Chapter – 1

TETRAD FORMALISMS IN RELATIVITY

1. Introduction

Let $X^i = X^i\ (s)$ be the world line of particle in a 4 – dimensional space – time of general theory of relativity, ~~when~~ Where s is the parameter. At each point of the world line a set of four basic vector fields can be ~~contracted~~ Constructed. These four basis vector fields form a tetrad. A vector field X^i of tetrad is said to be time like, space like or null according as $X^i X_i > 0$, $X^i X_i < 0$ and $X^i X_i = 0$.

A choice of the four basis vector fields in tetrad may be a combination of the time like vector fields, a space like vector field or a null vector field. However the choice of the tetrad vector fields has to be compatible with the ~~signal line~~ Signature (- - - +) of the metric. Accordingly, all the four vector fields may be null or one is null and three are space like or one is time like and three are space like or two are null and two are space like or any other combination.

2. Tetrad Formalism in General Relativity

1. One time like and three space like vector fields

For studying self gravitating perfect fluid distribution Eisenhart used a tetrad formalism in 1964. His tetrad consisted of one time like vector field and three space like vector fields. The same tetrad has been utilized for the study of definite magnetofluid by *Shaha (1974)*. To study the local behaviour of a space – like congruence and the magnetic field line *Date (1972-76). Date and Patil (1978)* utilized the same tetrad for the relativistic magneto- hydrodynamics. To maintain the orthonormality relations between the time like and three space like vector fields *Greenbergs (1970)* has introduced the three natural transport laws. The orthonormality relations between the time like vector field u^i and three space like vector field p^i, q^i, r^i are

$$u_i u^i = 1 , \quad p_i p^i = q_i q^i = r_i r^i = -1$$

and all other inner products are zero.

2. One null and three space like vector fields

A tetrad consisting of one null vector field and three space like vector fields has been designed by *Synge (1972)* for studying the three curvature of a dynamical null curve in the 4 – dimensional space time. He obtained the Serret Frenat formulae of a curve which is the world line of a massless particle.

3. Two null vector fields and two space like vector fields:

A tetrad consisting of two non-collinear null vector fields and two space like vector fields has been introduced by *Hall (1977)* for the classification of Ricci tensor in General Relativity.

4. All four vector fields of a tetrad are null:

Newman and Penrose (1962) has designed a tetrad which consisted of all null vector fields, l^i, n^i, m^i and $\bar{m}^i$ of which l^i and n^i are real null vector fields and m^i and $\bar{m}^i$ are complex conjugate of each other. This formalism is proved to be more powerful tetrad formalism amongst all the formalism in the general theory of relativity. For more than four decades much research in the field of relativity is carried with the help of Newman Penrose tetrad formalism. This tetrad formalism has found many applications especially in finding exact solution of Einstein's field equation *Kramer (1980).* In the theory of black holes *Chandrashekhar (1981).*

5. GHP Formalism:

Yet another formalism has been invented by *Geroch*, Held and Penrose (1973). It is generalization of Newman Penrose formalism which is a compromise between the fully covariant formalism and the spin coefficient formalism.

6. Rheotetrad Formalism:

By using a single time like vector field u^a and its intrinsic derivatives u'^a, u''^a, u'''^a together with the intrinsic scalars Viz., the curvature κ, the torsion τ and bitorsion B, *Radhakrishna (1993)* introduced a tetrad called as Rheotetrad which is specially suited for the ~~expansion~~ exploration of non-geodesic flow in relativistic continuum mechanics.

Mathematical artifact of rheotetrad was exploited by *Unde (1993)* to study the implication of regular relativistic thermodynamics of *Carter (1991), Katkar and Khairmode (2004)* obtained the NP concomitants rheotetrad and exploited it study the geometry of the world line of a particle in a given space time geometry.

