CHAPTER – III

GENERATING RELATION

CHAPTER - 3

GENERATING RELATION

(3.1) INTRODUCTION:-

Thakare and Madhekar [37] constructed polynomials $S_n(x\ ;\ k)$ and $T_n(x\ ;\ k)$ ($n=0,1,2,---$, $k= 1,3,5,----$) that are related to the Konhauser [7,8] biorthogonal pair of polynomials

$$Z_n^{\alpha}(x;k) \text{ and } Y_n^{\alpha}(x;k)$$

the following form.

$$S_{2n}(x;k) = \frac{(-1)^n 2^{2n} n!\,\Gamma\left(kn + \frac{k}{2}\right)}{\Gamma\left(kn + \frac{1}{2}\right)} Z_n^{-\frac{1}{2}}\left(x^{2k};k\right) \quad ---(3.1.1)$$

$$S_{2n+1}(x;k) = (-1)^n 2^{2n+1} n!\, x^k Z_n^{\frac{k}{2}}\left(x^2;k\right) \quad ---(3.1.2)$$

$$T_{2n}(x;k) = (-1)^n 2^{2n} n!\, Y_n^{-\frac{1}{2}}\left(x^2;k\right) \quad ---(3.1.3)$$

$$T_{2n+1}(x;k) = (-1)^n 2^{2n+1} n!\, x Y_n^{\frac{k}{2}}\left(x^2;k\right) \quad ---(3.1.4)$$

We express the biorthogonal polynomials corresponding to Konhauser biorthogonal polynomials of first kind in to the double hypergeometric functions.

(3.2) GENERATING RELATIONS:-

The first set

$$Z_n^\alpha(x;k)$$

of the Konhauser [17,p. 304] biorthogonal polynomials is

$$Z_n^\alpha(x;k)=\frac{\Gamma(kn+\alpha+1)}{n!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{x^{kj}}{\Gamma(kj+\alpha+1)} \quad \text{---(3.2.1)}$$

were $\alpha>-1$ and k is a positive integer. An immediate consequence of (3.2.1) is

$$Z_n^\alpha(x;k)=\frac{(1+\alpha)_{kn}}{n!}\,{}_1F_k\left[\begin{matrix}-n; & \left(\frac{x}{k}\right)^k \\ \Delta(k,\alpha+1); & \end{matrix}\right] \quad \text{---(3.2.2)}$$

where Δ (k,α) represents the array of parameters

$$\frac{\alpha}{k},\frac{\alpha+1}{k},\ldots,\frac{\alpha+k-1}{k}$$

The Kampe de Feriet's double hypergeometric function in the contracted notation of Burchnall and Chaundy [4, P.112] is defined by

$$F^{(2)}\left[\begin{matrix}(a):(b):(c); \\ (e):(g):(h);\end{matrix} x, y\right]$$

$$= \sum_{m,n=0}^{\infty} \frac{[(a)]_{m+n}[(b)]_m[(c)]_n x^m y^n}{[(e)]_{m+n}[(g)]_m[(h)]_n m!n!} \quad ---(3.2.3)$$

In this definition (a) and $[(b)]_n$ abbreviate the sequence of A parameters a_1 , a_2,..., a_A and the product $(a_1)_n (a_2)_n$-------$(a_A)_n$ respectively .Recently S. N. Singh and L.S. Singh [31,P.31(2.1)] deduced generating relations for the polynomials Z_n^{α} (x; k) in terms of double hypergeometric function $F^{(2)}$ (x; k)

$$\sum_{n=0}^{\infty} \frac{[(a)]_n Z_n^{\alpha}(x;k) y^n}{[(e)]_n (1+\alpha)_{kn}}$$

$$= F^{(2)}\left[\begin{matrix}(a);—;—; \\ (e);\frac{(\alpha+1)}{k}...\frac{\alpha+k}{k};—;\end{matrix} \quad y, -\left\{\left[\frac{x}{k}\right]^k y\right\}\right] \quad ---(3.2.4)$$

and

$$\sum_{n=0}^{\infty} \frac{[(a)]_n\left(1-\frac{\alpha+1}{k}\right)_n ---\left(1-\frac{\alpha+k}{k}\right)_n Z_n^{\alpha-kn}(x;k)y^n}{[(e)]_n(\alpha+1-kn)_{kn}}$$

$$= F^{(2)}\left[\begin{matrix}(a);-;1-\frac{\alpha+1}{k},.....,\left(1-\frac{\alpha+k}{k}\right) \\ (e);---;-------;\end{matrix} \quad y, -\left\{\left[-\frac{x}{k}\right]^k y\right\}\right] \quad ---(3.2.5)$$

