C H A P T E R – I

"BASIC CONCEPTS AND PREREQUISITS"

1. INTRODUCTION

2. TYPES OF CONGRUENCES

3. THE STRESS ENERGY TENSOR FOR FERROFLUID

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1. INTRODUCTION :

This chapter consists of mathematical conventions, prerequisites and basic mathematical tools which are required for the development of dissertation work. Hence this topic do not carry any original research problems.

The parameters associated with time like and space like congruences along, with their properties are presented in section 2 . A system of differential equations emerging out of gravitational interactions compatible with ferrofluid system are stated in section 3. The section 4 speakes about the form of stress energy tensor for ferrofluid system. The last section comprises of the transportation properties of prescribed Tensor fields.

2. TYPES OF GONGRUENCES :

I) Time Like Congruence :

The time congruence is the family of time like curves which is defined in the form of parametric relations

$$X^{a} = X^{a}\ (\xi^{\nu}, s), \quad \nu = 1,2,3$$

Where ξ^{ν} are lagrangian Co-ordinates and s is the parameter along one of the curves of the congruence. Let u^{a} be the unit 4-velocity vector tangential to the world line which is defined as

$$u^{a} = \frac{dX^{a}}{ds},\ (\xi^{\alpha},\ \text{fixed}) .$$

with normalizing condition

$$u^a u_a = 1.$$

This provides the result

$$u_{a;b} u^a = 0. \qquad .....(2.1)$$

Where semicolen denotes the covariant differentiation.

The 3-Space Projection Operator :

The 3-diamensional projection operator h_{ab} defined as

$$h_{ab} = g_{ab} - u_a u_b. \qquad .....(2.2)$$

The properties of this operator are

$$h^c_a h^b_c = h^b_a, \; h^a_a = 3, \; h_{ab} = h_{ba},$$

$$h_{ab} u^a = h_{ab} u^b = 0.$$

Kinematical Parameters :

The kinematical parameters of the time like congruences are given below (Greenberg, 1970)

a) Expansion scalar θ :

$$\theta = u^a_{\ ;a} \qquad .....(2.3)$$

b) Shear Tensor field :

$$\sigma_{ab} = u_{(a;b)} - \dot{u}_{(a;b)} - 1/3\, \theta\, h_{ab}, \qquad .....(2.4)$$

c) Rotation Tensor field :

$$W_{ab} = u_{[a;b]} - \dot{u}_{[a} u_{b]}. \qquad .....(2.5)$$

Where ( ) bracket around the indices denotes symetrization, [ ] bracket indicates antisymmetrization where as term $\dot{u}_a = u_{a;b} u^b$ is the acceleration.

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The gradient of flow is decomposed in terms of these parameters as

$$u_{a;b} = \sigma_{ab} + W_{ab} + \dot{u}_a u_b + 1/3\,\theta\, h_{ab}. \quad .....(2.6)$$

The properties of shear and rotation tensors :

We have the following relations (from definition)

$$\sigma_{ab} = \sigma_{ba},\ \theta^a{}_a = \theta\ ,$$

$$\sigma_{ab}u^a = W_{ab}u^a = \theta_{ab}u^a = \dot{u}^a u_a = 0\ . \quad .....(2.7)$$

So also the shear and rotation tensor are u- orthogonal.

II) SPACE LIKE CONGRUENCE :

The space like congruence is described through the parametric relations

$$X^a = X^a(s),$$

Where $X^a = (X^0, X^1, X^2, X^3)$ are world co-ordinates and s is the parameter along the curve of the congruence. The unit vector tangent to the fixed curve of space like congruence is given by

$$h^a = \frac{dX^a}{ds}.$$

Where $h^a h_a = -1$, ...(2.8)

and $X^a = X^a(\xi^\nu, s)$. (ξ^i, constant)

This yields that

$$h_{a;b}h^a = 0. \quad .....(2.9)$$

The 2-Space Projection Operator :

We define the 2-space orthogonal projection operator p_{ab} in the form

$$p_{ab} = g_{ab} - u_a u_b + h_a h_b. \qquad \text{.....(2.10)}$$

This has the properties

$$p_{ab} = p_{ba}, \; p^a_c \, p^c_b = p^a_b \, , \; p^a_a = 2,$$

$$p_{ab} u^a = p_{ab} h^a = 0.$$

Parameters Associated With The Space Like Congruence : (Greenberg, 1970) :

We define the parameters associated with the unit space like congruence $\bar{h}$ as follows

a) Expansion scalar $\overset{*}{\theta}$:

$$\overset{*}{\theta} = h^a{}_{;a} - h_{a;b} u^a u^b. \qquad \text{.....(2.11)}$$

b) Shear tensor field :

$$\overset{*}{\sigma}_{ab} = p^c_a p^d_b h_{(c;d)} - 1/2 \; p_{ab} \overset{*}{\theta}. \qquad \text{.....(2.12)}$$

c) Rotation Tensor field :

$$\overset{*}{W}_{ab} = p^c_a p^d_b \; h_{[c;d]}. \qquad \text{.....(2.13)}$$

The unitary space like congruence h^a is subject to natural transport laws

$$u^a{}_{;b} h^b = h^a{}_{;b} u^b - u^a h_{b;c} u^b u^c + h^a h_{b;c} h^c u^b. \qquad \text{.....(2.14)}$$