7. A Newman Penrose Type Tetrad Formalism with Torsion

Einstein Cartan Theory had its origin on non-Riemannian space due to *Cartan (1923, 1924)* and deals with spin of matter. This theory is popularized as U_4 theory by *Trautmann (1972, 73a, 73b)* and *Hehl (1973,74)*. In this theory the connections are not symmetric in general. *Trautmann (1973b)* has proved that spin and torsion avert gravitational singularities. Analogus to the 'amazingly useful' Newman Penrose tetrad formalism for Einstein theory of gravitation, *Jogia and Griffiths (1980)* have developed another tetrad formalism for space times with torsion. This formalism is applicable to Einstein Cartan theory and all other theories which involves torsion, *Gambini* and *Herrera (1980)* have also developed a null tetrad formalism for space times with torsion.

3. Newman Penrose Tetrad Formalism

In the present dissertation we have developed a Newman Penrose type formalism for a tetrad which consist of one time like and three space like congruences. Our tetrad formalism is similar to the Newman Penrose tetrad formalism, it is imperative to know Newman Penrose tetrad formalism. Hence a brief exposition of the Newman Penrose formalism is presented in this chapter.

At a point of four dimensional space time of general theory of relativity, a set of four null vector fields l^i, n^i, m^i and $\overline{m}^i$ is introduced. Here l^i and n^i are real null vector field while m^i and $\overline{m}^i$ are complex conjugate of each other,(An overhead bar indicate the complex conjugate). The vector fields of the tetrad are to satisfy the following restrictions on their inner product.

$$l_i n^i = 1,\ m_i \overline{m}^i = -1 \qquad \text{~~normally~~ normality conditions}$$

$$l_i m^i = l_i \overline{m}^i = n_i m^i = n_i \overline{m}^i = 0 \qquad \text{orthogonality conditions}$$

$$l_i l^i = n_i n^i = m_i m^i = m_i \overline{m}^i = 0 \text{ nullity conditions} \qquad (3.1)$$

Thus out of the ten inner products of the four null vector fields, as many as eight vanish, which provides enormous simplification in computations. It has been shown that the null formalism affords a saving of 60% of computer time. *Vide, Cambell and Wainwright (1977).*

The relation between the tetrad vector fields and the geometry of space time is given by,

$$g_{ij} = l_i n_j + n_i l_j - m_i \overline{m}_j - \overline{m}_i m_j \tag{3.2}$$

This relation is referred as the "Completeness relation".

Let us denote the tetrad at each point of the world line of a particle in the four dimensional space time of general theory of relativity by

$$\overline{Z_\alpha} \quad ;\alpha = 1,2,3,4$$

The covariant and contravariant components of $\overline{Z_\alpha}$ will be denoted by $Z_{\alpha i}$ and Z^i_α respectively. Thus the basis of contravariant vector fields associated to each point of space time is defined by ,

$$Z^i_\alpha = (l^i, n^i, m^i, m^{-i}) \tag{3.3}$$

Associated with the contravariant vector fields, the covariant vector field are defined by ,

$$Z_{\alpha i} = g_{ik} Z^k_\alpha$$

$$Z_{\alpha i} = (l_i, n_i, m_i, \overline{m}_i) \tag{3.4}$$

Also we define the inverse (Z^α_i) of the matrix (Z^i_α)

$$\text{As } Z^i_\alpha Z^\beta_i = \delta^\beta_\alpha \tag{3.5}$$

$$\text{and } Z^i_\alpha Z^\alpha_j = \delta^i_j \tag{3.6}$$

The matrix of the inner product of the basis vectors is given by

$$\eta_{\alpha\beta} = Z^i_\alpha Z_{\beta i} \tag{3.7}$$

Where $\eta_{\alpha\beta} = \eta_{\beta\alpha}$

Using the equation (3.4) we have from (3.7) that

$$\eta_{\alpha\beta} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Equation (3.7) can also be written as

$$\eta_{\alpha\beta} = g_{ik} Z^i_\alpha Z^k_\beta \tag{3.8}$$

The inverse matrix $\eta^{\alpha\beta}$ of the matrix $\eta_{\alpha\beta}$ is such that

$$\eta^{\alpha\beta}\eta_{\beta\nu} = \delta^\alpha_\nu \tag{3.9}$$

The metric tensor g_{ij} is used to raise or lower the ~~tetrad~~ tensor indices, while $\eta_{\alpha\beta}$ is used to raise or lower tetrad indices. Thus we have

