CHAPTER I

BASIC CONCEPTS AND SYSTEM EQUATIONS

1. INTRODUCTION :

The general idealization of space-time demands that it will be filled with relativistic perfect fluid described by well known stress-energy tensor (Eric A Lord, 1976),

$$T_{ab} = (\rho + P) u_a u_b - P g_{ab} . \qquad ---- \quad 1.1$$

Where ρ is matter energy density, P is the isotropic pressure and u_a is unit flow vector. Lichnerowicz (1967) has developed stress-energy tensor characterizing perfect fluid with infinite conductivity and constant magnetic permeability (μ).

According to Ray and Banarji (1980), in Ferrofluid the magnetic induction vector and polarization vector are linearly related and the magnetic permeability is variable quantity. They have considered the stress-energy tensor characterizing Ferrofluid with infinite electrical conductivity and variable magnetic permeability.

According to general article on physics of Ferrofluid by Mehta (1989), Ferrofluid is defined as the magnetically soft fluid. Here Ferrofluid means an infinitely conducting relativistic charged fluid with variable magnetic permeability. The Lichnerowicz's formalism deals with constant magnetic permeability where as the Ferrofluid deals with variable magnetic permeability.

The relativistic dust distribution (Eric A Lord, 1976) is used to discuss the physical implications of cosmological FRW models. On similar lines we want to study Relativistic Magneto-Dust Distribution with variable magnetic permeability. Moreover, the roll of magnetic field in the cosmological evolution of space-time will be given due importance.

This study is juncture of two streams. One of which is the roll of Conformal symmetry and its impact on the dynamical structure. The other one is to the restrict stress-energy tensor by Magneto-Dust Distribution and thereby to examine the geometrical properties of the space-time.

2. PRECURSORY NOTATIONS :

We mainly deal with four-dimensional manifold V_4 with Lorentzian metric of signature (-, -, -, +).

The various symbols used are as follows :

,	:	Partial derivative
;	:	Covariant derivative
$\dot{x}$	:	Covariant derivative of x with respect to time like vector.
$\underset{\xi}{L}$	:	Lie derivative along the vector ξ .

3. STRESS-ENERGY TENSOR FOR MAGNETO-DUST DISTRIBUTION :

A relativistic magnetohydrodynamical scheme consisting of a space-time filled with infinitely conducting charged fluid and infinite electrical conductivity and constant magnetic permeability $(\bar{\mu})$ characterized by stress-energy tensor is given as (Lichnerowicz, 1967)

$$T_{ab} = (\rho + P + \bar{\mu} h^2) u_a u_b - (P + \tfrac{1}{2} \bar{\mu} h^2) g_{ab} - \bar{\mu}\, h_a h_b \,, \quad ---- \quad 3.1$$

If the magnetic permeability is allowed to vary, then the new scheme (Cissoko, 1978), (Ray & Banarji, 1980) composed of infinitely conducting charged fluid and <u>variable</u> magnetic permeability (μ) is described as

$$T_{ab} = (\rho + P + \mu h^2)\, u_a u_b - (P + \tfrac{1}{2}\mu h^2)\, g_{ab} - \mu\, h_a h_b . \qquad \text{----} \quad 3.2$$

Where,

ρ : matter energy density

P : isotropic pressure

μ : variable magnetic permeability

u_a : time-like vector

h^a : space-like vector

$$u^a u_a = 1,\ h^a h_a = -h^2,\ u^a h_a = 0 \qquad \text{----} \quad 3.3$$

If this system (3.2) is free from isotropic pressure P, then we call it as Relativistic Magneto-Dust Distribution. This is presented by following stress-energy tensor

$$T_{ab} = (\rho + \mu h^2)\, u_a u_b - \tfrac{1}{2}\mu h^2\, g_{ab} - \mu\, h_a h_b . \qquad \text{----} \quad 3.4$$

Where μ is variable magnetic permeability. Through the dissertation this form (3.4) has been used.

