CHAPTER - II

COMMAN FIXED POINTS OF GENERALIZED CONTRACTION MAPPINGS IN HILBERT SPACE

2.1 INTRODUCTION :

Ishikawa [18] defined an iteration scheme (1.2.19) and proved that the sequence of Ishikawa iterates for a Lipschitzian pseudo-contractive mapping in a convex compact subset of Hilbert space must converge to a fixed point of this mapping. As a sequal to this research work Rhoades [44], Hicks and Kubicek [15], Naimpally and Singh [29] studied the convergence of sequences of Ishikawa iterates for various mappings. Further Liu Qihou [40-42] solved some open problems putforth by Naimpally and Singh [29] and extended their results. Also Das and Debata [12] have extended and generalized the results of Ishikawa [18] and others simultaneously by considering a more generalized iteration scheme involving a family of maps and secondly by taking less restrictive hemicontractive maps. Pathan [33] has extended the result of Das and Debata [12] and obtained the common fixed points of a family of more restrictive strictly pseudocontractive mappings. In this chapter we have obtained the common fixed points of a family of Lipschitzian generalized contraction mappings by considering the generalized contraction mapping and generalization of Ishikawa iteration scheme. Lastly we have established the generality of our result. The prerequisite for our investigations are (1.2.7), (1.2.10), (1.5.12), (1.5.14) and (1.5.15). Our result is as follows :

Theorem (2.1) : Let C be a convex and compact subset of a Hilbert space H and let $\{ T_i \}_{i=1}^{k}$ be a family of generalized contractive selfmappings of C and have atleast one common fixed point in C Let the sequence $\{ x_n \}$ in C be defined by

$x_1 \in E$

$$x_{n+1} = (1-\alpha_n)\, x_n + \alpha_n T_k\, u_{k-1}(n) \qquad .....(2.1.1)$$

where

$$u_0(n) = x_n \,,\; u_i(n) = (1-\beta_n)\, x_n + \beta_n T_i\, u_{i-1}(n) \qquad .....(2.1.2)$$

for $i = 1,2,\ldots, k$ and $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $[0,1]$ such that

i) $0 \leqslant \alpha_n \leqslant \beta_n \leqslant 1$, for $n = 1,2,\ldots$

ii) $\lim\limits_{n\to\infty} \beta_n = 0$

iii) $\sum\limits_{n=1}^{\infty} \alpha_n \beta_n^{k-1} = \infty$ for each $k \geqslant 2$.

Let the family of maps $\{T_i\}$ satisfy

$$\| T_i x - T_j y \| \leqslant L \| x - y \|$$

for all x , y in C and all pairs (i,j), L being a positive constant. Then the sequence $\{x_n\}$ converges to a common fixed point of the family of maps $\{T_i\}$.

Proof : According to the hypothesis, let p be a common fixed point of the family $\{T_i\}$. Since each T_i is a generalized contraction, we have for $i=1,2,\ldots,k$ and for any x,y in C , there exist constant a_j , $a_j \geqslant 0$ and $\sum\limits_{j=1}^{4} a_j < 1$ such that

$$\|T_i x - T_i y\|^2 \leqslant a_1\| x - y\|^2 + a_2\| T_i y - x \|^2 + a_3\|T_i x-y\|^2 + a_4\|(I-T_i)x-(I-T_i)y\|^2. \qquad .....(2.1.3)$$

Also by hypothesis

$$\| T_i x - T_j y \| \leqslant L \| x - y \|. \qquad .....(2.1.4)$$

For any x,y , z in a Hilbert space and a real number λ , the Ishikawa identity is

$$\| \lambda x+(1-\lambda)y-z\|^2 = \lambda \|x-z\|^2 + (1-\lambda)\| y-z\|^2 - \lambda(1-\lambda)\|x-y\|^2. \qquad .....(2.1.5)$$

Using (2.1.5) and (2.1.1) we have

$$\begin{aligned} \|x_{n+1}-p\|^2 &= \|\alpha_n T_k u_{k-1}(n) + (1-\alpha_n)x_n - p\|^2 \\ &= \alpha_n \|T_k u_{k-1}(n)-p\|^2 + (1-\alpha_n)\|x_n - p\|^2 - \\ &\quad - \alpha_n(1-\alpha_n)\|T_k u_{k-1}(n) - x_n\|^2. \end{aligned} \qquad .....(2.1.6)$$