Now we have

$$\sum_{n=0}^{\infty}\frac{[(a)]_n(-1)^n\Gamma(kn+\frac{1}{2})S_{2n}(x;k)y^n}{[(e)]_n(\frac{1}{2})_{kn}2^{2n}n!\Gamma(kn+k/2)}$$

$$=\sum_{n=0}^{\infty}\frac{[(a)]_n(-1)^n\Gamma(kn+\frac{1}{2})}{[(e)]_n(\frac{1}{2})_{kn}2^{2n}n!}\frac{(-1)^n2^{2n}n!\Gamma(kn+k/2)Z_n^{-1/2}(x^2;k)}{\Gamma(kn+\frac{1}{2})}$$

$$=\sum_{n=0}^{\infty}\frac{[(a)]_nZ_n^{-1/2}(x^2;k)}{[(e)]_n(\frac{1}{2})_{kn}}\text{---by..using(3.1.1)--(3.2.6)}$$

Replace x by x^2 and put $\alpha=-1/2$ in equation (3.2.4)

$$\sum_{n=0}^{\infty}\frac{[(a)]_nZ_n^{-1/2}(x^2;k)}{[(e)]_n(\frac{1}{2})_{kn}}=F^{(2)}\left[\begin{matrix}(a);—;—;\\(e);\frac{1}{2k};\frac{3}{2k};\ldots;\frac{k-\frac{1}{2}}{k};—;\end{matrix}\quad y,-\left\{\left(\frac{x^2}{k}\right)^k y\right\}\right]\text{--(3.2.7)}$$

Using equations (3.2.7) in right hand side of the equation (3.2.6) we get,

$$\sum_{n=0}^{\infty}\frac{[(a)]_n(-1)^n\Gamma(kn+\frac{1}{2})}{[(e)]_n(\frac{1}{2})_{kn}2^{2n}n!}\frac{s_{2n}(x;k)y^n}{\Gamma(kn+\frac{k}{2})}$$

$$=F^{(2)}\left[\begin{matrix}(a);—;—;\\(e);\frac{1}{2k};\frac{3}{2k};\ldots;\frac{k-\frac{1}{2}}{k};—;\end{matrix}\quad y,-\left\{\left(\frac{x^2}{k}\right)^k y\right\}\right]\text{---(3.2.8)}$$

Similarly

$$\sum_{n=0}^{\infty}\frac{[(a)]_n(-1)^n s_{2n+1}(x;k)y^n}{[(e)]_n\left(\frac{k}{2}+1\right)_{kn}2^{2n+1}n!}=\sum_{n=0}^{\infty}\frac{[(a)]_n x^k Z_n^{k/2}(x^2;k)y^n}{[(e)]_n\left(\frac{k}{2}+1\right)_{kn}} \quad ---(3.2.9)$$

by using (3.1.2)

Replace x by x^2 and put $\alpha = k/2$ in equation. (3.2.4)

$$\sum_{n=0}^{\infty}\frac{[(a)]_n \; Z_n^{k/2}(x^2;k)y^n}{[(e)]_n\left(\frac{k}{2}+1\right)_{kn}}$$

$$=F^{(2)}\left[\begin{matrix}(a);\text{—};\text{—};\\(e);\frac{\frac{k}{2}+1}{k},\frac{k/2+2}{k},\ldots\ldots\frac{k/2+k}{k};\text{—};\end{matrix}\; y,\left\{\left(\frac{x^2}{k}\right)^k y\right\}\right] \quad ---(3.2.10)$$

Using (3.2.10) in right hand side of (3.2.9), we get,

$$\sum_{n=0}^{\infty}\frac{[(a)]_n(-1)^n s_{2n+1}(x;k)y^n}{[(e)]_n\left(\frac{k}{2}+1\right)_{kn}2^{2n+1}n!}=x^k\sum_{n=0}^{\infty}\frac{[(a)]_n Z_n^{k/2}(x^2;k)y^n}{[(e)]_n\left(\frac{k}{2}+1\right)_{kn}}$$

$$=x^kF^{(2)}\left[\begin{matrix}(a);\text{—};\text{—};\\(e);\frac{\frac{k}{2}+1}{k},\frac{k/2+2}{k},\text{—}\frac{k/2+k}{k};\end{matrix}\; y,\left\{\left(\frac{x^2}{k}\right)^k y\right\}\right] \quad ---(3.2.11)$$

By equation (3.1.1) we have

$$S_{2n}(x;k)=\frac{(-1)^n 2^{2n} n!\Gamma\left(kn+\frac{k}{2}\right)Z_n^{-1/2}\left(x^2;k\right)}{\Gamma\left(kn+\frac{1}{2}\right)}$$

$$\frac{(-1)^n\Gamma\left(kn+\frac{1}{2}\right)S_{2n}(x;k)}{2^{2n}\cdot n!\cdot\Gamma\left(kn+\frac{k}{2}\right)}=Z_n^{-1/2}\left(x^2;k\right)$$

