Chapter – 3

Chapter - 3

SPLINE INTERPOLATION

3.1 Introduction

Newton'and Langrange's form are used to interpolate n^{th} order polynomial for n+1 data points. e.g. for 8 points 7^{th} order polynomial .

this curvewould capture all meandearing suggested by thecpoints . However there arexcases where these can ;ead to wrong result because of round off error and overshoot An alternative approach is to apply lower order polynomials to subset of data points such connecting polynomials are called Spline functions..

Third order curves employed to connect each pair of data points are called cubic splines These functions can be constructed so that the connections between adjacent cubic equentions are visually smooth. On the surface , it would seem that the third order approximation of spline would be inferior to the seventh order expression.

A visual representation of a situation where the splines are superior to higher order interplating polynomials

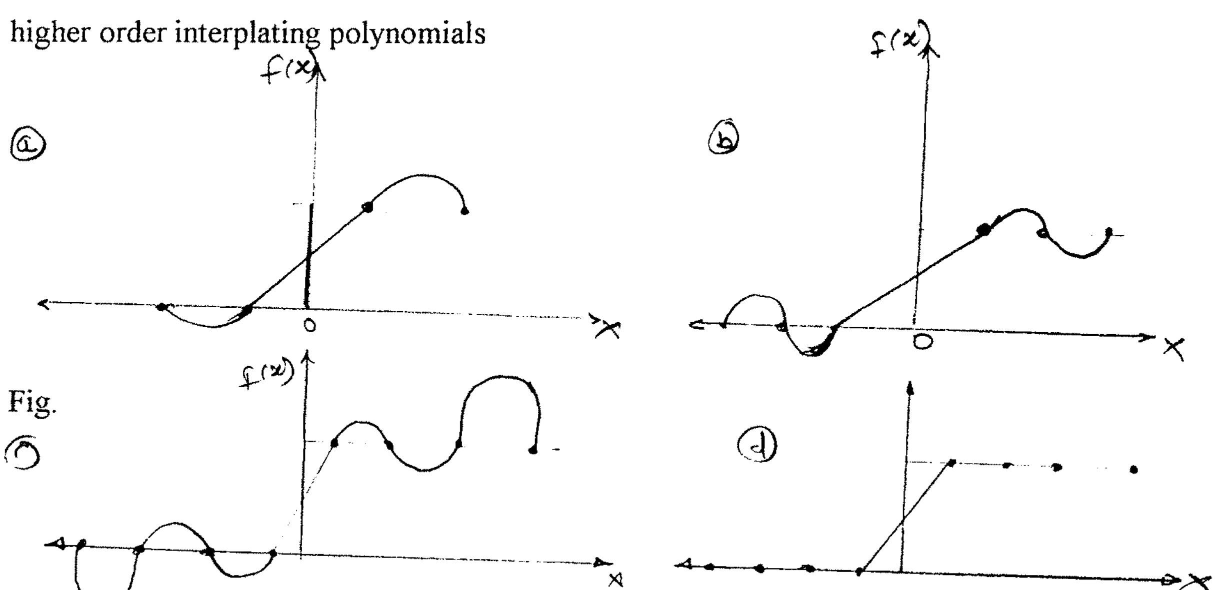


Fig.

Above fig. Illustrates a sitiuation where a spline performs better than a higher order polynomial.

This is the case where a function is generally smooth but undergoes an abrupt change somewhere along the region of interpolation . The step increse described in abve fig. Is an extreme example of such a change and serves to illustratethe point .

Fig (a) to(c) illlustrates how high order polynomials tend to swing through wild oscillations. In the vicinity of an abrupt change .In contrast the spline also connects the points but because it is limited to third order changes the oscillations are kept to a minimum The spline usually provides a superior approximation to the functions that have local abrupt changes behavior.

The concpt of the spline originated from the drafting technique of using a thin , flexible strip,[called a spline] to draw smooth curves through a set of points. The process is depicted in the following fig. For a series of five pins [data points] . In this technique , the drafter places paper cover a wooden board and hammers nails or pins into the paper [and board] at the location of the data points . A smooth cubic curve results from inter- wearing the strip between thee pins , hence the name " Cubic spline" has given for this polynomials.