Since each T_i is a selfmap on C and C is convex set, $u_i(n)$ and $T_i u_{i-1}(n)$ are in C for each i and n, then we have the following system of relations for i=1,2,...,k-1. Using (2.1.2) and (2.1.5) we obtain

$$\begin{aligned} \|u_{k-i}(n)-p\|^2 &= \|\beta_n T_{k-i} u_{k-i-1}(n)+(1-\beta_n)x_n-p\|^2 \\ &= \beta_n \|T_{k-i} u_{k-i-1}(n)-P\|^2 + (1-\beta_n)\|x_n - P\|^2 - \\ &\quad -\beta_n(1-\beta_n) \|T_{k-i} u_{k-i-1}(n)-x_n\|^2 \end{aligned} \qquad .....(2.1.7)$$

and

$$\begin{aligned} \|u_{k-i}(n)-T_{k-i+1} u_{k-i}(n)\|^2 &= \| \beta_n T_{k-i} u_{k-i-1}(n)+ \\ &\quad +(1-\beta_n)x_n-T_{k-i+1} u_{k-i}(n)\|^2 \\ &= \beta_n \|T_{k-i}u_{k-i-1}(n)-T_{k-i+1}u_{k-i}(n)\|^2 + \\ &\quad + (1-\beta_n)\| x_n-T_{k-i+1}u_{k-i}(n)\|^2 \\ &\quad - \beta_n(1-\beta_n)\|T_{k-i}u_{k-i-1}(n)-x_n\|^2. \end{aligned}$$

$$.....(2.1.8)$$

Now T_i is a generalized contraction mapping, for each i, we have

$$\|T_{k-i+1}u_{k-i}(n)-P\|^2 = \|T_{k-i+1}\,u_{k-i}(n)-T_{k-i+1}\,P\|^2$$

$$\leq a_1\|u_{k-i}\,(n)-p\|^2+$$

$$+\,a_2\|T_{k-i+1}\,p-u_{k-i}(n)\,\|^2+$$

$$+\,a_3\|T_{k-i+1}\,u_{k-i}(n)-P\|^2+$$

$$+\,a_4\|(I-T_{k-i+1})u_{k-i}(n)-(I-T_{k-i+1})p\|^2$$

$$=\,(a_1+a_2)\|u_{k-i}(n)-P\|^2+$$

$$+\,a_3\|T_{k-i+1}\,u_{k-i}(n)-P\|^2\,+$$

$$+\,a_4\|(I-T_{k-i+1}\,)u_{k-i}(n)-(I-T_{k-i+1})P\|^2$$

or

$$(1-a_3)\|T_{k-i+1}\,u_{k-i}(n)-P\|^2 \leq (a_1+a_2)\|u_{k-i}(n)-P\|^2+$$

$$+\,a_4\|u_{k-i}(n)-T_{k-i+1}\,u_{k-i}(n)\|^2$$

i.e.,

$$\|T_{k-i+1}\,u_{k-i}(n)-P\|^2 \leq \left(\frac{a_1+a_2}{1-a_3}\right)\|\,u_{k-i}(n)-P\|^2+$$

$$+\left(\frac{a_4}{1-a_3}\right)\|u_{k-i}(n)-T_{k-i+1}\,u_{k-i}(n)\|^2.$$

.....(2.1.9)

Now multiplying relations (2.1.7), (2.1.8), (2.1.9) and adding each one of them over i, we respectively get relations (2.1.10), (2.1.11), (2.1.12) as follows :

$$\sum_{i=1}^{k-1}\beta_n^{i-1}\|u_{k-i}(n)-P\|^2 = \beta_n\sum_{i=1}^{k-1}\beta_n^{i-1}\,\|T_{k-i}\,u_{k-i-1}\,(n)-P\|^2+$$

$$+(1-\beta_n)\|x_n-\,P\|^2\,\sum_{i=1}^{k-1}\beta_n^{i-1}\;-$$

$$-\beta_n(1-\beta_n)\sum_{i=1}^{k-1}\beta_n^{\,i-1}\,\|T_{k-i}u_{k-i-1}(n)-x_n\|^2.$$