Multiplying by

$$\frac{[(a)]_n\left(1-\frac{kn+\frac{1}{2}}{k}\right)_n ---\left(1-\frac{kn-\frac{1}{2}+k}{k}\right)_n y^n}{[(e)]_n\left(\frac{1}{2}\right)_{kn}}$$

and taking summation on both the sides over n=o, 1,2,...., we get,

$$\sum_{n=0}^{\infty}\frac{[(a)]_n\left(1-\frac{kn+\frac{1}{2}}{k}\right)_n ---\left(1-\frac{kn-\frac{1}{2}+k}{k}\right)_n (-1)^n\Gamma\left(kn+\frac{1}{2}\right)S_{2n}(x;k)y^n}{[(e)]_n\left(\frac{1}{2}\right)_{kn}2^{2n}n!\Gamma\left(kn+\frac{k}{2}\right)}$$

$$=\sum_{n=0}^{\infty}\frac{[(a)]_n\left(1-\frac{kn+\frac{1}{2}}{k}\right)_n ---\left(1-\frac{kn-\frac{1}{2}+k}{k}\right)_n Z_n^{-1/2}\left(x^2;k\right)y^n}{[(e)]_n\left(\frac{1}{2}\right)_{kn}} \qquad ---(3.2.12)$$

Replace x by x^2 and put $\alpha = kn-1/2$ in equation (3.2.5)

$$\sum_{n=0}^{\infty} \frac{[(a)]_n \left(1-\frac{kn-\frac{1}{2}+1}{k}\right)_n --- \left(1-\frac{kn-\frac{1}{2}+k}{k}\right)_n Z_n^{-1/2}(x^2;k) y^n}{[(e)]_n \left(\frac{1}{2}\right)_{kn}}$$

$$= F^{(2)}\left[\begin{matrix}(a);-;;\frac{1-kn-\frac{1}{2}+1}{k},\ldots\ldots;\frac{1-kn-\frac{1}{2}+k}{k} \\ (e);-;-;\end{matrix}\; y, -\left\{\left(-x^2/k\right)^k y\right\}\right] \qquad ---(3.2.13)$$

Using equation (3.2.13) in right side of the equation (3.2.12) we get.

$$\sum_{n=0}^{\infty} \frac{[(a)]_n \left(1-\frac{kn+\frac{1}{2}}{k}\right)_n --- \left(1-\frac{kn-\frac{1}{2}+k}{k}\right)_n (-1)^n \Gamma\left(kn+\frac{1}{2}\right) S_{2n}(x;k) y^n}{[(e)]_n \left(\frac{1}{2}\right)_{kn} 2^{2n} n! \Gamma\left(kn+\frac{k}{2}\right)}$$

$$= F^{(2)}\left[\begin{matrix}(a);-;1-\frac{kn+\frac{1}{2}}{k},\ldots\ldots;1-\frac{kn-\frac{1}{2}+k}{k} \\ (e);-;-;\end{matrix}\; y, -\left\{\left(-x^2/k\right)^k y\right\}\right] \qquad ----(3.2.14)$$

We recall (3.1.2)

$$S_{2n+1}(x;k) = (-1)^n 2^{2n+1} n! x^k Z_n^{k/2}(x^2;k)$$

$$\frac{(-1)^n S_{2n+1}(x;k)}{2^{2n+1} n!} = x^k Z_n^{k/2}(x^2;k)$$

Multiplying this equation by

$$\frac{[(a)]_n \left(1-\frac{k\left(n+\frac{1}{2}\right)+1}{k}\right)_n --- \left(1-\frac{k\left(n+3/2\right)}{k}\right)_n y^n}{[(e)]_n (1+k/2)_{kn}}$$

and taking summation on both the side over n = 0, 1, 2, ...

$$\sum_{n=0}^{\infty} \frac{(-1)^k [(a)]_n \left(1-\frac{k\left(n+\frac{1}{2}\right)+1}{k}\right)_n --- \left(1-\frac{k\left(n+3/2\right)}{k}\right)_n S_{2n+1}(x;k) y^n}{[(e)]_n (1+k/2)_{kn} 2^{2n+1} n!}$$

$$= \sum_{n=0}^{\infty} \frac{x^k [(a)]_n \left(1-\frac{k\left(n+\frac{1}{2}\right)+1}{k}\right)_n --- \left(1-\frac{k\left(n+3/2\right)}{k}\right)_n Z_n^{k/2}(x^2;k) y^n}{[(e)]_n (1+k/2)_{kn}}$$

$$= x^k \sum_{n=0}^{\infty} \frac{[(a)]_n \left(1-\frac{k\left(n+\frac{1}{2}\right)+1}{k}\right)_n --- \left(1-\frac{k\left(n+3/2\right)}{k}\right)_n Z_n^{k/2}(x^2;k) y^n}{[(e)]_n (1+k/2)_{kn}} \quad ---(3.2.15)$$

By equation (3.2.5)

$$\sum_{n=0}^{\infty} \frac{[(a)]_n \left(1-\frac{\alpha+1}{k}\right)_n --- \left(1-\frac{\alpha+k}{k}\right)_n Z_n^{\alpha-kn}(x;k) y^n}{[(e)]_n (\alpha+1-kn)_{kn}}$$

$$= F^{(2)}\left[\begin{matrix}(a);\text{———};1-\frac{\alpha+1}{k},\text{—},\left(1-\frac{\alpha+k}{k}\right); \\ (e);\text{———};\text{————————};\end{matrix}\; y, -\left\{\left(-\tfrac{x}{k}\right)^k y\right\}\right]$$