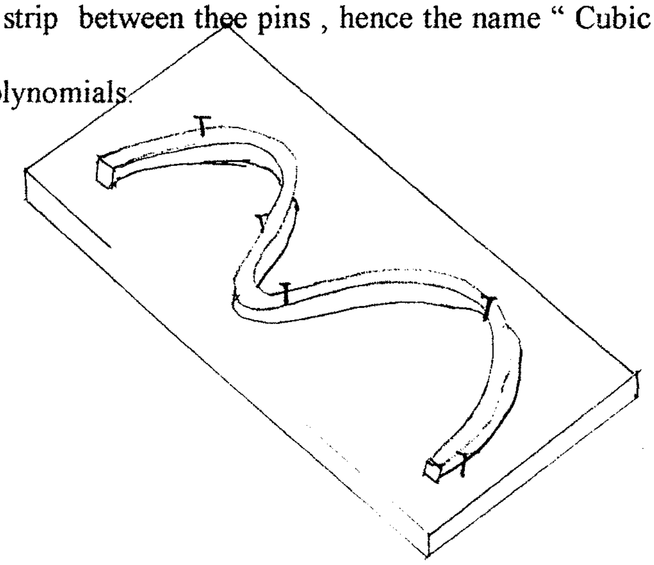


FIG.

The drafting technique of using a spline to draw smooth curves through a series of points. At the end point the spline straightens out . This is called a " Natural Spline" .

3.2 Linear Spline Intrpolation :-

The first order splines (two points are connected by a straight line) for a group of ordered data points can be defined as a set of Linear functions .

$f(x) = f(x_0)+m_0 (x-x_0)$ $x_0 \le$

$x \le x_1$

$f(x) = f(x_1)+m_1 (x-x_1)$ $x_1 \le$

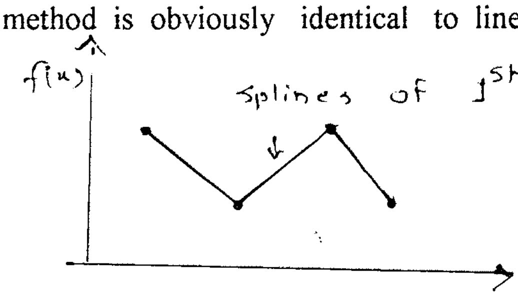
$x \le x_2$

$f(x) = f(x_{n-1})+ m_{n-1}(x-x_{n-1})$ $x_{n-1} \le x \le x_n$

Where m_i is the slope of the st. line connecting the pts.

$$M_I = \left[\frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_I}\right]$$

The eq^{ns} can be used to evaluate the fu^n at any points bet^n x_0 and x_n by first locating the interval within which the point lies .Then the appropriate eq^n is used to determine the fu^n value within the interval . The method is obviously identical to linear interpolation.



Above fi^g indicates that the primary disadvantage of first order spline is that they are not smooth. In essence at the data points where two splines meet (called a knot) the sploe changes arbruptly. The first derivative of the fu^n is discontinuous at these points . This deficiency is over come by using higher order polynomial splines that ensure smoothness at the knots by equating derivatives at these points as discussed in the next part.

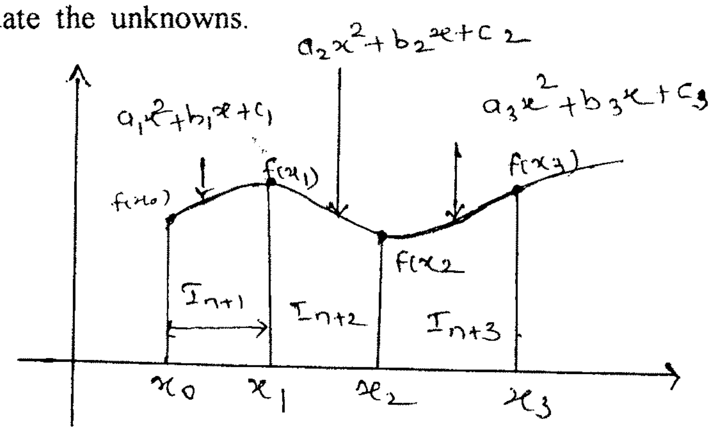
3.3 Quadratic spline Interpolation :-

Spline interpolation using second order polynomial called Quadratic splines have continuous first derivatives at the knots these do not ensure equal second derivatives at the knots.