$$Z_{\alpha i} = \eta_{\alpha\beta} Z^\beta_i \quad \text{and}$$

$$Z^\beta_i = \eta^{\alpha\beta} Z_{\alpha i}$$

$$\begin{pmatrix} Z^1_i \\ Z^2_i \\ Z^3_i \\ Z^4_i \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} l_i \\ n_i \\ m_i \\ \overline{m}_i \end{pmatrix}$$

i.e. $Z^\alpha_i = (n_i, l_i, -\overline{m}_i, -m_i)$

and $Z^{\alpha i} = (n^i, l^i, -\overline{m}^i, -m^i)$ (3.10)

Thus from equation (3.4) and (3.7) we find the relation between g_{ij} and $\eta_{\alpha\beta}$ is given by

$$g_{ij} = \eta_{\alpha\beta} Z_i^{\alpha} Z_j^{\beta} \qquad (3.11)$$

$$\eta_{\alpha\beta} = g_{ij} Z_{\alpha}^{i} Z_{\beta}^{j} \qquad (3.12)$$

Ricci's Coefficients of Rotations :

The Ricci's coefficient of rotations are denoted $\gamma^{\alpha}{}_{\beta\delta}$ and are defined by

$$\gamma^{\alpha}_{\beta\delta} = -Z^{\alpha}_{i;j} Z^{i}_{\beta} Z^{j}_{\delta} \qquad (3.13)$$

These coefficients of rotations are skew symmetric in the first indices

$$\gamma_{\alpha\beta\delta} = -\gamma_{\beta\alpha\delta} \qquad (3.14)$$

In Newman Penrose tetrad formalism the role of these coefficients is taken by 12 complex spin coefficients which are defined (3.13) as follows.

$$\kappa = -\gamma_{311} = \gamma_{131} = l_{i;j} m^i l^j,$$
$$\tau = \gamma_{132} = l_{i;j} m^i \overline{m}^j,$$
$$\sigma = \gamma_{133} = l_{i;j} m^i m^j,$$
$$\rho = \gamma_{134} = l_{i;j} m^i \eta^j,$$
$$\pi = -\gamma_{241} = -\eta_{i;j} m^{-i} l^j$$
$$\upsilon = -\gamma_{242} = -\eta_{i;j} \overline{m}^i n^j$$
$$\mu = -\gamma_{243} = -n_{i;j} \overline{m}^i m^j$$

$$
\begin{aligned}
\lambda &= -\gamma_{244} = -n_{i;j}\,\overline{m}^{i}\,\overline{m}^{j} \\
\in &= \frac{1}{2}(\gamma_{121} - \gamma_{341}) = \frac{1}{2}(l_{i;j}n^{i}l^{j} - m_{i;j}\overline{m}^{i}l^{j}) \\
\nu &= \frac{1}{2}(\gamma_{122} - \gamma_{342}) = \frac{1}{2}(l_{i;j}n^{i}n^{j} - m_{i;j}\overline{m}^{i}n^{j}) \\
\beta &= \frac{1}{2}(\gamma_{123} - \gamma_{343}) = \frac{1}{2}(l_{i;j}n^{i}m^{j} - m_{i;j}\overline{m}^{i}m^{j}) \\
\alpha &= \frac{1}{2}(\gamma_{124} - \gamma_{344}) = \frac{1}{2}(l_{i;j}n^{i}\overline{m}^{j} - m_{i;j}\overline{m}^{i}\overline{m}^{j})
\end{aligned}
\tag{3.15}
$$

The amazing usefulness of the Newman Penrose tetrad formalism in simplifying the differential relation is rendered possible because of the fact that the covariant derivative of tetrad vector fields is expressible as an algebraic combination of the four null vector fields of the tetrad $Z_{\alpha i}$.