Replace x by x^2 and put $\alpha = kn + k/2$

$$\sum_{n=0}^{\infty} \frac{[(a)]_n \left(1-\frac{kn+k/2+1}{k}\right)_n \cdots \left(1-\frac{kn+k/2+k}{k}\right)_n Z_n^{k/2}(x^2;k)y^n}{[(e)]_n (1+k/2)_{kn}}$$

$$= F^{(2)}\left[\begin{matrix}(a);\text{——};1-\frac{kn+\frac{k}{2}+1}{k},\ldots,1-\frac{kn+\frac{k}{2}+k}{k}; \\ (e);\text{———};\text{————————};\end{matrix}\; y, -\left\{\left(-x^2/k\right)^k y\right\}\right] \text{-----}(3.2.16)$$

By equation (3.2.15) and (3.2.16) we get.

$$\sum_{n=0}^{\infty}(-1)^n \frac{[(a)]_n \left(1-\frac{kn+k/2+1}{k}\right)_n \cdots \left(1-\frac{kn+k/2+k}{k}\right)_n S_{2n+1}(x;k)y^n}{[(e)]_n (1+k/2)_{kn} 2^{2n+1} n!}$$

$$= x^k F^{(2)}\left[\begin{matrix}(a);\text{———};\left(1-\frac{kn+k/2+1}{k}\right),\text{—},\left(1-\frac{kn+k/2+k}{k}\right); \\ (e);\text{———};\text{————————};\end{matrix}\; y, -\left\{\left(-x^2/k\right)^k y\right\}\right] \text{---}(2.3.17)$$

(3.3) Generating Relations of Biorthogonal Polynomials for the Szego Hermite weight functions:-

The Szego Hermite polynomials $H_n^{\mu}(x)$ are orthogonal with respect to the Szego Hermite weight function $|x|^{2\mu}\exp(-x^2)$ for $\mu > -1/2$ over the interval $(-\infty, \infty)$. For $\mu = 0$ these polynomials are just the classical Hermite polynomials..

Thakare and Madhekar [38] have introduced the following biorthogonal pair.

$$S_{2n}^{\mu}(x;k)=\frac{(-1)^n 2^{2n} n!\Gamma(kn+\mu+k/2)\cdot n!}{\Gamma(kn+\mu+\frac{1}{2})} Z_n^{\mu-1/2}(x^2;k) \text{---}(3.1.1)$$

$$S_{2n+1}^{\mu}(x;k)=(-1)^n 2^{2n+1} n! x^k Z_n^{\mu+k/2}(x^2;k) \qquad \text{---}(3.1.2)$$

$$T_{2n}^{\mu}(x;k)=(-1)^n 2^{2n} n! Y_n^{\mu-1/2}(x^2;k) \qquad \text{---}(3.1.3)$$

$$T_{2n+1}^{\mu}(x;k)=(-1)^n 2^{2n+1} n! x Y_n^{\mu+k/2}(x^2;k) \text{---}(3.1.4)$$

where

$Z_n^{\alpha}(x;k)$ and $Y_n^{\alpha}(x;k)$

is a pair of Konhauser biorthogonal polynomials w.r.t. the Laguerre weight function

$x^{\alpha}\exp(-x)$ over $(0,\infty)\, \mu > -\frac{1}{2}, \alpha > -1$

and k is positive integer.

We express $S_n^{\mu}(x;k)$ in double hypergeometric functions. Replace x by x^2 and put $\alpha = \mu - 1/2$ in equation . (3.2.4)

$$\sum_{n=0}^{\infty}\frac{[(a)]_n Z_n^{\mu-\frac{1}{2}}(x^2;k)y^n}{[(e)]_n\left(\mu+\frac{1}{2}\right)_{kn}} = F^{(2)}\left[\begin{matrix}(a);\text{———};\text{———}; \\ (e);\frac{\mu+\frac{1}{2}}{k},\frac{\mu+\frac{3}{2}}{k},\ldots\frac{\mu-\frac{1}{2}+k}{k};\text{——};\end{matrix}\; y, -\left\{\left(\frac{x^2}{k}\right)^k y\right\}\right] \quad \text{---}(3.3.5)$$

By (3.3.1)

$$S_{2n}^{\mu}(x;k) = \frac{(-1)^n 2^{2n} n!\,\Gamma\left(kn+\mu+\frac{k}{2}\right)}{\Gamma\left(kn+\mu+\frac{1}{2}\right)} Z_n^{\mu-\frac{1}{2}}\left(x^2;k\right)$$

$$\frac{(-1)^n\Gamma\left(kn+\mu+\frac{k}{2}\right)S_{2n}^{\mu}(x;k)}{2^{2n} n!\,\Gamma\left(kn+\mu+\frac{k}{2}\right)} = Z_n^{\mu-\frac{1}{2}}\left(x^2;k\right)$$