.....(2.1.10)

$$\sum_{i=1}^{k-1} \beta_n^{i-1} ||u_{k-i}(n)-T_{k-i+1}u_{k-i}(n)||^2 = \beta_n \sum_{i=1}^{k-1} \beta_n^{i-1} ||T_{k-i}u_{k-i-1}(n)-T_{k-i+1}u_{k-i}(n)||^2 + (1-\beta_n) \sum_{i=1}^{k-1} \beta_n^{i-1} ||x_n-T_{k-i+1}u_{k-i}(n)||^2 - \beta_n(1-\beta_n) \sum_{i=1}^{k-1} \beta_n^{i-1} ||T_{k-i}u_{k-i-1}(n)-x_n||^2 \quad .....(2.1.11)$$

and

$$\sum_{i=1}^{k-1} \beta_n^{i-1} ||T_{k-i+1}u_{k-i}(n)-P||^2 \leqslant \left(\frac{a_1+a_2}{1-a_3}\right) \sum_{i=1}^{k-1} \beta_n^{i-1} ||u_{k-i}(n)-P||^2 + \left(\frac{a_4}{1-a_3}\right) \sum_{i=1}^{k-1} \beta_n^{i-1} ||u_{k-i}(n)-T_{k-i+1}u_{k-i}(n)||^2. \quad .....(2.1.12)$$

Using the fact that, T_1 is generalized contraction mapping, from inequality (2.1.3) we have

$$||T_1x_n - P||^2 = ||T_1x_n - T_1p||^2$$
$$\leqslant a_1||x_n - p||^2 + a_2||T_1p-x_n||^2 + a_3||T_1x_n-p||^2 + a_4||(I-T_1)x_n - (I-T_1)p||^2$$
$$= (a_1+a_2)||x_n - p||^2 + a_3||T_1x_n - p||^2 + a_4||x_n - T_1x_n||^2$$

or

$$||T_1x_n-p||^2 \leqslant \left(\frac{a_1+a_2}{1-a_3}\right)||x_n-p||^2 + \left(\frac{a_4}{1-a_3}\right)||x_n - T_1x_n||^2. \quad .....(2.1.13)$$

Now from the definition (2.1. 3) we have

$$a_1 + a_2 + a_3 + a_4 < 1$$

or

$$a_1 + a_2 + a_4 < 1 - a_3$$

$$\Rightarrow \quad \left(\frac{a_1 + a_2}{1 - a_3}\right) + \left(\frac{a_4}{1-a_3}\right) < 1$$

or

$$h + \gamma < 1 \text{ , where } h = \frac{a_1 + a_2}{1-a_3} \text{ , } \gamma = \frac{a_4}{1-a_3} \qquad (2.1.14)$$

Also

$$a_i \geqslant 0 \quad (i=1,\ 2,\ 3,\ 4).$$

$$\Rightarrow \quad h < 1 \text{ , } \gamma < 1 \quad \text{and} \quad h^2 < h.$$

Introducing (2.1.10) and (2.1.11) in (2.1.12) and simplifying we obtain

$$\sum_{i=1}^{k-1} \beta_n^{i-1} ||T_{k-i+1}u_{k-i}(n)-p||^2 \leqslant \beta_n h \sum_{i=1}^{k-1} \beta_n^{i-1} ||T_{k-i}u_{k-i-1}(n)-p||^2 +$$

$$+h(1-\beta_n)||x_n-p||^2 \sum_{i=1}^{k-1} \beta_n^{i-1} -$$

$$-h\beta_n(1-\beta_n) \sum_{i=1}^{k-1} \beta_n^{i-1} ||T_{k-i}u_{k-i-1}(n)-x_n||^2 +$$

$$+\gamma\beta_n \sum_{i=1}^{k-1} \beta_n^{i-1} ||T_{k-i}u_{k-i-1}(n)-T_{k-i+1}u_{k-i}(n)||^2 -$$

$$+\gamma(1-\beta_n) \sum_{i=1}^{k-1} \beta_n^{i-1} ||x_n-T_{k-i+1}u_{k-i}(n)||^2 -$$

$$\gamma\beta_n(1-\beta_n) \sum_{i=1}^{k-1} \beta_n^{i-1} ||T_{k-i}u_{k-i-1}(n)-x_n||^2$$