Consider for example the covariant derivative of the real null vector field l_i can be expressed as a linear combination of the null tetrad vector fields.

$$
\begin{aligned}
l_{i;j} = {} & A l_i l_j + B l_i n_j + C l_i m_j + D l_i \overline{m}_j + \\
& + A_1 n_i l_j + B_1 n_i n_j + C_1 n_i m_j + D_1 n_i \overline{m}_j + \\
& + A_2 m_i l_j + B_2 m_i n_j + C_2 m_i m_j + D_2 m_i \overline{m}_j + \\
& + A_3 \overline{m}_i l_j + B_3 \overline{m}_i n_j + C_3 \overline{m}_i n_j + C_3 \overline{m}_i m_j + D_3 \overline{m}_i \overline{m}_j
\end{aligned}
\tag{3.16}
$$

By transvecting ~~with~~ this expression with null vector fields, we obtain the coefficient $A, B \ldots\ldots\ldots C_3, D_3$ as

$$
\begin{aligned}
A &= l_{i;j} n^i n^j = (\gamma + \bar{\gamma}) \\
B &= l_{i;j} n^i l^j = (\varepsilon + \bar{\varepsilon}) \\
C &= l_{i;j} n^i \bar{m}^j = -(\alpha + \bar{\beta}) \\
D &= l_{i;j} n^i m^j = -(\bar{\alpha} + \beta) \\
A_2 &= l_{i;j} \bar{m}^i n^j = -\bar{\tau} \\
B_2 &= l_{i;j} \bar{m}^i l^j = -\bar{\kappa} \\
C_2 &= l_{i;j} \bar{m}^i \bar{m}^j = \bar{\sigma} \\
D_2 &= l_{i;j} \bar{m}^i m^j = \bar{\rho} \\
A_3 &= l_{i;j} m^i n^j = -\tau \\
B_3 &= l_{i;j} m^i l^j = -\kappa \\
C_3 &= l_{i;j} m^i \bar{m}^j = \rho \\
D_3 &= l_{i;j} m^i m^j = \sigma
\end{aligned}
\tag{3.17}
$$

Thus, we have

$$
\begin{aligned}
l_{i,j} = {} & (\gamma + \bar{\gamma}) l_i l_j + (\varepsilon + \bar{\varepsilon}) l_i n_j - (\alpha + \bar{\beta}) l_i m_j - (\bar{\alpha} + \beta) l_i \bar{m}_j - \\
& - \bar{\tau} m_i l_j - \bar{\kappa} m_i n_j + \bar{\sigma} m_i m_j + \bar{\rho} m_i \bar{m}_j - \tau \bar{m}_i l_j - \kappa \bar{m}_i n_j \\
& + \rho \bar{m}_i m_j + \sigma \bar{m}_i \bar{m}_j
\end{aligned}
\tag{3.18}
$$

Similarly, we can readily obtain the expressions for the covariant derivations of the complex null tetrad as,

$$
\begin{aligned}
n_{i;j} = {} & -(\gamma + \bar{\gamma}) n_i l_j - (\varepsilon + \bar{\varepsilon}) n_i n_j + (\alpha + \bar{\beta}) n_i m_j + (\bar{\alpha} + \beta) n_i \bar{m}_j + \\
& + \nu m_i l_j + \pi m_i n_j - \lambda m_i m_j - \mu m_i \bar{m}_j + \bar{\nu} \bar{m}_i l_j + \bar{\pi} \bar{m}_i n_j - \\
& - \bar{\mu} \bar{m}_i m_j - \bar{\lambda} \bar{m}_i \bar{m}_j .
\end{aligned}
\tag{3.19}
$$

$$
\begin{aligned}
m_{i;j} = {} & \bar{\nu} l_i l_j + \bar{\pi} l_i n_j - \bar{\mu} l_i m_j - \bar{\lambda} l_i \bar{m}_j - \tau n_i l_j - \kappa n_i n_j + \\
& + \rho n_i m_j + \sigma n_i \bar{m}_j + (\gamma - \bar{\gamma}) m_i l_j + (\varepsilon - \bar{\varepsilon}) m_i n_j - \\
& - (\alpha - \bar{\beta}) m_i m_j + (\bar{\alpha} - \beta) m_i \bar{m}_j .
\end{aligned}
\tag{3.20}
$$

The Ricci Scalars :