4. EIGEN VALUES FOR (3.4) :

i) If we transvect (3.4) with u^a, we obtain :

$$T_{ab}\, u^a = (\rho + \mu h^2)\, u^a u_a u_b - \tfrac{1}{2}\mu h^2\, u^a g_{ab} - \mu\, u^a h_a h_b ,$$

i.e. $$T_{ab}\, u^a = (\rho + \mu h^2)\, u_b - \tfrac{1}{2}\mu h^2\, u_b , \qquad \because \text{vide (3.3)}$$

i.e. $$T_{ab}\, u^a = (\rho + \tfrac{1}{2}\mu h^2)\, u_b . \qquad \text{----} \quad 4.1$$

ii) Transvecting (4.1) with u^b, yields

$$T_{ab}\, u^a u^b = (\rho + \tfrac{1}{2}\mu h^2)\, u^b u_b ,$$

$$T_{ab}\, u^a u^b = (\rho + \tfrac{1}{2}\mu h^2) . \qquad \text{----} \quad 4.2$$

$\because u^b u_b = 1$

iii) On multiplying (3.4) with h^a, we obtain

$$T_{ab}\, h^a = (\rho + \mu h^2)\, h^a u_a u_b - \tfrac{1}{2}\mu h^2 h^a g_{ab} - \mu h^a h_a h_b ,$$

i.e. $T_{ab}\, h^a = -\tfrac{1}{2}\mu h^2 h_b + \mu h^2 h_b ,$ vide (3.3)

i.e. $T_{ab}\, h^a = \tfrac{1}{2}\mu h^2 h_b .$ ---- 4.3

iv) Further contracting (4.3) with h^b, gives

$$T_{ab}\, h^a h^b = \tfrac{1}{2}\mu h^2 h^b h_b ,$$

i.e. $T_{ab}\, h^a h^b = -\tfrac{1}{2}\mu h^4 .$ ---- 4.4

$\because h^b h_b = -h^2$

v) On multiplying (4.1) by h^b, we obtain

$$T_{ab}\, u^a h^b = (\rho + \tfrac{1}{2}\mu h^2)\, h^b u_b ,$$

i.e. $T_{ab}\, u^a h^b = 0$ ---- 4.5

$\because h^b u_b = 0$

vi) If we contract (3.4) with g^{ab}, yields

$T_{ab}\, g^{ab} = (\rho + \mu h^2)\, g^{ab} u_a u_b - \tfrac{1}{2}\mu h^2 g^{ab} g_{ab} - \mu\, g^{ab} h_a h_b ,$---- 4.6

Using $u^a u_a = 1$, $h^a h_a = -h^2$, $g^{ab} g_{ab} = 4$ in (4.6), we get

$$T_{ab}\, g^{ab} = \rho + \mu h^2 - 2\mu h^2 + \mu h^2 ,$$

i.e. $T_{ab}\, g^{ab} = T = \rho .$ ---- 4.7

Thus stress-energy tensor (3.4) involves two eigen vectors one of which is the time-like eigen vector u^a with eigen value $e_1 = \rho + \tfrac{1}{2}\mu h^2$ and the other is space-like eigen vector h^a with the eigen value

$e_2 =$ $= \frac{1}{2}\mu h^2$

Also, we have trace of (3.4), $T = \rho$ which represent active gravitational mass density for Magneto-Dust Distribution.