Asgekar (1979). In ferrofluid system Neuringer and Rosensweig (1964) have reported the synthesis of a classical ferrofluid. Yodzis (1971) and Mason (1976) have describes the rate of growth of the magnetic energy density during the gravitational collapse. The generalised version of the scheme for ferrofluid is suggested by Cissoko (1978). According to Roy and Banerji (1980) there do not exist exact behaviour of the perfectly conducting ferrofluid at the last stage of gravitational collapse when very high magnetic fields and temperature are expected to be produced. They have also established the rate of growth of the magnetic energy density for the ferrofluid. The variation of the magnetic permeability accelerates the growth of the magnetic energy density.

The Stress-energy tensor for the ferrofluid is characterized by (Cissoko, 1978)

$$T_{ab} = (\rho + p + \mu H^2)\, u_a u_b - (P + 1/2\, \mu H^2)\, g_{ab} - \mu H_a H_b. \qquad .....(3.1)$$

Where ρ = The proper energy density,

p = The isotropic pressure,

μ = The variable magnetic permeability,

u^a = The 4- velocity vector,

H^a = The space like magnetic field vector.

The time like flow vector $\bar{u}$ and space like magnetic field vector $\bar{H}$ satisfied the conditions

$$u^a H_a = 0, \; H^a H_a = -H^2. \qquad .....(3.2)$$

REMARK : If μ is constant then we recover the Lichnerowicz's magnetofluid system.

It follows from Einstien's field equations

$$R_{ab} = - K \, [T_{ab} - 1/2 \quad T \quad g_{ab}],$$

$$\text{i.e., } R_{ab} = - K \, [(\rho + p + \mu H^2) \, u_a u_b - (p + 1/2 \, \mu H^2) g_{ab} -$$

$$- \mu H_a H_b - 1/2 \, (\rho - 3p) \, g_{ab}], \quad \text{(vide (3.1))}$$

$$\text{i.e., } R_{ab} = - K \, [(\rho + p + \mu H^2) \, u_a u_b - 1/2 \, (\rho - p + \mu H^2) g_{ab} -$$

$$- \mu H_a H_b]. \qquad .....(3.3)$$

This is the Ricci Tensor expression for ferrofluid system.

4. MAXWELL'S EQUATIONS :

For ferrofluid system the conductivity is infinite and hence the ideal approximation that electric field vanishes seems to be resonable. Hence the only valid set of Maxwell's equation is given by

$$[\mu \, (H^a u^b - H^b u^a)]_{;b} = 0, \qquad .....(4.1)$$

$$\text{i.e., } \mu \, (H^a{}_{;b} u^b + H^a u^b{}_{;b} - u^a{}_{;b} H^b - u^a H^b{}_{;b}) +$$

$$+ \mu_{;b} (H^a u^b - u^a H^b) = 0. \qquad .....(4.2)$$

On contracting this equation with H_a and noting that relations $H^a u_a = 0$, $H^a H_a = - H^2$ and $u^a{}_{;a} = \theta$ we get

$$\mu[1/2(\dot{H})^2 + H^2\theta + u_{a;b} H^a H^b] + \dot{\mu} H^2 = 0 \quad \ldots\ldots(4.3)$$

Again transverting equation (4.2) with u_a we get after simplification

$$\mu(u_a H^a{}_{;b} u^b - u_a u^a{}_{;b} H^b - H^b{}_{;b}) - \mu_{;b} H^b = 0,$$

i.e., $\mu(u_a H^a{}_{;b} u^b - H^b{}_{;b}) - \mu_{;b} H^b = 0,$ (Since $u^a{}_{;b} u_a = 0$)

i.e., $\mu(-u_{a;b} H^a u^b - H^b{}_{;b}) - \mu_{;b} H^b = 0.$

Hence finally we get

$$\mu\, \dot{u}_a H^a + (\mu H^b)_{;b} = 0 \quad \ldots\ldots(4.4)$$

This implies that $\mu_{;b} H^b = 0$ if $\dot{u}_a H^a = H^b{}_{;b} = 0$.

Hence we conclude in particular that μ is preserved along $\bar{H}$ if the divergence free magnetic lines are normal to acceleration vector.