$$= \quad h(1-\beta_n)||x_n-p||^2 \sum_{i=1}^{k-1} \beta_n^{i-1} +$$

$$+ h\beta_n \sum_{i-1}^{k-1} \beta_n^{i-1} ||T_{k-i}u_{k-i-1}(n)-p||^2 -$$

$$-h\beta_n(1-\beta_n) \sum_{i=1}^{k-1} \beta_n^{i-1} ||T_{k-i}u_{k-i-1}(n)-x_n||^2 +$$

$$+\gamma\beta_n \sum_{i=1}^{k-1} \beta_n^{i-1} ||T_{k-i}u_{k-i-1}(n)-T_{k-i+1}u_{k-i}(n)||^2 +$$

$$+\gamma(1-\beta_n)[||x_n-T_k u_{k-1}(n)||^2-$$

$$-\beta_n^{k-1}\,||x_n-T_1x_n||^2] \;, \qquad ......(2.1.15)$$

(since $\sum_{i=1}^{k-1}\beta_n^{i-1}||x_n-T_{k-i+1}u_{k-i}(n)||^2-\beta_n\sum_{i=1}^{k-1}\beta_n^{i-1}||x_n-T_{k-i}u_{k-i-1}(n)||^2=$

$$=[||x_n-T_ku_{k-1}(n)||^2-\beta_n^{k-1}||x_n-T_1x_n||^2]).$$

Also we note that

$$\sum_{i=2}^{k-1}\beta_n^{i-1}\,||T_{k-i+1}u_{k-i}(n)-p||^2=\beta_n\sum_{i=1}^{k-2}\beta_n^{i-1}||T_{k-i}u_{k-i-1}(n)-p||^2$$

$$.....(2.1.16)$$

Hence using (2.1.13) we obtain

$$\sum_{i=1}^{k-1}\beta_n^{i-1}||T_{k-i+1}u_{k-i}(n)-p||^2\leqslant h||x_n-p||^2\sum_{i=1}^{k-1}\beta_n^{i-1}-h\beta_n||x_n-p||^2\sum_{i=1}^{k-1}\beta_n^{i-1}+$$

$$+h\beta_n\beta_n^{k-2}||T_1u_o(n)-p||^2+$$

$$+h\beta_n\sum_{i=1}^{k-2}\beta_n^{i-1}||T_{k-i}u_{k-i-1}(n)-p||^2-$$

$$-h\beta_n(1-\beta_n)\sum_{i=1}^{k-1}\beta_n^{i-1}||T_{k-i}u_{k-i-1}(n)-x_n||^2+$$

$$+\gamma\beta_n\sum_{i=1}^{k-1}\beta_n^{i-1}||T_{k-i}u_{k-i-1}(n)-T_{k-i+1}u_{k-i}(n)||^2+$$

$$+\gamma(1-\beta_n)\;[||x_n-T_ku_{k-1}(n)||^2-\beta_n^{k-1}||x_n-T_1x_n||^2]$$

$$\leqslant h\;||x_n-p||^2+\;h||x_n-p||^2\sum_{i=2}^{k-1}\beta_n^{i-1}\quad -$$

$$- h\beta_n^{k-1}||x_n-p||^2 - h\beta_n||x_n-p||^2 \sum_{i=1}^{k-2} \beta_n^{i-1}$$

$$+ h^2\beta_n^{k-1}||x_n-p||^2 + h\gamma\beta_n^{k-1}||x_n-T_1x_n||^2 +$$

$$+ h\beta_n \sum_{i=1}^{k-2} \beta_n^{i-1}||T_{k-i}u_{k-i-1}(n)-p||^2 -$$

$$- h\beta_n(1-\beta_n) \sum_{i=1}^{k-1} \beta_n^{i-1}||T_{k-i}u_{k-i-1}(n)-x_n||^2 +$$

$$+\gamma\beta_n \sum_{i=1}^{k-1} \beta_n^{i-1}||T_{k-i}u_{k-i-1}(n)-T_{k-i+1}u_{k-i}(n)||^2 +$$

$$+ \gamma(1-\beta_n)\,[||x_n-T_ku_{k-1}(n)||^2 - \beta_n^{k-1}||x_n-T_1x_n||^2].$$

.....(2.1.17)