Note : Theorem (Hawking & Ellis, 1973)

"Any stress-energy tensor with distinct eigen values has distinct eigen vectors orthogonal to each other." This Theorem holds for Magneto-dust Distribution. ---- vide (4.2), (4.4)

5. *ENERGY CONDITIONS :*

According to Hawking & Ellis (1973), the stress-energy tensor (3.4) has to satisfy following energy conditions :

i) Weak energy conditions :

It is stated through

$$T_{ab}\, u^a u^b \geq 0 \qquad \text{---- } 5.1$$

i.e. $$\rho + \tfrac{1}{2}\mu h^2 \geq 0 \qquad \text{vide (4.2) ---- } 5.2$$

ii) Strong energy conditions :

This is given by

$$T_{ab}\, u^a u^b - \tfrac{1}{2} T \geq 0 , \qquad \text{---- } 5.3$$

i.e. $$\rho + \tfrac{1}{2}\mu h^2 - \tfrac{1}{2}\rho \geq 0 ,$$

i.e. $$\rho + \mu h^2 \geq 0 . \qquad \text{vide (4.2), (4.7) ---- } 5.4$$

iii) Dominant energy conditions :

$\rho + \tfrac{1}{2}\mu h^2 \geq 0$ and $(\rho + \tfrac{1}{2}\mu h^2)\, u_b$ which is time-like vector.

It justifies that stress-energy tensor (3.4) is physically transparent.

6. GEOMETRICAL SYMMETRIES :

Some definitions leading geometrical symmetries are stated below :

i) A group of conformal motions :

A one parameter group of continuous infinitesimal transformation described by

$$\bar{x}^a = x^a + \xi^a \delta t \qquad \text{----} \quad 6.1$$

is said to exhibit a group of conformal motions if

$$\underset{\xi}{L}\, g_{ab} = 2\,\psi\, g_{ab} \qquad \text{---} \quad 6.2$$

where ψ is scalar function of co-ordinates and $\underset{\xi}{L}$ is Lie derivative along vector ξ.

ii) A group of Conharmonic conformal motions :

(Abdussattar & Babita Dwivedi, 1998)

A group exhibited in equation (6.2) is said to exhibit a group of Conharmonic Conformal motions if

$$\underset{\xi}{L}\, g_{ab} = 2\,\psi\, g_{ab}\,,\ \psi_{;ab}\, g^{ab} = 0 \qquad \text{----} \quad 6.3$$

iii) Dynamical Inheritance :

The transformations (6.1) lead to Dynamical Inheritance if

$$\underset{\xi}{L}\, T_{ab} = 2\,\alpha\, T_{ab} \qquad \text{---} \quad 6.4$$

where α is scalar function of co-ordinates.

iv] A vector ξ is said to be conformal killing vector if $\xi_{a;b} + \xi_{b;a} = 2\psi g_{ab}$.

Note :

i) Equation (6.2) leads to group of isometric motions if $\psi = 0$

ii) Equation (6.2) leads to group of special Conformal motions

if $\psi_{;ab} = 0$, $\psi_{;a} \neq 0$.

iii) Equation (6.4) leads to Dynamical collineations if $\alpha = 0$.

7. FIELD EQUATIONS GOVERNING FERRO-FLUID :

We study the field equations that are necessary for describing the geometrodynamical features of Ferro-fluid. These are mainly Einstein field equations for gravitation and Maxwell field equations for electromagnetism. The coupled Einstein-Maxwell field equations are stated below :

We find Einstein field equations have the form,

$$R_{ab} - \tfrac{1}{2} R\, g_{ab} = -k\, T_{ab} \qquad \text{----} \quad 7.1$$

Where T_{ab} is stress-energy tensor.

These are 16 non-linear partial differential equations of which 10 are independent .

Transvecting (7.1) with g^{ab}, yields

$$R = k\, T \qquad \text{----} \quad 7.2$$

Hence from (7.2), (7.1) can be written as

$$R_{ab} - \tfrac{1}{2} k\, T\, g_{ab} = -k\, T_{ab} \ . \qquad \text{----} \quad 7.3$$

If we introduce cosmological constant Λ in (7.1), we have the following alternative form of gravitational field equations

$$R_{ab} - \tfrac{1}{2} R\, g_{ab} + \Lambda\, g_{ab} = -k\, T_{ab} \ . \qquad \text{----} \quad 7.4$$