Equation of Continuity :

The equation of continuity is provided by the local energy balance equation

$$T^{ab}{}_{;b} = 0, \quad \ldots\ldots(4.5)$$

i.e., $(\rho + p + \mu H^2)_{;b} u^a u^b + (\rho + p + \mu H^2) u^a{}_{;b} u^b +$

$+ (\rho + p + \mu H^2) u^a u^b{}_{;b} - (p + 1/2\, \mu H^2)_{;b} g^{ab} -$

$- \mu_{;b} H^a H^b - \mu H^a{}_{;b} H^b - \mu H^a H^b{}_{;b} = 0,$ (vide 3.1)

$\ldots\ldots(4.6)$

Hence $u_a T^{ab}{}_{;b} = 0$

i.e., $(\rho + 1/2\,\mu H^2)_{;b} u^b + (\rho + p + \mu H^2)\,\theta - \mu H^a{}_{;b} H^b u_a = 0.$ .....(4.7)

On using the relation (4.3) the equation (4.7) reduces to

$$\dot{\rho} + (\rho + p)\,\theta - 1/2\,\dot{\mu} H^2 = 0. \qquad .....(4.8)$$

This is the equation of continuity.

To write the equation of Streamlines we utilize the continuity equation (4.8) in the conservation equation (4.6). So that after simplification we get

$$(\rho + p + \mu H^2)\dot{u}^a + (p + 1/2\,\mu H^2)_{;b} h^{ab} -$$

$$-(\mu H^b)_{;b} H^a = 0. \qquad .....(4.9)$$

This equation exhibits the factors causing the deviation of path lines from geodesic path

On transvecting equation (4.6) with H_a We get after simplification

$$(\rho + p + \mu H^2)\,\dot{u}^a H_a - (p + 1/2\,\mu H^2)_{;b} H^b +$$

$$+ 1/2\,\mu (H^2)_{;b} H^b + (\mu H^b)_{;b} H^2 = 0. \qquad .....(4.10)$$

5. SOME TRANSPORT EQUATIONS :

a) **Lie-Transport** : The Lie derivative to tensor field along the time like congruence is given by (Trautman, 1964)

$$\underset{u}{\pounds} X_{ab} = \dot{X}_{ab} + X_{ac}u^{c}{}_{;b} + X_{cb}u^{c}{}_{;a}. \qquad .....(5.1)$$

The tensor field X_{ab} is said to be Lie transported along the time like congruence u^{a} iff

$$\underset{u}{\pounds} X_{ab} = 0. \qquad .....(5.1)$$

This is the most significant transport, since it is independent of the affine connections. Lie-transport along the space-like congruence $\bar{h}$ can similarly be written in the form

$$\underset{h}{\pounds} X_{ab} = X'_{ab} + X_{ac}h^{c}{}_{;b} + X_{cb}h^{c}{}_{;a} \qquad .....(5.2)$$

b) Jaumann Transport : The concept of Jaumann Transport is introduced by (Katkar et, al., 1981) through the Jaumann derivative of tensor field in the form

$$\underset{u}{J} X_{ab} = \dot{X}_{ab} + X_{ac}W^{c}{}_{.b} + X_{cb}W^{c}{}_{.a}. \qquad .....(5.3)$$

c) Truesdell Transport : A Truesdell derivative of a tensor field X_{ab} along the time like congruence is defined as (Walwadkar, 1983)

$$\underset{u}{T} X^{ab} = \dot{X}^{ab} + X^{ac}u^{b}{}_{;c} + X^{cb}u^{c}{}_{;a} - X^{ab}\theta. \qquad .....(5.4)$$

Hence the tensor field X_{ab} is said to Truesduall trasported iff

$$\underset{u}{T} X_{ab} = 0. \qquad .....(5.4)$$

d) Convective-Transport : This transport is described by Ehlers (1973) through the Convective derivative of a tensor field X_{ab} measuring the change of the tensor relative to the body. The expression for the Convective derivative of a tensor field X_{ab} along the time like congruence is given by

$$\underset{u}{\overset{C}{}} X_{ab} = \dot{X}_{ab} + X_{cb}(u^{c}{}_{;a} - \dot{u}^{c}u_{a}) + + X_{ac}(u^{c}{}_{;b} - \dot{u}^{c}u_{b}). \qquad .....(5.5)$$

Accordingly the convective transport is defined as

$$\underset{u}{\overset{C}{}} X_{ab} = 0,$$

i.e., $\dot{X}_{ab} + X_{cb}(u^{c}{}_{;a} - \dot{u}^{c}u_{a}) + X_{ac}(u^{c}{}_{;b} - \dot{u}^{c}u_{b}) = 0.$

e) Fermi Transport : The Fermi derivative of a Tensor field X_{ab} is defined along the time like vector u^{a} as follows (Radhakrishna and Bhosale,1975)

$$\underset{u}{\overset{F}{}} X^{ab} = \dot{X}^{ab} + X^{cb}(u^{a}\dot{u}_{c} - \dot{u}^{a}u_{c}) + X^{ac}(u^{b}\dot{u}_{c} - - \dot{u}^{b}u_{c}). \qquad .....(5.6)$$

A tensor field X^{ab} is said to be Fermi transported if and only if

$$\underset{u}{\overset{F}{}} X^{ab} = 0$$