The tetrad components of the Ricci Tensor R_{ij} and the Ricci Scalar R in the Newman Penrose formalism are given by

$$\phi_{00} = -\frac{1}{2}R_{ij}l^i l^j$$

$$\phi_{01} = -\frac{1}{2}R_{ij}l^i m^j$$

$$\phi_{10} = -\frac{1}{2}R_{ij}l^i \bar{m}^j$$

$$\phi_{11} = -\frac{1}{4}R_{ij}(l^i n^j + m^i \bar{m}^j)$$

$$\phi_{02} = -\frac{1}{2}R_{ij}m^i m^j$$

$$\phi_{20} = -\frac{1}{2}R_{ij}\bar{m}^i \bar{m}^j$$

$$\phi_{12} = -\frac{1}{2}R_{ij}n^i m^j$$

$$\phi_{21} = -\frac{1}{2}R_{ij}n^i \bar{m}^j$$

$$\phi_{22} = -\frac{1}{2}R_{ij}n^i n^j$$

$$\wedge = \frac{1}{24}R \tag{3.21}$$

The Weyl Scalars:

The free gravitational part of the curvature tensor R_{hijk} is the Weyl tensor C_{hijk} and is defined by ,

$$C_{hijk} = R_{hijk} - \frac{1}{2}(g_{hk}R_{ik} - g_{hj}R_{ik} + g_{ij}R_{hk} - g_{ik}R_{hj}) + \frac{R}{6}(g_{hk}g_{ij} - g_{hj}g_{ik})$$

In Newman Penrose formalism, the curvature tensor has five independent component and are given by,

$$\psi_0 = -C_{hijk} l^h m^i l^j n^k$$
$$\psi_1 = -C_{hijk} l^h n^i l^j m^k$$
$$\psi_2 = -\frac{1}{2} C_{hijk} (l^h n^i l^j n^k - l^h n^i m^j \overline{m}^k)$$
$$\psi_3 = -C_{hijk} n^h l^i n^j \overline{m}^k$$
$$\psi_4 = -C_{hijk} n^h \overline{m}^i n^j \overline{m}^k$$

(3.22)

Einstein Field Equation:

For any vector field of a tetrad (Z_i^α) the Ricci Identities are given by

$$Z^\alpha_{i;jk} - Z^\alpha_{i;kj} = Z^\alpha_h R^h_{ijk} \tag{3.23}$$

The translation of this tensor equation into the 18 Newman Penrose field equations originally derived by Newman and Penrose (1962) is accomplished in the following equations ,