By using (2.1.16) and noting that $h^2 < h$, we have

$$\sum_{i=1}^{k-1} \beta_n^{i-1}||T_{k-i+1}u_{k-i}(n)-p||^2 \leqslant h||x_n-p||^2 + h\gamma\beta_n^{k-1}||x_n-T_1x_n||^2 +$$

$$+ h\beta_n \sum_{i=1}^{k-1} \beta_n^{i-1} ||T_{k-i}u_{k-i-1}(n)-p||^2 -$$

$$- h\beta_n(1-\beta_n) \sum_{i=1}^{k-1} \beta_n^{i-1} ||T_{k-i}u_{k-i-1}(n)-x_n||^2 +$$

$$+\gamma\beta_n \sum_{i=1}^{k-1} \beta_n^{i-1}||T_{k-i}u_{k-i-1}(n)-T_{k-i+1}u_{k-i}(n)||^2$$

$$+\gamma(1-\beta_n)[||x_n-T_ku_{k-1}(n)||^2 - \beta_n^{k-1}||x_n-T_1x_n||^2].$$

$$\leqslant h\,||x_n-p||^2+[h\gamma-\gamma(1-\beta_n)]\,\beta_n^{k-1}||x_n-T_1x_n||^2+$$

$$+h\beta_n\sum_{i=1}^{k-2}\beta_n^{i-1}||T_{k-i}u_{k-i-1}(n)-p||^2-$$

$$-\;h\beta_n^{k-1}(1-\beta_n)||T_1x_n-x_n||^2-$$

$$-\;h\beta_n(1-\beta_n)\sum_{i=1}^{k-2}\beta_n^{i-1}||T_{k-i}u_{k-i-1}(n)-x_n||^2+$$

$$+\gamma\beta_n\sum_{i=1}^{k-1}\beta_n^{i-1}||T_{k-i}u_{k-i-1}(n)-T_{k-i+1}u_{k-i}(n)||^2+$$

$$+\;\gamma(1-\beta_n)||x_n-T_ku_{k-1}(n)||^2.$$

$$=\;h||x_n-p||^2+\;[h\gamma\;-\gamma(1-\beta_n)-h(1-\beta_n)]\;\beta_n^{k-1}||x_n-T_1x_n||^2+$$

$$+h\sum_{i=2}^{k-1}\beta_n^{i-1}||T_{k-i+1}u_{k-i}(n)-p||^2-$$

$$-h\beta_n(1-\beta_n)\sum_{i=1}^{k-2}\beta_n^{i-1}||T_{k-i}u_{k-i-1}(n)-x_n||^2+$$

$$+\gamma\beta_n\sum_{i=1}^{k-1}\beta_n^{i-1}||T_{k-i}u_{k-i-1}(n)-T_{k-i+1}u_{k-i}(n)||^2+$$

$$+\;\gamma\,(1-\beta_n)||x_n-T_ku_{k-i}(n)||^2$$

$$\leqslant\;h\;||x_n-p||^2-(1-\beta_n-\gamma\,h\beta_n)\,\beta_n^{k-1}\;||x_n-T_1x_n||^2+$$

$$+\sum_{i=2}^{k-1}\beta_n^{i-1}||T_{k-i+1}u_{k-i}(n)-p||^2\;-$$

$$-h\beta_n(1-\beta_n)\sum_{i=1}^{k-2}\beta_n||T_{k-i}u_{k-i-1}(n)-x_n||^2\;+$$

$$+\gamma\beta_n\sum_{i=1}^{k-1}\beta_n^{i-1}||T_{k-i}u_{k-i-1}(n)-T_{k-i+1}u_{k-i}(n)||^2+$$

$$+\gamma(1-\beta_n)\ ||x_n-T_ku_{k-1}(n)||^2, \quad \ldots\ldots(2.1.18)$$

since $h<1$ and $\beta_n<1$.

Using (2.1.16) in L.H.S.of above equation and simplifying we obtain

$$||T_ku_{k-1}(n)-p||^2 \leqslant h||x_n-p||^2-(1-\beta_n-\gamma h\beta_n)\,\beta_n^{k-1}||x_n-T_1x_n||^2-$$

$$-h\beta_n(1-\beta_n)\sum_{i=1}^{k-2}\beta_n^{k-1}||T_{k-i}u_{k-i-1}(n)-x_n||^2+$$