The objective in quadratic splines is to derive a second order polynomial for each interval bet^n data points .The polynomial for each interval can be represented generally as

$f_I(x) = a_i x^2 + b_i x + c_I$ .... (1)

Consider n + 1 data points (i =0,1,.........n)there are n intervals and 3n unknown constants therefore 3n equ[ns] or conditions are required to evaluate the unknowns.



To get 3n equ[ns] we use following

1] The fun. Values of adjacent polynomials must be equal at the interior knots. This condition can be represented as

$a_{i-1}x^2_{i-1} + b_{i-1}x_{i-1} + c_{i-1} = f(x_{i-1})$

....(2)

$a_ix^2_{I-1} + b_i x_{i-1} + c_i \neq f(x_{i-1})$

.... (3)

for i = 2 to n because only interior knots are used. eq[ns] 2 & 3 each provide (n – 1) for a total of 2n-2 cond[n]

2] The first & last fun[ns] must pass through the end points this adds two additional eq[ns]

$a_1n_0^2 + b_1x_o + c_1= f(x_0)$

....(4)

$a_nx_n^2 + b_nx_n + c_n = f(x_n)$

....(5)

$\Rightarrow$ Total = 2n-2+2=2n cond^{ns}

3] The first derivatives at the interior knots must be equal. The first derivatite of eq^n [1] is

$f(x) = 2_{ax+b}$

Therefore the cond^n. can be represental generally as

$2a_{i-1}x_{i-1} + b_{i-1} = 2a_i x_{i-1} + b_i$ ....(6)

For i =2 to n This provides another n-1 cond^{ns} for total

$2n+n-1 = 3_{n-1}$

4] Assume that the second derivative is zero because the second derivative of eq^n [1] is $2a_i$ this cond^n can be expressed mathematically as $a_i = 0$

....[7]

(3n unknowns = 3n conditions).

3.4 Cubic Spline

This spline proves to be an efficient tool for approximation and interpolation.

Def^n :- Cubic spline is defined as , It is a flexible curve passing through each of the given points but goes smoothly from each interval to the next .so having the following properties.

1] The cubics and their first and second derivatives are continuous

2] The third derivatives of the cubic usually have jump discontinuities at the junction points .

The objective in cubic splines is to derive a third order polynomial for each interval bet^n knots. As in

$f_i(x) = a_i x^3 + b_i x^2 + c_i x + d_i$ ....(A)

Thus for n+1 data points (i=0,1,2.....n) there are n intervals and consequently 4n unknown constants to evaluate so 4n eq^{ns} are required .

These are

1] The fu^n. values must be equal at the interior knots (2n-2) conditions.

2] The first and last fu^n. must pass through the end points (2) cond^{ns}

3] The first derivatives at the interior knots must be equal (n-1) conditions.

4] The second derivatives at the interior knots must be equal (n-1)

5] The second derivatives at the end knots are zero .(2 $cond^{ns}$) i.e. the fu^n becomes a st. line at the end knots and called natural spline.

This name is given because the drafting spline naturally behaves in this fashion. If the value of the second derivative at the end knots is non-zero i.e. there is some curvature this information can be used alternatively to supply the two final conditions.

These conditions provide the total of 4n equations required to solve for the 4n coefficients.