Multiplying both the side by

$$\frac{[(a)]_n y^n}{[(e)]_n\left(\mu+\frac{1}{2}\right)_{kn}}$$

and taking summation on both the sides over n=0, 1,2---

$$\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma\left(kn+\mu+\frac{1}{2}\right)\cdot S_{2n}^{\mu}(x;k)\cdot[(a)]_n y^n}{2^{2n} n!\,\Gamma\left(kn+\mu+\frac{k}{2}\right)\cdot[(e)]_n\left(\mu+\frac{1}{2}\right)_{kn}} = \sum_{n=0}^{\infty}\frac{[(a)]_n Z_n^{\mu-\frac{1}{2}}\left(x^2;k\right)y^n}{[(e)]_n\left(\mu+\frac{1}{2}\right)_{kn}} \quad \text{---}(3.3.6)$$

Using (3.3.5) in right hand side of equation (3.3.6)

$$\sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(kn+\mu+\tfrac{1}{2}) S^{\mu}_{2n}(x;k) [(a)]_n y^n}{2^{2n} n! \Gamma(kn+\mu+\tfrac{k}{2}) [(e)]_n (\mu+\tfrac{1}{2})_{kn}}$$

$$= F^{(2)}\left[\begin{matrix} (a); \text{———}; \text{———}; \\ (e); \frac{\mu+\frac{1}{2}}{k}, \frac{\mu+\frac{3}{2}}{k}, \text{——}, \frac{\mu-\frac{1}{2}+k}{k}, \text{———}; \end{matrix}\; y, -\left\{\left(\tfrac{x^2}{k}\right)^k y\right\}\right] \quad \text{---}(3.3.7)$$

Replace x by x^2 and put $\alpha = \mu + k/2$ in equation (3.2.4)

$$\sum_{n=0}^{\infty} \frac{[(a)]_n Z_n^{\mu+k/2}(x^2;k) y^n}{[(e)]_n (\mu+\tfrac{k}{2}+1)_{kn}}$$

$$= F^{(2)}\left[\begin{matrix} (a); \text{———}; \text{———}; \\ (e); \frac{\mu+\frac{k}{2}+1}{k}, \frac{\mu+\frac{k}{2}+2}{k}, \text{——}, \frac{\mu+\frac{k}{2}+k}{k}; \text{———}; \end{matrix}\; y, -\left\{\left(\tfrac{x^2}{k}\right)^k y\right\}.\right] \quad \text{---}(3.3.8)$$

By (3.3.2)

$$S^{\mu}_{2n+1}(x;k) = (-1)^n 2^{2n+1} n! x^k Z_n^{\mu+k/2}(x^2;k)$$

$$\frac{(-1)^n S^{\mu}_{2n+1}(x;k)}{2^{2n+1} n!} = x^k Z_n^{\mu+k/2}(x^2;k)$$

Multiplying this equation by

$$\frac{[(a)]_n y^n}{[(e)]_n (\mu+\tfrac{k}{2}+1)_{kn}}$$

and taking summation over $n = 0, 1, 2, \ldots.$

$$\sum_{n=0}^{\infty}\frac{(-1)^n[(a)]_n S_{2n+1}^{\mu}(x;k)y^n}{[(e)]_n\left(\mu+\frac{k}{2}+1\right)_{kn}2^{2n+1}n!}=\sum_{n=0}^{\infty}\frac{x^k[(a)]_n Z_n^{\mu+k/2}(x^2;k)y^n}{[(e)]_n\left(\mu+\frac{k}{2}+1\right)_{kn}}$$

$$=x^k\sum_{n=0}^{\infty}\frac{[(a)]_n Z_n^{\mu+k/2}(x^2;k)y^n}{[(e)]_n\left(\mu+\frac{k}{2}+1\right)_{kn}} \qquad ---(3.3.9)$$

Using (3.3. 9) in right hand side of equation (3.3.8)

$$\sum_{n=0}^{\infty}\frac{(-1)^n[(a)]_n S_{2n+1}^{\mu}(x;k)y^n}{[(e)]_n\left(\mu+\frac{k}{2}+1\right)_{kn}2^{2n+1}n!}$$

$$=x^kF^{(2)}\left[\begin{matrix}(a);\text{———};\text{———};\\(e);\frac{\mu+\frac{k}{2}+1}{k},\frac{\mu+\frac{k}{2}+2}{k},\text{——},\frac{\mu+\frac{k}{2}+k}{k};\text{———};\end{matrix}\;y,-\left\{\left(x^2/k\right)^k y\right\}\right] \qquad ---(3.3.10)$$