$$+\gamma(1-\beta_n)\ ||x_n-T_ku_{k-1}(n)||^2+$$

$$+\gamma\beta_n\sum_{i=1}^{k-1}\beta_n^{i-1}||T_{k-i}u_{k-i-1}(n)-T_{k-i+1}u_{k-i}(n)||^2.$$

$$\ldots\ldots(2.1.19)$$

Substituting (2.1.19) in (2.1.6) we have

$$||x_{n+1}-p||^2 \leqslant h\ \alpha_n||x_n-p||^2-\ \alpha_n(1-\beta_n-\gamma\ h\beta_n)\ \beta_n^{k-1}\ ||x_n-T_1x_n||^2-$$

$$-h\alpha_n\beta_n(1-\beta_n)\sum_{i=1}^{k-2}\beta_n^{i-1}||T_{k-i}u_{k-i-1}(n)-x_n||^2+$$

$$+\gamma\alpha_n\beta_n\sum_{i=1}^{k-1}\beta_n^{i-1}\ ||T_{k-i}u_{k-i-1}(n)-T_{k-i+1}u_{k-i}(n)||^2+$$

$$+\gamma\alpha_n(1-\beta_n)||x_n-T_ku_{k-1}(n)||^2+(1-\alpha_n)||x_n-p||^2\ -$$

$$-\ \alpha_n(1-\alpha_n)||x_n-T_ku_{k-1}(n)||^2$$

$$=\ [1-\alpha_n(1-h)]||\ x_n-p||^2-\alpha_n(1-\beta_n-\gamma h\beta_n)\ \beta_n^{k-1}\ ||x_n-\ T_1x_n||^2-$$

$$- h \alpha_n \beta_n (1-\beta_n) \sum_{i=1}^{k-2} \beta_n^{i-1} ||T_{k-i}u_{k-i-1}(n)-x_n||^2 +$$

$$+\gamma\alpha_n\beta_n \sum_{i=1}^{k-1} \beta_n^{i-1} ||T_{k-i}u_{k-i-1}(n)-T_{k-i+1}u_{k-i}(n)||^2 -$$

$$- [\gamma(\beta_n-\alpha_n)+\alpha_n(1-\alpha_n)]||x_n-T_ku_{k-1}(n)||^2 \quad \ldots\ldots(2.1.20)$$

$$\leqslant ||x_n-p||^2 - \alpha_n(1-\beta_n-\gamma h \beta_n) \beta_n^{k-1} ||x_n-T_1x_n||^2 +$$

$$+ \gamma\alpha_n\beta_n \sum_{i=1}^{k-1} \beta_n^{i-1}||T_{k-i}u_{k-i-1}(n)-T_{k-i+1}u_{k-i}(n)||^2,$$

Since $0 \leqslant \alpha_n \leqslant \beta_n \leqslant 1$ and $h < 1$.

Thus

$$||x_{n+1}-p||^2 \leqslant ||x_n-p||^2 - \alpha_n(1-\beta_n-\gamma h \beta_n) \beta_n^{k-1}||x_n-T_1x_n||^2 +$$

$$+\gamma \alpha_n\beta_n \sum_{i=1}^{k-1} \beta_n^{i-1} ||T_{k-i}u_{k-i-1}(n)-T_{k-i+1}u_{k-i}(n)||^2. \quad \ldots\ldots(2.1.21)$$

Using relations (2.1.2) and (2.1.4) we have for each $i=1,2,\ldots,k-1$,

$$||T_{k-i+1}u_{k-i}(n)-T_{k-i}u_{k-i-1}(n)|| \leqslant L||u_{k-i}(n)-u_{k-i-1}(n)||$$

$$=L \beta_n||T_{k-i-1}(n)-T_{k-i-1}u_{k-i-2}(n)||.$$

By induction on k we get

$$||T_{k-i+1}u_{k-i}(n)-T_{k-i}u_{k-i-1}(n)||^2 \leqslant (L\beta_n)^{2(k-i)}||T_1x_n-x_n||^2,$$

for $i=1,2,\ldots,k-1$. Hence, substituting this in (2.1.21)we obtain

$$||x_{n+1}-p||^2 \leqslant ||x_n-p||^2 - \alpha_n(1-\beta_n-\gamma h\beta_n)\beta_n^{k-1}||x_n-T_1x_n||^2 +$$

$$+\gamma\alpha_n\beta_n\sum_{i=1}^{k-1}\beta_n^{i-1}(L\beta_n)^{2(k-i)}||T_1x_n-x_n||^2$$

