-\alpha) - \sum_{n=p}^{\infty} (2\xi\alpha + 2n\xi - n - p)\,|a_n|\,z^n} \right| < \beta$$

As $[\,\mathrm{Re}(z)\,] \leq |z|$ for all z we have

$$\mathrm{Re}\left\{ \frac{\sum_{n=p}^{\infty} (n-p)\,|a_n|\,z^n}{2\xi(p-\alpha) - \sum_{n=p}^{\infty} (2\xi\alpha + 2n\xi - n - p)\,|a_n|\,z^n} \right\} < \beta$$

choose values of z on real axis such that $\frac{zf'(z)}{f(z)}$ is real and clearing denominator of above expression and letting $z \to 1$ through real values we get

$$\sum_{n=p}^{\infty} (n+p)+\beta(2\xi\alpha+2n\xi-n-p)\,|a_n| \le 2\beta\xi(p-\alpha)$$

Theorem 2

Let $f(z) = \frac{1}{z^p} - \sum_{n=p}^{\infty} |a_n| z^n$ be in $\sigma p^*(\alpha,\beta,\xi)$

Then $f(z) = \frac{1}{c}\left[(c+1)f(z) + zf'(z)\right] \quad c > 0$

$$= \frac{1}{z^p} - \sum_{n=p}^{\infty} \frac{c+n+1}{c} |a_n| z^n$$

Belongs to $\sigma^*(\delta,\beta,\xi)$

for $0 < |z| < r = r(\alpha,\beta,\xi,\delta)$

where

$$r(\alpha,\beta,\xi,\delta) = \operatorname{Inf}_n \left[\frac{c2\beta\xi(p-\delta)\,[n-p+\beta(2\xi\alpha+2n\xi-n-1)]}{2\beta\xi(p-\alpha)\,[n+p-\beta(2\xi\delta+2n\xi-n-p)]\,(c+n+1)}\right]^{1/n+p}$$

$n = 1,2,3$

The result is sharp for the functions

$$f_n(z) = \frac{1}{z^p} - \frac{2\beta\xi(p-\alpha)}{[n+p+\beta(2\xi\alpha-2n\xi-n-p)]} z^n \qquad (n = 1, 2, 3, \ldots)$$

Proof It is sufficient to prove that

$$\left| \frac{zf' + fp}{\beta(2\xi(zf'+\delta p) - (zf'+fp)} \right| < 1$$

$$= \left| \frac{\sum_{n=p}^{\infty} (n+p) \frac{c+n+1}{c} |a_n| z^{n-p}}{\beta(2\xi(p-\delta) - \sum_{n=p}^{\infty} (2\xi\delta+2n\xi-n-p) \frac{c+n+1}{c} |a_n| z^{n-p}} \right|$$

$$\leq \frac{\sum_{n=p}^{\infty} (n+p) \frac{c+n+1}{c} a_n |z|^{n-p}}{2\beta\xi(p-\delta) - \beta\sum_{n=p}^{\infty} (2\xi\delta-2n\xi-n-p) |a_n| |z|^{n-p} \frac{c+n+1}{c}}$$

Last expression is bounded above by one if

$$\sum_{n=p}^{\infty} \frac{n+p+\beta(2\xi\delta+2n\xi-n-p)}{2\beta\xi(p-\delta)} \frac{c+n+1}{c} |a_n| |z|^{n-p} \leq 1 \quad \ldots\ldots(1)$$

By theorem (1) we have

$$\sum_{n=p}^{\infty} \frac{n+p+\beta(2\xi\alpha+2n\xi-n-p)}{2\beta\xi(p-\alpha)} |a_n| \leq 1$$

Hence (1) will be satisfied if

$$\frac{[n+p+\beta(2\xi\delta+2n\xi-n-p)]\,(c+n-1)}{2\beta\xi(p-\delta)c}\,|z|^{n-p} \le \frac{[n+p+\beta(2\xi\alpha+2n\xi-n-p)]}{2\beta\xi(p-\alpha)}$$

$n = 1,2,3,\ldots\ldots$

solving for $|z|$ we get

$$|z| < \left[\frac{c\,2\beta\xi(p-\delta)\,[\,n+p+\beta(2\xi\alpha-2n\xi-n-p)\,]}{2\beta\xi(p-\alpha)\,[\,n+p+\beta(2\xi\delta+2n\xi-n-p)\,]\,(c+n+1)}\right]^{1/n-p}$$

writting $|z| = r(\alpha,\beta,\xi,\delta)$ the desired result follows we note the following known case.