We recall (3.2.5)

$$\sum_{n=0}^{\infty}\frac{[(a)]_n\left(1-\frac{\alpha+1}{k}\right)_n ----\left(1-\frac{\alpha+k}{k}\right)_n Z_n^{\alpha-kn}(x;k)y^n}{[(e)]_n(\alpha+1-kn)_{kn}}$$

$$=F^{(2)}\left[\begin{matrix}(a);\text{———};\left(1-\frac{\alpha+1}{k}\right),\text{——};\left(1-\frac{\alpha+k}{k}\right),\\(e);\text{———};\text{——————————};\end{matrix}\;y,-\left\{\left(-\frac{x}{k}\right)^k y\right\}\right]$$

Replace x by x^2 and put $\alpha = kn + \mu - 1/2$

$$\sum_{n=0}^{\infty}\frac{[(a)]_n\left(1-\frac{kn+\mu+\frac{1}{2}}{k}\right)_n\left(1-\frac{kn+\mu+\frac{3}{2}}{k}\right)_n--\left(1-\frac{kn+\mu-\frac{1}{2}}{k}\right)_n Z_n^{\mu-\frac{1}{2}}(x^2;k)}{[(e)]_n\left(\mu+\frac{1}{2}\right)_{kn}}$$

$$=F^{(2)}\left[\begin{matrix}(a);\text{———};\left(1-\frac{kn+\mu+\frac{1}{2}}{k}\right),\text{——},\left(1-\frac{kn+\mu+k-\frac{1}{2}}{k}\right); \\ (e);\text{———};\text{———————};\end{matrix}\; y,-\left\{\left(-\frac{x}{k}^2\right)^k y\right\}\right] \qquad ---(3.3.11)$$

By (3.3.1)

$$S_{2n}^{\mu}(x;k)=\frac{(-1)^n 2^{2n} n!\,\Gamma\left(kn+\mu+\frac{k}{2}\right)Z_n^{\mu-\frac{1}{2}}(x^2;k)}{\Gamma\left(kn+\mu+\frac{1}{2}\right)}$$

$$\frac{(-1)^n\Gamma\left(kn+\mu+\frac{1}{2}\right)S_{2n}^{\mu}(x;k)}{2^{2n} n!\,\Gamma\left(kn+\mu+\frac{k}{2}\right)}=Z_n^{\mu-\frac{1}{2}}(x^2;k)$$

Multiplying both the sides of this equation by

$$\frac{[(a)]_n\left(1-\frac{kn+\mu+\frac{1}{2}}{k}\right)_n\left(1-\frac{kn+\mu+\frac{3}{2}}{k}\right)_n--\left(1-\frac{kn+\mu-\frac{1}{2}+k}{k}\right)_n}{[(e)]_n\left(\mu+\frac{1}{2}\right)_{kn}}$$

and taking summation over n=0,1,2,---

$$\sum_{n=0}^{\infty}\frac{[(a)]_n\left(1-\frac{kn+\mu+\frac{1}{2}}{k}\right)_n\cdots\left(1-\frac{kn+\mu-\frac{1}{2}+k}{k}\right)_n(-1)^n\Gamma\left(kn+\mu+\frac{1}{2}\right)}{[(e)]_n\left(\mu+\frac{1}{2}\right)_{kn}2^{2n}n!\,\Gamma\left(kn+\mu+\frac{k}{2}\right)}S_{2n}^{\mu}\left(x^2;k\right)y^n$$

$$=\sum_{n=0}^{\infty}\frac{[(a)]_n\left(1-\frac{kn+\mu+\frac{1}{2}}{k}\right)_n\cdots\left(1-\frac{kn+\mu-\frac{1}{2}+k}{k}\right)_n}{[(e)]_n\left(\mu+\frac{1}{2}\right)_{kn}}Z_n^{\mu-1/2}\left(x^2;k\right)y^n\ ---(3.3.12)$$

Using (3.3.11) in right hand side of (3.3.12)

$$\sum_{n=0}^{\infty}\frac{[(a)]_n\left(1-\frac{kn+\mu+\frac{1}{2}}{k}\right)_n\cdots\left(1-\frac{kn+\mu-\frac{1}{2}+k}{k}\right)_n(-1)^n\Gamma\left(kn+\mu+\frac{1}{2}\right)}{[(e)]_n\left(\mu+\frac{1}{2}\right)_{kn}2^{2n}n!\,\Gamma\left(kn+\mu+\frac{k}{2}\right)}S_{2n}^{\mu}\left(x^2;k\right)\cdot y^n$$

$$=F^{(2)}\left[\begin{matrix}(a);\text{———};1-\frac{kn+\mu+\frac{1}{2}}{k},---,1-\frac{kn+\mu+k-\frac{1}{2}}{k};\\(e);\text{———};\text{——————————};\end{matrix}\ y,-\left\{\left(-\frac{x}{k}^{2}\right)^{k}y\right\}\cdot\right]----(3.2.13)$$

By (3.2.5)

$$\sum_{n=0}^{\infty}\frac{[(a)]_n\left(1-\frac{\alpha+1}{k}\right)_n----\left(1-\frac{\alpha+k}{k}\right)_n z_n^{\alpha-kn}(x;k)y^n}{[(e)]_n(\alpha+1-kn)_{kn}}$$

$$=F^{(2)}\left[\begin{matrix}(a);\text{———};\left(1-\frac{\alpha+1}{k}\right),\text{——},\left(1-\frac{\alpha+k}{k}\right);\\(e);\text{———};\text{——————————};\end{matrix}\ y,-\left\{\left(-\frac{x}{k}\right)^{k}y\right\}\cdot\right]$$