$$= ||x_n-p||^2 -$$

$$-\alpha_n\beta_n^{k-1}\{1-\beta_n-\gamma h\beta_n-\gamma L^2\beta_n^2[(L^2\beta_n)^{k-1}-1]/(L^2\beta_n-1)\}\cdot$$

$$\cdot||T_1x_n-x_n||^2 \qquad .....(2.1.22)$$

Letting

$$V_n = 1-\beta_n-\gamma h\beta_n-\gamma L^2\beta_n^2\,[(L^2\beta_n)^{k-1}-1]/\,(L^2\beta_n-1)$$

the relation (2.1.22) reduces to

$$||x_{n+1}-p||^2 \leqslant ||x_n-p||^2 - \alpha_n\beta_n^{k-1}V_n||T_1x_n-x_n||^2$$

or

$$\alpha_n\beta_n^{k-1}V_n||T_1x_n-x_n||^2 \leqslant ||x_n-p||^2 - ||x_{n+1}-p||^2 \quad ....(2.1.23)$$

By hypothesis $\beta_n \to 0$ as $n\to\infty$, we have $V_n \to 1$ as $n\to\infty$. Hence there exists a number N such that $V_m > 1/2$ for all integers $m \geqslant N$. Now adding the above inequalities (2.1.23) from m=l,l+1,....n we obtain

$$\sum_{m=l}^{n}\alpha_m\beta_m^{k-1}V_m||T_1x_m-x_m||^2 \leqslant ||x_l-p||^2-||x_{n+1}-p||^2,$$

and hence

$$1/2\sum_{m=l}^{n}\alpha_m\beta_m^{k-1}||T_1x_m-x_m||^2 \leqslant \sum_{m=l}^{n}\alpha_m\beta_m^{k-1}V_m||T_1x_m-x_m||^2$$

$$\leqslant ||x_l-p||^2 - ||x_{n+1}-p||^2$$

$$.....(2.1.24)$$

Since C is bonded, the right hand side of the above inequality is bounded. Hence the series on left hand side is bounded and also by hypothesis

$$\sum_{n=1}^{\infty} \alpha_n \beta_n^{k-1} = \infty \quad \text{for each } k \geqslant 2.$$

This implies that

$$\lim_{n \to \infty} \inf ||T_1 x_n - x_n||^2 = 0$$

From the compactness of C, it follows that there exist a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to q$ as $j \to \infty$ and $||T_1 q-q|| = 0$.

We claim that q is a common fixed point of the family $\{T_i\}$.

Indeed, for

$$||T_i q-q|| \leqslant ||T_i q-T_i x_{n_j}|| + ||T_i x_{n_j} - T_1 q|| , \quad \because T_1 q=q$$

$$\leqslant L\, ||q-x_{n_j}|| + L\, ||x_{n_j} - q|| , \text{ by (2.1.4)}$$

$$= 2L\, ||x_{n_j} - q||$$

$$\to 0 \text{ as } j \to \infty ,$$

Which implies that $||T_i q - q|| = 0$ and hence $T_i q=q$ for each i.

Now from relation (2.1.23)we see that

$$||x_{n+1} - q|| \leqslant ||x_n - q||$$

and q is common fixed point of $\{T_i\}$, this implies that $\{x_n - q\}$ is monotonically decreasing sequence and hence convergent. This along with the fact that $x_{n_j} \to q$ as $j \to \infty$ implies that $x_n \to q$ as $n \to \infty$. This completes the proof of the theorem.

Discussion and conclusions :

From the definition of generalized contraction (1.2.10) and the table (1.2.11), we have the following conclusions :

i) If we take k=2, $T_1 = T_2$, $a_2 = a_3 = 0$, $a_1 = a_4 = 1 = r = h$ in our result (2.1), we have the theorem (1.5.12) of Ishikawa [18].

ii) For y=p=Tp and $a_1 + a_2 + a_3 = 1$, $a_3 + a_4 = 1$ i.e., $r = h = 1$, our result (2.1) reduces to the result (1.5.14) of Das and Debata [12].

iii) For $r = \lambda < 1$ and h=1, our result gives the theorem (1.5.15) of Pathan [33].

iv) Some results of Liu Qihou [40-42] may be seen as immediate corollaries to our result.