$$
\begin{aligned}
&D\rho-\bar{\delta}\kappa=\rho^{2}+\sigma\bar{\sigma}+(\varepsilon+\bar{\varepsilon})\rho-\bar{\kappa}\tau-\kappa(3\alpha+\bar{\beta}-\pi)+\phi_{00}\\
&D\sigma-\delta k=(\rho+\bar{\rho})\sigma+(3\varepsilon-\bar{\varepsilon})\sigma-(\tau-\bar{\pi}+\bar{\alpha}+3\beta)\kappa+\psi_{0}\\
&D\tau-\Delta\kappa=(\tau+\bar{\pi})\rho+(\varepsilon-\bar{\varepsilon})\tau-(3\gamma+\bar{\gamma})\kappa+\psi_{1}+(\bar{\tau}+\pi)\sigma+\phi_{01}\\
&D\alpha-\bar{\delta}\varepsilon=(\rho+\bar{\varepsilon}-2\varepsilon)\alpha+\beta\bar{\sigma}-\bar{\beta}\varepsilon-\kappa\lambda-\bar{\kappa}\gamma+(\varepsilon+\rho)\pi+\phi_{10}\\
&D\beta-\delta\varepsilon=(\alpha+\pi)\sigma+(\bar{\rho}-\bar{\varepsilon})\beta-(\mu+\gamma)\kappa-(\bar{\alpha}-\bar{\pi})\varepsilon+\psi_{1}\\
&D\gamma-\Delta\varepsilon=(\tau+\bar{\pi})\alpha+(\bar{\tau}+\pi)\beta-(\varepsilon+\bar{\varepsilon})\gamma-(\gamma+\bar{\gamma})\varepsilon+\tau\pi-\nu k+\psi_{2}-\wedge+\phi_{11}\\
&D\lambda-\bar{\delta}\pi=\rho\lambda+\bar{\sigma}\mu+\pi^{2}+(\alpha-\bar{\beta})\pi-\nu\bar{\kappa}-(3\varepsilon-\bar{\varepsilon})\lambda+\phi_{20}\\
&D\mu-\delta\pi=\bar{\rho}\mu-\nu k+\sigma\lambda+\pi\bar{\pi}-(\varepsilon+\bar{\varepsilon})\mu-\pi(\bar{\alpha}-\beta)+\psi_{2}+2\wedge\\
&D\gamma-\Delta\pi=(\pi+\bar{\tau})\mu+(\bar{\pi}+\tau)\lambda+(\nu-\bar{\nu})\pi-(3\varepsilon+\bar{\varepsilon})\nu+\psi_{3}+\phi_{21}\\
&\Delta\lambda-\bar{\delta}\nu=-(\mu+\bar{\mu})\lambda-(3\nu-\bar{\nu})\lambda+(3\alpha+\bar{\beta}+\pi-\bar{\tau})\nu-\psi_{4}\\
&\delta\rho-\bar{\delta}\sigma=\rho(\bar{\alpha}+\beta)-\sigma(3\alpha-\bar{\beta})+(\rho-\bar{\rho})\tau+(\mu-\bar{\mu})\kappa-\psi_{1}+\phi_{01}\\
&\delta\alpha-\bar{\delta}\beta=\mu\rho+\alpha\bar{\alpha}+\beta\bar{\beta}-2\alpha\beta+\gamma(\rho-\bar{\rho})+\varepsilon(\mu-\bar{\mu})-\lambda\sigma-\psi_{2}+\wedge+\phi_{11}\\
&\delta\lambda-\bar{\delta}\mu=(\rho-\bar{\rho})\nu+(\mu-\bar{\mu})\pi+\mu(\alpha+\bar{\beta})+\lambda(\bar{\alpha}-3\beta)-\psi_{3}+\phi_{21}\\
&\delta\nu-\Delta\mu=\mu^{2}+\lambda\bar{\lambda}+(\gamma+\bar{\gamma})\mu-\bar{\nu}\pi+(\tau-3\beta-\bar{\alpha})\nu+\phi_{22}\\
&\delta\gamma-\Delta\beta=(\tau-\bar{\alpha}-\beta)\gamma+\mu\tau-\sigma\nu-\varepsilon\bar{\nu}-\beta(\gamma-\bar{\gamma}-\mu)+\alpha\bar{\lambda}+\phi_{12}\\
&\delta\tau-\Delta\sigma=\mu\sigma+\bar{\lambda}\rho+(\tau+\beta-\bar{\alpha})\tau-(3\gamma-\bar{\gamma})\sigma-\kappa\bar{\nu}+\phi_{02}\\
&\Delta\rho-\bar{\delta}\tau=-\rho\bar{\mu}-\sigma\lambda+(\bar{\beta}-\alpha-\bar{\tau})\tau+(\gamma+\bar{\gamma})\rho+\nu\kappa-\psi_{2}-2\wedge\\
&\Delta\alpha-\bar{\delta}\gamma=(\rho+\varepsilon)\nu-(\tau+\beta)\lambda+(\bar{\gamma}-\bar{\mu})\alpha+(\bar{\beta}-\bar{\tau})\gamma-\psi_{3}
\end{aligned}
\tag{3.24}
$$

Bianchi Identities:

In Newman Penrose spin coefficient formalism the eleven equations equivalent to the 24 Binachi Identities Viz, $R_{hijk;l} + R_{hikl;j} + R_{hilj;k} = 0$ are cited below

$$\begin{aligned}&\bar{\delta}\psi_0 - D\psi_1 + D\phi_{01} - \delta\phi_{00} = (4\alpha - \pi)\psi_0 - 2(2\rho + \varepsilon)\psi_1 + \\ &+3\kappa\psi_2 + (\bar{\pi} - 2\bar{\alpha} - 2\beta)\phi_{00} + 2(\varepsilon + \bar{\rho})\phi_{01} + 2\sigma\phi_{10} - \\ &-2\kappa\phi_{11} - \bar{\kappa}\phi_{02}\end{aligned} \tag{3.25}$$

$$\begin{aligned}&\Delta\psi_0 - \delta\psi_1 + D\phi_{02} - \delta\phi_{01} = (4\gamma - \mu)\psi_0 - 2(2\tau + \beta)\psi_1 + \\ &+3\sigma\psi_2 - \bar{\lambda}\phi_{00} + 2(\bar{\pi} - \beta)\phi_{01} + 2\sigma\phi_{11} + (2\varepsilon - 2\bar{\varepsilon} + \bar{\rho})\phi_{02} - \\ &-2\kappa\phi_{12}\end{aligned} \tag{3.26}$$

$$\begin{aligned}&3(\bar{\delta}\psi_1 - D\psi_2) + 2(D\phi_{11} - \delta\phi_{10}) + \bar{\delta}\bar{\phi}_{00} - \Delta\phi_{00} = 3\lambda\psi_0 - \\ &-9\rho\psi_2 + 6(\alpha - \pi)\psi_1 + 6\kappa\psi_3 + (\bar{\mu} - 2\mu - 2\gamma - 2\bar{\gamma})\phi_{00} + \\ &+(2\alpha + 2\pi + 2\bar{\tau})\phi_{01} + 2(\tau - 2\bar{\alpha} + \bar{\pi})\phi_{10} + 2(2\bar{\rho} - \rho)\phi_{11} + \\ &+2\sigma\phi_{20} - \bar{\sigma}\phi_{02} - 2\bar{\kappa}\phi_{12} - 2\kappa\phi_{21}\end{aligned} \tag{3.27}$$

$$\begin{aligned}&3(\Delta\psi_1 - \delta\psi_2) + 2(D\phi_{12} - \delta\phi_{11}) + (\bar{\delta}\phi_{02} - \Delta\phi_{01}) = \\ &= 3\gamma\psi_0 + 6(\gamma - \mu)\psi_1 - 9\tau\psi_2 + 6\sigma\psi_3 - \bar{\nu}\phi_{00} + \\ &+2(\bar{\mu} - \mu - \gamma)\phi_{01} - 2\bar{\lambda}\phi_{10} + 2(\tau + 2\bar{\pi})\phi_{11} + \\ &+(2\alpha + 2\pi + \bar{\tau} - 2\bar{\beta})\phi_{02} + (2\bar{\rho} - 2\rho - 4\bar{\varepsilon})\phi_{12} + \\ &+2\sigma\phi_{21} - 2\kappa\phi_{22}.\end{aligned} \tag{3.28}$$

$$\begin{aligned}&3(\bar{\delta}\psi_2 - D\psi_3) + D\phi_{21} - \delta\phi_{20} + 2(\bar{\delta}\phi_{11} - \Delta\phi_{10}) = \\ &= 6\lambda\psi_1 - 9\pi\psi_2 - 6(\varepsilon - \rho)\psi_3 + 3\kappa\psi_4 - 2\nu\phi_{00} + 2\lambda\phi_{01} + \\ &+2(\bar{\mu} - \mu - 2\bar{\gamma})\phi_{10} + (2\pi + 4\bar{\tau})\phi_{11} + (2\beta + 2\tau + \bar{\pi} - 2\bar{\alpha})\phi_{20} - \\ &-2\bar{\sigma}\phi_{12} + 2(\bar{\rho} - \rho - \varepsilon)\phi_{21} - \bar{\kappa}\phi_{22}.\end{aligned} \tag{3.29}$$