Replace x by x^2 and put $\alpha = kn + \mu + k/2$

$$\sum_{n=0}^{\infty} \frac{[(a)]_n \left(1-\frac{kn+\mu+\frac{k}{2}+1}{k}\right)_n \cdots \left(1-\frac{kn+\mu+\frac{k}{2}+k}{k}\right)_n}{[(e)]_n \left(\mu+\frac{k}{2}+1\right)_{kn}} Z_n^{\mu+k/2}\left(x^2;k\right)\cdot y^n$$

$$= F^{(2)}\left[\begin{matrix} (a);\text{———};1-\frac{kn+\mu+\frac{k}{2}+1}{k},\text{—},1-\frac{kn+\mu+\frac{k}{2}+k}{k}; \\ (e);\text{———};\text{——————————}; \end{matrix} y,-\left\{\left(-\frac{x}{k}^2\right)^k y\cdot\right\}\right] \text{---} (3.3.14)$$

By .....(3.3.2)

$$S_{2n+1}^{\mu}(x;k) = (-1)^n 2^{2n+1} n! x^k Z_n^{\mu+k/2}\left(x^2;k\right)$$

$$\frac{(-1)^n S_{2n+1}^{\mu}(x;k)}{2^{2n+1} n!} = .x^k Z_n^{\mu+k/2}\left(x^2;k\right)$$

Multiply this equation by

$$\frac{[(a)]_n \left(1-\frac{kn+\mu+\frac{k}{2}+1}{k}\right)_n \text{ --- } \left(1-\frac{kn+\mu+\frac{k}{2}+k}{k}\right)_n}{[(e)]_n \left(\mu+\frac{k}{2}+1\right)_{kn}} y^n$$

and.taking.summation.over.n = 0,1,2...

$$\sum_{n=0}^{\infty}(-1)^n \frac{[(a)]_n \left(1-\frac{kn+\mu+\frac{k}{2}+1}{k}\right)_n \cdots\cdots \left(1-\frac{kn+\mu+\frac{k}{2}+k}{k}\right)_n}{[(e)]_n 2^{2n+1} n! \left(\mu+\frac{k}{2}+1\right)_{kn}} S_{2n+1}^{\mu}(x;k) y^n$$

$$= \sum_{n=0}^{\infty} \frac{x^k [(a)]_n \left(1-\frac{kn+\mu+\frac{k}{2}+1}{k}\right)_n \cdots \left(1-\frac{kn+\mu+\frac{3k}{2}}{k}\right)_n}{[(e)]_n (\mu+\frac{k}{2}+1)_{kn}} Z_n^{\mu+k/2}(x^2;k) y^n$$

$$= x^k F^{(2)}\left[\begin{matrix}(a); \text{------}; 1-\frac{kn+\mu+\frac{k}{2}+1}{k}, \text{---}, 1-\frac{kn+\mu+\frac{3k}{2}}{k}; \\ (e); \text{------}; \text{----------------}; \end{matrix} y, -\left\{-\frac{x}{k}^2\right\}^k y\right] \quad \text{---(3.3.15)}$$

Particular cases: For $\mu=0$, the equations, (3.3.7), (3.3.10), (3.3.13) and (3.3.15) reduces to (3.2.8), (3.2.11), (3.2.14) and (3.2.17) respectiely.

(3.4) Generating Relation of Biorthogonal polynomials for Generalized Hermite polynomials:

For, $\beta = -\frac{1}{2}k\ldots$, $k = 1, 2, 3, \ldots$, $n=0, 1, 2, 3, \ldots$ .and $\ell = 1,3,5\ldots$.

Andhare and Jagtap [2] constructed following pair of Biorthogonal polynomials.

$$S_{2n}(x;k,\ell) = \frac{(-1)^n n!(1+n)_n}{(1+\beta)_n} Z_n^{\beta}(x^{2k};\ell) \quad \text{---(3.4.1)}$$

$$S_{2n+1}(x;k,\ell) = \frac{(-1)^n n!\left(\frac{3}{2}\right)_n 2^{2n+1} x^{\ell} . Z_n^{-\beta\ell}(x^{2k};\ell)}{(1-\beta)_n} \quad \text{---(3.4.2)}$$

$$T_{2n}(x;k,\ell) = \frac{(-1)^n n!(1+n)_n}{(1+\beta)_n} Y_n^{\beta}(x^{2k};\ell) \quad \text{---(3.4.3)}$$

$$T_{2n+1}(x;k,\ell) = \frac{(-1)^n n! 2^{2n+1}\left(\frac{3}{2}\right)_n x.}{(1-\beta)_n} Y_n^{-\beta\ell}(x^{2k};\ell) \quad \text{---(3.4.4)}$$

where $Z_n^{\alpha}(x;k)$ and $Y_n^{\alpha}(x;k)$ is a pair of Konhauser biorthogonal polynomials w.r.t Laguerre weight function x^{α} exp.(-x) over (0,∞)