We know that in empty space stress-energy tensor is zero. Therefore (7.1) reduces to

$$R_{ab} - \tfrac{1}{2} R\, g_{ab} = 0 \ .$$

And on contracting this with g^{ab} gives

$$R = 0$$

Equation (7.3) can be rearranged as

$$R_{ab} = -k\,(T_{ab} - \tfrac{1}{2} T\, g_{ab}) \qquad \text{---- } 7.5$$

Hence dynamical expression for Ricci tensor compatible with Ferro-fluid exhibited by (3.4) can be written as

$$R_{ab} = -k\,[\,(\rho + \mu h^2)\, u_a u_b - \tfrac{1}{2}\,\mu h^2 g_{ab} - \mu\, h_a h_b - \tfrac{1}{2} T\, g_{ab}\,], \qquad \text{---- } 7.6$$

i.e. $$R_{ab} = -k\,[\,(\rho + \mu h^2)\, u_a u_b - \tfrac{1}{2}\,\mu h^2 g_{ab} - \mu\, h_a h_b - \tfrac{1}{2} \rho\, g_{ab}\,], \quad \text{vide (4.7)} \qquad \text{---- } 7.7$$

i.e. $$R_{ab} = -k\,[\,(\rho + \mu h^2)\, u_a u_b - \tfrac{1}{2}\,(\rho + \mu h^2)\, g_{ab} - \mu\, h_a h_b\,]. \qquad \text{---- } 7.8$$

8. MAXWELL EQUATIONS :

Electromagnetic field has to satisfy the Maxwell equations that are valid for condition of infinite conductivity which are in the form of (Lichnerowicz, 1967),

$$[\,\mu\,(h^a u^b - u^a h^b)]_{;b} = 0 \qquad \text{---- } 8.1$$

i.e. $$\mu\,(h^a{}_{;b} u^b + h^a u^b{}_{;b} - u^a{}_{;b} h^b - u^a h^b{}_{;b}) + (h^a u^b - u^a h^b)\, \mu_{;b} = 0 . \qquad \text{---- } 8.2$$

On multiplying equation (8.2) with h_a, we obtain

$$\mu\,(h^a{}_{;b} h_a u^b + h^a h_a u^b{}_{;b} - u^a{}_{;b} h_a h^b - u^a h_a h^b{}_{;b}) + (h_a h^a u^b - u^a h_a h^b)\, \mu_{;b} = 0 , \qquad \text{---- } 8.3$$

If we put $u^a{}_{;a} = \theta$, $h^a h_a = -h^2$, $u^a h_a = 0$, $\mu = \dot{\mu}_{;b} u^b$ and $h^2{}_{;b} u^b = (h^2)^{\cdot}$ in equation (8.3), we get

$$\mu \left(h^a{}_{;b} h_a u^b - h^2 u^b{}_{;b} - u^a{}_{;b} h_a h^b \right) + (-h^2)\, \mu_{;b} u^b = 0 , \quad \text{----} \quad 8.4$$

i.e. $$\mu \left(h^a{}_{;b} h_a u^b - h^2 \theta - u^a{}_{;b} h_a h^b \right) - \dot{\mu} h^2 = 0 ,$$

i.e. $$\mu \left(h^2 \theta - h^a{}_{;b} h_a u^b + u^a{}_{;b} h_a h^b \right) + \dot{\mu} h^2 = 0 , \quad \text{----} \quad 8.5$$

i.e. $$\mu \left(h^2 \theta + \tfrac{1}{2} (h^2)_{;b} u^b + u^a{}_{;b} h_a h^b \right) + \dot{\mu} h^2 = 0 , \quad \text{----} \quad 8.6$$

$$\because h^a{}_{;b} h_a = - \tfrac{1}{2} h^2{}_{;b}$$

i.e. $$\mu \left(h^2 \theta + \tfrac{1}{2} (h^2)^{\cdot} + u^a{}_{;b} h_a h^b \right) + \dot{\mu} h^2 = 0 ,$$

i.e. $$\mu \left[h^2 \theta + \tfrac{1}{2} (h^2)^{\cdot} + u_{a;b} h^a h^b \right] + \dot{\mu} h^2 = 0 . \quad \text{----} \quad 8.7$$