Theorem (3)

If $f(z) = \dfrac{1}{z^p} - \sum\limits_{n=p}^{\infty} |a_n| z^n$ belongs to $\sigma^*p(\alpha,\beta,\xi)$, $c > 0$

Then $f(z) = c\int_0^1 u^c f(uz)\,du = \dfrac{1}{z^p} - \sum\limits_{n=p}^{\infty} \dfrac{c}{c+n+1} |a_n| z^n$

belongs to

$$\sigma^*\left[\frac{p[1+p-2\beta\xi\alpha+2\beta\xi-\beta+2\beta\xi-\beta\,]-[\alpha c+\alpha cp+2\beta\xi\alpha c-\beta c\alpha-3cp\alpha+2\beta\xi\alpha cp+p^2-\beta p^2 c+2\beta\xi p\alpha-p^2\beta]}{[c+1+p+2\beta\xi c-\beta c+2\beta\xi\alpha+\beta p+2\beta\xi-\beta]+[pc+p+2\beta\xi\alpha c-\beta pc+2\beta\xi\alpha-\beta p+\beta pc2\xi-\beta c2\alpha\xi]}\right]$$

in $0 < |z| < 1$. The result is sharp for

$$f(z) = \frac{1}{z^p} - \frac{2\beta\xi(p-\alpha)}{[\,n+p+\beta(2\xi\alpha+2n\xi-n-p)]}\,z$$

Proof

Since we have $f(z) \in \sigma^* p(\alpha, \beta, \xi)$

$$\sum_{n=p}^{\infty} [\, n + p + \beta(2\xi\alpha + 2n\xi - n - p)] \; a_n \leq 2\beta\xi(p - \alpha)$$

Now we make use of theorem 1 with α replaced by σ we shall find out the largest value of σ for which

$$\sum_{n=p}^{\infty} \frac{c(n + p + \beta(2\xi\sigma + 2n\xi - n - p)}{2\beta\xi(p - \sigma)(c+n-1)} |a_n| \leq 1$$

It is sufficient to find the range of values of σ for which

$$\frac{c(n + p + \beta(2\xi\sigma + 2n\xi - n - p)}{2\beta\xi(p - \sigma)(c-n+1)} \leq \frac{(n - p + \beta(2\xi\alpha + 2n\xi - n - p)}{2\beta\xi(p - \alpha)}$$

for each n solving this we obtain

$$\therefore \sigma \leq \frac{n^2 p + np\,[1+p-2\beta\xi\alpha - 2\beta\xi - \beta - 2n\xi\beta - \beta n] + [\,\alpha cn + \alpha cp + \beta c 2n\xi\alpha - \beta cn\alpha - \beta cp\alpha + 2\beta\xi\alpha cp + p^2 - \beta p^2 c + 2\xi\alpha\beta p - p^2\beta]}{n^2 + n\,[c+1+p+2\beta\xi c - \beta c + 2\beta\xi\alpha - \beta p + 2\beta\xi - \beta] + [pc + p + 2\beta\xi\alpha c - \beta pc + 2\beta\xi\alpha - \beta p + \beta pc 2\xi - \beta c 2\xi\alpha]}$$

for each fixed (α, β, ξ) and c let

$$f(n) = \frac{n^2 p + np\,[1+p-2\beta\xi\alpha - 2\beta\xi - \beta - 2n\xi\beta - \beta n] + [\,\alpha cn + \alpha cp + \beta c 2n\xi\alpha - \beta cn\alpha - \beta cp\alpha + 2\beta\xi\alpha cp + p^2 - \beta p^2 c + 2\xi\alpha\beta p - p^2\beta]}{n^2 + n\,[c+1-p-2\beta\xi c - \beta c - 2\beta\xi\alpha - \beta p - 2\beta\xi - \beta] + [pc + p + 2\beta\xi\alpha c - \beta pc + 2\beta\xi\alpha - \beta p + \beta pc 2\xi - \beta c 2\xi\alpha]}$$

Then

$$f(n+1)-f(n) = \frac{(2n+1)p + p[2\beta\xi - \beta] + [\alpha c + 2\beta\xi\alpha c - \beta c\alpha]}{(n+1)^2 + (n+1)[c+1+p+2\beta\xi c-\beta c+2\beta\xi\alpha-\beta p+2\beta\xi-\beta] + [pc+p+2\beta\xi\alpha c-\beta pc+2\beta\xi\alpha-\beta p+\beta pc2\xi-\beta c2\xi\alpha]} > 0$$

Hence $f(n)$ is an increasing function of n since

$$f(1) = \frac{p[1+p-2\beta\xi\alpha+2\beta\xi-2\beta+2\beta\xi] + [\alpha c+\alpha cp+2\beta\xi\alpha c^2-\beta c\alpha-\beta cp\alpha+2\beta\xi\alpha cp+p^2-\beta pc^2+2\xi\alpha\beta p-p^2\beta]}{[c+1+p+2\beta\xi c-\beta c+2\beta\xi\alpha-\beta p-2\beta\xi-\beta] + [pc+p+2\beta\xi\alpha c-\beta pc+2\beta\xi\alpha-\beta p+\beta pc2\xi-\beta c2\xi\alpha]}$$

The result follows

The result is sharp for

$$f(z) = \frac{1}{z^p} - \frac{2\beta\xi(p-\alpha)}{[n+p+\beta(2\xi\alpha+2n\xi-n-p)]} z$$