CHAPTER - III

FIXED POINTS OF GENERALIZED NONEXPANSIVE MAPPINGS IN GENERALIZED HILBERT SPACE

3.1 INTRODUCTION

Precupanu [36-39] studied the H-locally convex linear topological spaces (i.e., the locally convex spaces whose generating family of seminorms satisfies the parallelogram law). Hicks and Huffman [14] have considered such H-locally convex spaces for completeness and termed them as generalized Hilbert spaces (GHS) and they extended some fundamental results of Browder [3], Browder and Petryshyn [8], Z.Opial [32], et.al. for nonexpansive mappings in generalized Hilbert spaces (GHS). Further Mukherjee and T.Som [27] have studied generalized nonexpansive and contraction mappings in generalized Hilbert spaces and extended the result of Hicks and Huffman [14]. In this chapter we have proved the results concerned the convexity of fixed point set, demiclosedness of a mapping & construction of fixed points by using the generalized contraction mappings. The result thus obtained generalize those of Browder [3], Browder and Petryshyn [8], Z. Opial [32], Hicks and Huffman [14], Mukherjee and T.Som [27-28] et.al. Now we prove the result about convexity of fixed point set as follows :

Theorem (3.1.1) Let X be a Hausdorff H-locally convex space and T be a generalized contraction selfmapping of a convex subset C of X. Let F(T) be the nonempty fixed point set of T. Then F(T) is convex.

Proof : Let $x,y \in F(T)$ and $0 < t < 1$. Suppose $z=tx+(1-t)y$ and $z \notin F(T)$ i.e. $Tz \neq z$.

Now, using the parallelogram law and the property of T (1.412), we obtain

$$
\begin{aligned}
\rho^2(x-y) &= \rho^2(\,(x-Tz)-(y-Tz))\\
&\leqslant \rho^2(x-Tz) + \rho^2(y-Tz)\\
&= \rho^2(Tx-Tz) + \rho^2(Ty-Tz)\\
&\leqslant a_1\,\rho^2(x-z) + a_2\,\rho^2(Tz-x) + a_3\rho^2(Tx-z) +\\
&\quad + a_4\,\rho^2[(I-T)x-(I-T)z] + a_1\,\rho^2(y-z) +\\
&\quad + a_2\,\rho^2(Tz-y) + a_3\,\rho^2(Ty-z) +\\
&\quad + a_4\,\rho^2[(I-T)y-(I-T)z].\\
&= a_1\,\rho^2(x-z) + a_2\,\rho^2(x-Tz) + a_3\,\rho^2(x-z) +\\
&\quad + a_4\,\rho^2[(x-z)-(x-Tz)] + a_1\rho^2(y-z) +\\
&\quad + a_2\,\rho^2(Tz-y) + a_3\,\rho^2(y-z) +\\
&\quad + a_4\,\rho^2[(y-z)-(y-Tz)]\\
&\leqslant a_1\rho^2(x-z) + a_2\,\rho^2(x-Tz) + a_3\,\rho^2(x-z) +\\
&\quad + a_4\,[\,\rho^2(x-z) + \rho^2(x-Tz)] + a_1\,\rho^2(y-z) +\\
&\quad + a_2\,\rho^2(y-Tz) + a_3\,\rho^2(y-z) +\\
&\quad + a_4\,[\,\rho^2(y-z) + \rho^2(y-Tz)]\\
&= (a_1+a_4)\,[\,\rho^2(x-z) + \rho^2(z-y)] +\\
&\quad + (a_2+a_4)\,[\,\rho^2(x-Tz) + \rho^2(Tz-y)] +\\
&\quad + a_3[\,\rho^2(x-z) + \rho^2(z-y)\,]\\
&\leqslant (a_1+a_4)\,\rho^2(x-z+z-y) +\\
&\quad + (a_2+a_4)\,\rho^2(x-Tz+Tz-y) +\\
&\quad + a_3\,\rho^2(x-z+z-y)\\
&= (a_1+a_2+a_3+2a_4)\,\rho^2(x-y)\\
&< \rho^2(x-y),
\end{aligned}
$$

since $a_1 + a_2 + a_3 + 2a_4 < 1$. Thus $\rho^2(x-y) < \rho^2(x-y)$ and this contradiction implies that $z \in F(T)$ and $F(T)$ is convex. This completes the proof.

For the following result we need definition (1.4.12) and theorem (1.6.5) of Hicks and Huffman [14].

The result runs as follows :

Theorem (3.2.1) : Let C be a closed convex subset of a GHS X and T be a generalized contraction mapping of C into X. Then (I-T) is demiclosed.