Also contracting (8.2) with u_a and using $u^a h_a = 0$, $u^a{}_{;b} u_a = 0$, $h^a{}_{;b} u^b = \dot{h}^a$, we obtain

$$\mu \left(\dot{u}_b h^b + h^b{}_{;b} \right) + \mu_{;b} h^b = 0 . \quad \text{----} \quad 8.8$$

- **Conclusion** : The equation (8.8) implies that the magnetic permeability is conserved along the divergence free magnetic lines iff 4-acceleration is normal to magnetic lines.

9. EQUATIONS OF MOTIONS :

The well known contracted Bianchi identities provide the local conservation laws through the conservation equation $T^{ab}_{;b} = 0$. These equations for T^{ab} expression (3.4) yield the following equation :

$$\left[(\rho + \mu h^2) u^a u^b - \tfrac{1}{2} \mu h^2 g^{ab} - \mu h^a h^b \right]_{;b} = 0 , \quad \text{----} \quad 9.1$$

i.e. $(\rho + \mu h^2)_{;b} u^a u^b + (\rho + \mu h^2)[u^a{}_{;b} u^b + u^a u^b{}_{;b}]$

$- \frac{1}{2}[\mu_{;b} g^{ab} h^2 + \mu h^2{}_{;b} g^{ab}] - \mu_{;b} h^a h^b - \mu\, h^a{}_{;b} h^b$

$- \mu\, h^a h^b{}_{;b} = 0 .$ ---- 9.2

On contracting (9.2) with u_a, we get

$(\rho + \mu h^2)_{;b} u_a u^a u^b + (\rho + \mu h^2)[u^a{}_{;b} u_a u^b + u_a u^a u^b{}_{;b}]$

$- \frac{1}{2}[\mu_{;b} g^{ab} u_a h^2 + \mu h^2{}_{;b} g^{ab} u_a] - \mu_{;b} h^a u_a h^b - \mu\, h^a{}_{;b} u_a h^b$

$- \mu\, h^a u_a h^b{}_{;b} = 0$ ---- 9.3

If we use $u^a u_a = 1$, $u^a h_a = 0$, $\rho_{;b} u^b = \dot{\rho}$, $\mu_{;b} u^b = \dot{\mu}$ and $u^a{}_{;b} u_a = 0$ in (9.3), we obtain

$(\rho + \mu h^2)_{;b} u^b + (\rho + \mu h^2)(\theta)$

$- \frac{1}{2}[\dot{\mu} h^2 + \mu (h^2)^{\bullet}] - \mu\, h^a{}_{;b} u_a h^b = 0$ ---- 9.4

i.e. $\dot{\rho} + (\mu h^2)^{\bullet} + \rho\,\theta + \mu h^2 \theta - \frac{1}{2}\dot{\mu} h^2 - \frac{1}{2}\mu (h^2)^{\bullet} - \mu\, h^a{}_{;b} u_a h^b = 0$

---- 9.5

i.e. $\dot{\rho} + (\mu h^2)^{\bullet} + \rho\,\theta + \mu h^2 \theta - \frac{1}{2}(\mu h^2)^{\bullet} + \mu\, u_{a;b} h^a h^b = 0$, ---- 9.6

i.e. $\dot{\rho} + \rho\,\theta + \frac{1}{2}(\mu h^2)^{\bullet} + \mu h^2 \theta + \mu\, u_{a;b} h^a h^b = 0$, ---- 9.7

i.e. $\dot{\rho} + \rho\,\theta + \frac{1}{2}\dot{\mu} h^2 + \frac{1}{2}\mu (h^2)^{\bullet} + \mu h^2 \theta + \mu\, u_{a;b} h^a h^b = 0$, ---- 9.8