Derivation of cubic spline

Since each paire of knots is connected by a cubic , the second derivative within each interval is a straight line .[A] can be differentiated twice to verify this observation. On this basis, the second derivatives can be represented by a first order lagrange interpolating polynomial

$$f_i''(x) = f_i''(x_i-1)\frac{x-x_L}{x_{i-1}x_I} + f_i''(x_i)\frac{(x-x_{i-1})}{(x-x_{i-1})} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Where f''_i is the value of the second derivative at any point x within the i^{th} interval. This is a straight line eq_n connecting the second derivative at the first knot $f'(x_{i-1})$ with the second derivative at the second knot $f''(x_i)$.

eq^n [1] can be integrated twice to yield an expression for $f_i(x)$ which contains two unknown constants of integration. These consts can be evaluated by using the the fu^n. equality conditions $f(x)$ must equal $f(x_{i-1})$ at x_{i-1} and must equal $f(x_i)$ at x_I

Using this we get the following cubic eq^n.

$$f_i(x) = \frac{f_i''(x_{i-1})}{6(x_i-x_{i-1})} (x_i-x^3) + \frac{f_i''(x_i)}{6(x_i-x_{i-1})} (x-x_{i-1})^3$$

$$+ \left[\frac{f(x_{i-1})}{(x_i-x_{i-1})} - \frac{f''(x_{i-1})\,(x_i-x_{i-1})}{6}\right] (x_i-x)$$

$$+ \left[\frac{f(x_i)}{(x_i-x_{i-1})} - \frac{f''(x_i)\,(x_i-x_{i-1})}{6}\right] (x-x_{i-1}) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots [2]$$

Which is a much more complicated expression for the cubic spline for the i^{th} interval than [A] However it contains only two known "coefficients" the second derivatives at the beginning .and the end of the interval $f''(x_{i-1})$ and $f''(x_i)$ so if we can determine the proper second derivative at each knot [2] is a third order polynomial that can be used to interpolate within the interval.

The second derivatives can be evaluated by in voking the cond^n that the first derivatives at the knots must be continuous.

$f'_{i-1}(x_i) = f_i''(x_i)$ ....[3]

$$(x_i-x_{i-1})\, f''(x_{i-1})+2(x_{i+1}-x_{i-1})\, f''(x_i)+(x_{i+1}-x_i)f''(x_{i+1})$$

$$= \frac{6}{x_{i+1}-x_i}\,[f(x_{i+1})-f(x_i)\,]+\frac{6}{x_i-x_{i-1}}\,[f(x_{i-1})-f(x_i)]$$

................[4]

if eq^n [4] is written for all interior knots, [n-1] simultaneous equations result with [n+1] unknown second derivatives. However since this is a nutural cubic spline, the second order derivatives at the end knots are zero and the problem reduces to (n-1) equations with (n-1) unknowns. In addition the system will reduces to tridiagonal which is extremely easy to solve.

We get the following cubic equ^n for each interval

$$f_i(x) = \frac{f''(x_{i-1})}{6(x_i-x_{i-1})}\,(x_i-x^3) + \frac{f_i''(x_i)}{6(x_i-x_{i-1})}\,(x-x_{i-1})^3$$

$$+ \frac{f(x_{i-1})}{(x_i-x_{i-1})} - \frac{f''(x_{i-1})\ (x_i-x_{i-1})}{6} \quad (x_i-x)$$

$$+ \frac{f(xi)}{xi-xi-1} - \frac{f''(xi)\ (xi-xi-1)}{6} \quad (x-xi-1) \qquad \text{....................................[5]}$$

This eqn contains only two unknowns . The second derivatives at the end of each interval . These unknowns can be evaluated using the following equation

,

$$(x_i-x_{i-1})\ f''(x_{i-1})+2(x_{i+1}- x_{i-1})\ f''(x_i)+(x_{i+1}- x_i)f''(x_{i+1})$$

$$= \frac{6}{x_{i+1}-x_I} \quad f(x_{i+1}) - f(x_i)+ \frac{6}{x_i-x_{i-1}} \quad [f(x_{i-1}) - f\,(x_i)]$$

............................[6]

If this eqn is written for all interior knots , (n-1) simultaneous eqns result with (n-1) unknowns .(since the second derivative at end knots are zero.)

Features of splines

1. polynomial splines are relatively smooth functions

2. Polynomial splines are finite dimensional and are with very convenient basis .
3. Splines are easy t o store , evaluate and manipulate on a digital computer.
4. They have very nice zero properties
5. In the sline function we are getting tridiagonal matrix .
6. Derivatives & antiderivatives of splines are spline .
7. Low degree splines are very much flexible and do not exhibit oscillations associated with polynomials.