Proof : Let $\{x_n\}$ be a sequence in C which is weakly convergent to an element x_o in C and let $x_o \in F(T)$. Let the sequence $\{x_n - Tx_n\}$ converge to an element y_o in X. Now, as T is generalized contraction mapping and X is a GHS, we obtain

$$\begin{aligned}
\rho^2(Tx_n-Tx_o) &\leqslant a_1 \rho^2(x_n-x_o) + a_2 \rho^2(Tx_o-x_n) + \\
&+ a_3 \rho^2(Tx_n-x_o) + a_4 \rho^2((I-T)x_n-(I-T)x_o) \\
&= (a_1+a_2) \rho^2(x_n-x_o) + a_3 \rho^2(Tx_n-Tx_o) + \\
&+ a_4 \rho^2(x_n-x_o)-(Tx_n-Tx_o) \\
&= (a_1+a_2) \rho^2(x_n-x_o) + a_3 \rho^2(Tx_n-Tx_o) + \\
&+ a_4[\rho^2(x_n-x_o) + \rho^2(Tx_n-Tx_o) - \\
&- 2\rho(x_n-x_o-Tx_n+Tx_o)] \\
&\leqslant (a_1+a_2+a_4) \rho^2(x_n-x_o) + \\
&+ (a_3+a_4) \rho^2(Tx_n-Tx_o)
\end{aligned}$$

or

$$\rho^2(Tx_n-Tx_o) \leqslant \frac{(a_1+a_2+a_4)}{(1-a_3-a_4)} \rho^2(x_n-x_o) \qquad \text{.....(3.2.2)}$$

But from the Definition(1.4.12) of T,

$$a_1 + a_2 + a_3 + 2a_4 < 1$$

or

$$a_1 + a_2 + a_4 < 1 - a_3 - a_4$$

or

$$\frac{(a_1 + a_2 + a_4)}{(1 - a_3 - a_4)} < 1.$$

Hence the above inequality (3.2.2) reduces to

$$\rho^2 (Tx_n - Tx_o) < \rho^2 (x_n - x_o)$$

or

$$\rho (x_n - x_o) > \rho (Tx_n - Tx_o).$$

Taking the limit inf. of both sides as $n \to \infty$, we have

$$\lim_{n \to \infty} \inf \rho (x_n - x_o) > \lim_{n \to \infty} \inf \rho (Tx_n - Tx_o)$$
$$= \lim \inf \rho (x_n - y_o - Tx_o).$$

Hence by applying Theorem (1.6.5), it follows that (I-T) is demiclosed and the proof is complete.

Remark (3.2.3) :

As immediate corollaries to our result (3.2.1) we have the Theorem (1.6.6) of Hicks and Huffman [14], Theorem (1.5.9) of Z.Opial[32].

3.3 :

A result concerning the construction of fixed points for generalized nonexpansive mappings (1.4.13) :

According to the definition (1.4.13), T is called generalized nonexpansive mapping if

$$\rho^2(Tx-Ty) \leqslant a_1\rho^2(x-y)+a_2\rho^2(Ty-x)+a_3\rho^2(Tx-y) + \\ + a_4\rho^2[(I-T)x-(I-T)y] \qquad .....(3.3.1)$$

For all $x,y \in C$ and $a_i \geqslant 0$, $i=1,2,3,4$ with $a_1 + a_2 + a_3 + 2a_4 \leqslant 1$. The above inequality (3.3.1) can be written as follows :

$$\rho^2(Tx-Ty) \leqslant a_1\rho^2(x-y)+a_2\rho^2(Ty-x)+a_3\rho^2(Tx-y) + \\ +a_4\rho^2(Tx-x) + a_4\rho^2(Ty-y) \qquad .....(3.3.2)$$

The prerequisites for our result are Theorem (1.6.1) of Hicks and Huffman [14] and (3.3.2)

Theorem (3.3.3) Let C be a closed, bounded, convex and weakly sequentially compact subset of a Hausdorff generalized Hilbert space X. Suppose $\{T_j\}$ be a sequence of generalized nonexpansive selfmappings of C with

$$\rho(T_jx-Tx) \to 0 \text{ as } j \to \infty \text{ for all } x \in C, \qquad .....(3.3.4)$$

where T is a generalized nonexpansive selfmapping of X. Then T has atleast one fixed point.

Proof : For $0 < j < 1$, let

$$T_j(x) = j\,T(x) + (1-j)\,V_o,$$

where V_o is a fixed point of C. Then the existence of a fixed point u_j for the generalized nonexpansive mapping T_j can easily be proved by following the proof of Theorem (1.6.2) of Hicks and Huffman [14]. Since C is weakly sequentially compact, the sequence $\{u_j\}$ has a subsequence $\{u_{j_k}\}$ such that $\{u_{j_k}\}$ converges weakly to a point u_o in C, i.e. $u_{j_k} \to u_o$ weakly as $k \to \infty$. Hence from (3.3.4) it follows that

$$\rho(Tv_k - T_k v_k) = \rho(Tv_k - v_k) \to 0 \text{ as } k \to \infty ,$$

where $v_k = u_{j_k}$

Now , T is generalized nonexpansive mapping, from (3.3.2)we have

$$\rho^2(Tv_k - Tu_o) \leqslant a_1\rho^2(v_k - u_o) + a_2\rho^2(Tu_o - v_k) + $$
$$+ a_3\rho^2(Tv_k - u_o) + a_4\rho^2(Tv_k - v_k) + a_4\rho^2(Tu_o - u_o). \quad .....(3.3.5)$$