i.e. $\dot{\rho} + \rho\,\theta + \mu[h^2 \theta + \frac{1}{2}(h^2)^{\bullet} + u_{a;b} h^a h^b] + \frac{1}{2}\dot{\mu} h^2 = 0$, ---- 9.9

i.e. $\dot{\rho} + \rho\,\theta + (-\dot{\mu} h^2) + \frac{1}{2}\dot{\mu} h^2 = 0$, vide (8.7)

i.e. $\dot{\rho} + \rho\,\theta - \frac{1}{2}\dot{\mu} h^2 = 0 .$ ---- 9.10

This equation (9.10) is called continuity equation for Ferrofluid.

Note : If the flow lines are expansion free ($\theta = 0$) then the equation (9.10) implies that $\dot{\rho} = 0 \Leftrightarrow \dot{\mu} = 0$

Hence we have a claim,

The matter energy density is conserved along the expansion free flow iff the magnetic permeability is kept invariant along these lines.

Further multiplying equation (9.2) with h_a, we get

$$(\rho + \mu h^2)_{;b} h_a u^a u^b + (\rho + \mu h^2)[u^a{}_{;b} h_a u^b + h_a u^a u^b{}_{;b}]$$
$$- \tfrac{1}{2}[\mu_{;b} g^{ab} h_a h^2 + \mu h^2{}_{;b} g^{ab} h_a] - \mu_{;b} h^a h_a h^b$$
$$- \mu h^a{}_{;b} h_a h^b - \mu h^a h_a h^b{}_{;b} = 0, \quad \text{---- } 9.11$$

Using $u^a u_a = 1$, $u^a h_a = 0$, $h^a h_a = -h^2$ in (9.11), we have

$$(\rho + \mu h^2)[\dot{u}^a h_a] - \tfrac{1}{2}[\mu_{;b} h^b h^2 + \mu h^2{}_{;b} h^b]$$
$$+ h^2 \mu_{;b} h^b - \mu h^a{}_{;b} h_a h^b + \mu h^2 h^b{}_{;b} = 0, \quad \text{---- } 9.12$$

i.e. $$\rho \dot{u}^a h_a + \mu h^2 [\dot{u}^a h_a + h^b{}_{;b}] + \tfrac{1}{2} h^2 \mu_{;b} h^b - \tfrac{1}{2} \mu h^2{}_{;b} h^b$$
$$- \mu h^a{}_{;b} h_a h^b = 0, \quad \text{---- } 9.13$$

As $h^a{}_{;b} h_a = -\tfrac{1}{2} h^2{}_{;b}$, hence from (9.13), we have

$$\rho \dot{u}^a h_a + \mu h^2 [\dot{u}^a h_a + h^b{}_{;b}] + \tfrac{1}{2} h^2 \mu_{;b} h^b - \tfrac{1}{2} \mu h^2{}_{;b} h^b$$
$$+ \mu \tfrac{1}{2} h^2{}_{;b} h^b = 0, \quad \text{---- } 9.14$$

i.e. $$\rho \dot{u}^a h_a + [\mu(\dot{u}^b h_b + h^b{}_{;b}) + \tfrac{1}{2} \mu_{;b} h^b] h^2 = 0, \quad \text{---- } 9.15$$

i.e. $$\rho \dot{u}_b h^b + [-\mu_{;b} h^b + \tfrac{1}{2} \mu_{;b} h^b] h^2 = 0, \quad \text{vide (8.8)}$$

i.e. $$\rho \dot{u}_b h^b = \tfrac{1}{2} \mu_{;b} h^b h^2 . \quad \text{---- } 9.16$$

This implies that

$$\dot{u}_b \perp h^b \Leftrightarrow \mu_{;b}h^b = 0. \qquad \text{---- } 9.17$$

- **Conclusion :** The acceleration is normal to magnetic lines iff magnetic permeability is conserved along magnetic lines.