$$3(\Delta\psi_2-\delta\psi_3)+D\phi_{22}-\delta\phi_{21}+2(\bar{\delta}\phi_{12}-\Delta\phi_{11})=$$
$$6\nu\psi_1-9\mu\psi_2+6(\beta-\tau)\overline{\psi}_3+3\sigma\psi_4-2\nu\phi_{01}--2\overline{\gamma}\phi_{10}+$$
$$+2(2\overline{\mu}-\mu)\phi_{11}+2\lambda\phi_{02}-\overline{\lambda}\phi_{20}+2(\tau+\beta+\overline{\pi})\phi_{21}+ \qquad (3.30)$$
$$+2(\pi+\overline{\tau}-2\overline{\beta})\phi_{12}+2(\beta+\overline{\pi})\phi_{21}+(\overline{\rho}-2\varepsilon-2\overline{\varepsilon}-2\rho)\phi_{22}.$$

$$\overline{\delta}\psi_3-D\psi_4+\overline{\overline{\delta}\phi}_{21}-\Delta\phi_{20}=3\lambda\psi_2-2(\alpha+2\pi)\psi_3+$$
$$+(4\in-\rho)\psi_4-2\pi\phi_{10}+2\pi\phi_{11}+(2\gamma-2\overline{\gamma}+\overline{\mu})\phi_{20}+ \qquad (3.31)$$
$$+2(\overline{\tau}-\alpha)\phi_{21}-\overline{\sigma}\phi_{22}.$$

$$\Delta\psi_3-\delta\psi_4+\overline{\delta}\phi_{22}-\Delta\phi_{21}=3\nu\overline{\psi}_2-2(\gamma+2\mu)\psi_3+$$
$$+(4\beta-\tau)\psi_4-2\gamma\phi_{11}-\overline{\gamma}\phi_{20}+2\lambda\phi_{12}+ \qquad (3.32)$$
$$+2(\gamma+\overline{\mu})\phi_{21}+(\overline{\tau}-2\overline{\beta}-2\alpha)\phi_{22}.$$

$$D\phi_{11}-\delta\phi_{10}-\overline{\delta}\phi_{01}+\Delta\phi_{00}+3D\wedge=$$
$$=(2\gamma-\mu+2\overline{\gamma}-\overline{\mu})\phi_{00}+(\pi-2\alpha-2\overline{\tau})\phi_{01}+$$
$$+(\overline{\pi}-2\overline{\alpha}-2\tau)\phi_{10}+2(\rho+\overline{\rho})\phi_{11}+\overline{\sigma}\phi_{02}+ \qquad (3.33)$$
$$+\sigma\phi_{20}-\overline{\kappa}\phi_{12}-\kappa\phi_{21}.$$

$$D\phi_{12}-\delta\phi_{11}-\overline{\delta}\phi_{02}+\Delta\phi_{01}+3\delta\wedge=$$
$$=(2\gamma-\mu-2\overline{\mu})\phi_{01}+\overline{\nu}\phi_{00}+2(\overline{\pi}-\tau)\phi_{11}+ \qquad (3.34)$$
$$+(\pi+2\overline{\beta}-2\alpha-\overline{\tau})\phi_{02}+(2\rho+\overline{\rho}-2\overline{\varepsilon})\phi_{12}+\sigma\phi_{21}-\kappa\phi_{22}.$$

$$D\phi_{22}-\delta\phi_{21}-\overline{\delta}\phi_{12}+\Delta\phi_{11}+3\Delta\wedge=\nu\phi_{01}+$$
$$+\overline{\nu}\phi_{10}-2(\mu+\overline{\mu})\phi_{11}-\lambda\phi_{02}-\overline{\lambda}\phi_{20}+$$
$$+(2\pi-\overline{\tau}+2\overline{\beta})\phi_{12}+(2\beta-\tau-2\overline{\pi})\phi_{21}+ \qquad (3.35)$$
$$+(\rho+\overline{\rho}-2\in-2\overline{\varepsilon})\phi_{22}$$

The Newman Penrose language is not only the best-suited language for the task, but many working relativists understood it. The formalism is very standard now and its exposition is readily available in many books, eg. *Carmelli (1977), Flaherty (1979), Hawking and Israel (1979), Frolov (1979), Held(1980), Kramer, Stephani, Herlt, Macmillan (1980)* and *Chandrasekhar (1983)* hence all other details are not presented here.

In the chapter.1 no originality is claimed.