Also

$$\rho^2(v_k - Tu_o) = \rho^2((Tv_k - Tu_o) - (Tv_k - v_k))$$
$$\leqslant \rho^2(Tv_k - Tu_o) + \rho^2(Tv_k - v_k). \quad ....(3.3.6)$$

substituting (3.3.5) in (3.3.6), we obtain

$$\rho^2(v_k - Tu_o) \leqslant \rho^2(Tv_k - v_k) + a_1\rho^2(v_k - u_o) + a_2\rho^2(Tu_o - v_k) +$$
$$+ a_3\rho^2(Tv_k - u_o) + a_4\rho^2(Tv_k - v_k) + a_4\rho^2(Tu_o - u_o)$$
$$= \rho^2(Tv_k - v_k) + a_1\rho^2(v_k - u_o) + a_2\rho^2(Tu_o - v_k) +$$
$$+ a_3\rho^2((v_k - u_o) - (v_k - Tv_k)) + a_4\rho^2(Tv_k - v_k) +$$
$$+ a_4\rho^2((v_k - u_o) - (v_k - Tu_o))$$
$$\leqslant \rho^2(Tv_k - v_k) + a_1\rho^2(v_k - u_o) + a_2\rho^2(Tu_o - v_k) +$$
$$+ a_3\rho^2(v_k - u_o) + a_3\rho^2(v_k - Tv_k) + a_4\rho^2(Tv_k - v_k) +$$
$$+ a_4\rho^2(v_k - u_o) + a_4\rho^2(v_k - Tu_o),$$

Equivalently

$$(1 - a_2 - a_4)\rho^2(v_k - Tu_o) \leqslant (1 + a_3 + a_4)\rho^2(Tv_k - v_k) + (a_1 + a_3 + a_4)\rho^2(v_k - u_o)$$

or

$$\rho^2(v_k - Tu_o) \leqslant \frac{(1 + a_3 + a_4)}{(1 - a_2 - a_4)}\rho^2(Tv_k - v_k) + \frac{(a_1 + a_3 + a_4)}{(1 - a_2 - a_4)}\rho^2(v_k - u_o) \quad .....3.3.7)$$

But from definition (1.4.13), it follows that

$$a_1 + a_2 + a_3 + 2a_4 \leqslant 1$$

or

$$\frac{(a_1 + a_3 + a_4)}{(1 - a_2 - a_4)} \leqslant 1 .$$

Hence (3.3.7) takes the form

$$\rho^2(v_k - Tu_o) - \rho^2(v_k - u_o) \leqslant \frac{(1+a_3+a_4)}{(1-a_2-a_4)} \rho^2(Tv_k - v_k) \quad \text{.....(3.3.8)}$$

Now , suppose that $u_o \neq Tu_o$. Then there exists $\alpha \in \Delta$ such that

$$\rho_\alpha(u_o - Tu_o) > 0 ,$$

By theorem (1.6.1), we have

$$\lim_k [\rho_\alpha^2(v_k - Tu_o) - \rho_\alpha^2(v_k - u_o)] = \rho_\alpha^2(Tu_o - u_o) > 0$$

Hence there exists j such that $k \geqslant j$ implies

$$\rho_\alpha^2(v_k - Tu_o) - \rho_\alpha^2(v_k - u_o) > 0$$

or

$$[\rho_\alpha(v_k - Tu_o) - \rho_\alpha(v_k - u_o)] . [\rho_\alpha(v_k - Tu_o) + \rho_\alpha(v_k - u_o)] > 0$$

Thus, for $k \geqslant j$, from (3.3.8) we obtain

$$0 < \rho_\alpha^2(v_k - Tu_o) - \rho_\alpha^2(v_k - u_o) \leqslant \frac{(1+a_3+a_4)}{(1-a_2-a_4)} \rho^2(Tv_k - v_k)$$

$$\rightarrow 0 \text{ as } k \rightarrow \infty .$$

Hence

$$0 < \lim_k [\rho_\alpha^2(v_k - Tu_o) - \rho_\alpha^2(v_k - u_o) = \rho_\alpha^2(Tu_o - u_o) \leqslant 0$$

which is a contradiction. Hence we must have $Tu_0 = u_0$ and the assertion of the theorem is proved.

Remark (3.3.9) : Several results may be seen to follow as immediate corollaries to theorem (3.3.3). Some of them are as follows :

Theorem (1.6.2) of Hicks and Huffman [14], Theorem (1.5.1) of Browder and Petryshyn [8], Theorem (1.6.3) of Mukherjee and T.Som [27-28].