By (3.2.4) we write.

$$\sum_{n=0}^{\infty}\frac{[(a)]_n Z_n^{\alpha}(x;k)y^n}{[(e)]_n(\alpha+1)_{kn}} = F^{(2)}\left[\begin{matrix}(a);\text{———};\text{———};\\ (e);\text{—}\frac{\alpha+1}{k},\text{——},\frac{\alpha+k}{k};\text{—};\end{matrix} y,-\left\{\left(\tfrac{x}{k}\right)^k y\right\}\right]$$

Replace x by x^{2k}, k by ℓ and put $\alpha = \beta$ we get,

$$\sum_{n=0}^{\infty}\frac{[(a)]_n Z_n^{\beta}(x^{2k};\ell)y^n}{[(e)]_n(\beta+1)_{n\ell}} = F^{(2)}\left[\begin{matrix}(a);\text{———};\text{———};\\ (e);\text{—}\frac{\beta+1}{\ell},\text{——},\frac{\beta+\ell}{\ell};\text{—};\end{matrix} y,-\left\{\left(\tfrac{x^{2k}}{\ell}\right)^{\ell} y\right\}\right] \quad ---(3.4.5)$$

From equation (3.4.1)

.

$$\frac{(-1)^n(1+\beta)_n S_{2n}(x;k,\ell)}{(1+n)_n\, n!} = Z_n^{\beta}(x^{2k};\ell)$$

Multiply above equation by

$$\frac{[(a)]_n y^n}{[(e)]_n(\beta+1)_{\ell n}}$$

and taking summation on both the side over n=0,1,2,----

$$\sum_{n=0}^{\infty}\frac{(-1)^n[(a)]_n(1+\beta)_n S_{2n}(x;k,\ell)y^n}{n![(e)]_n(1+\beta)_{n\ell}(1+n)_n}$$

$$= \sum_{n=0}^{\infty}\frac{[(a)]_n Z_n^{\beta}(x^{2k};\ell)y^n}{[(e)]_n(1+\beta)_{n\ell}} \quad ---(3.4.6)$$

By equation (3.4.5) and (3.4.6) we get

$$\sum_{n=0}^{\infty}\frac{(-1)^n[(a)]_n(1+\beta)_n S_{2n}(x;k,\ell)y^n}{n![(e)]_n(\beta+1)_{\ell n}(1+n)_n}=F^{(2)}\left[\begin{matrix}(a);------;-------;\\(e);-\frac{(\beta+1)}{\ell},---\frac{(\beta+\ell)}{\ell};---;\end{matrix}\quad y,-\left\{\left(\frac{x^{2k}}{\ell}\right)^{\ell}y\right\}\right]\text{--(3.4.7)}$$

We recall the relation (3.2.4)

$$\sum_{n=0}^{\infty}\frac{[(a)]_n Z_n^{\alpha}(x;k)y^n}{[(e)]_n(1+\alpha)_{kn}}=F^{(2)}\left[\begin{matrix}(a);\text{———};\text{———};\\(e);\frac{(\alpha+1)}{k}\cdots\cdots\frac{(\alpha+k)}{k};\text{—};\end{matrix}\quad y,-\left\{\left(\frac{x}{k}\right)^{k}y\right\}\right]$$

Replace x by x^{2k}, k by ℓ and put $\alpha=-\beta\ell$

$$\sum_{n=0}^{\infty}\frac{[(a)]_n Z_n^{-\beta\ell}(x^{2k};\ell)\cdot y^n}{[(e)]_n(1-\beta\ell)_{\ell n}}=F^{(2)}\left[\begin{matrix}(a);\text{———};\text{———};\\(e);\frac{(1-\beta\ell)}{\ell}\cdots\cdots,\frac{(\ell-\beta\ell)}{\ell};\text{—};\end{matrix}\quad y,-\left\{\left(\frac{x^{2k}}{\ell}\right)^{\ell}y\right\}\right]\text{---(3.4.8)}$$

By equation (3.4.2)

$$S_{2n+1}(x;k,\ell)=\frac{(-1)^n n!\left(\frac{3}{2}\right)_n 2^{2n+1}x^{\ell}.Z_n^{-\beta\ell}(x^{2k};\ell)}{(1-\beta)_n}$$

$$\frac{(-1)^n(1-\beta)_n S_{2n+1}(x;k,\ell)}{n!\left(\frac{3}{2}\right)_n 2^{2n+1}}=x^{\ell}.Z_n^{-\beta\ell}(x^{2k};\ell)$$

Multiplying both the sides by

$$\frac{[(a)]_n y^n}{[(e)]_n(1-\beta\ell)_{\ell n}}$$

and taking summation over n=0,1,2,.... we get,

$$\sum_{n=0}^{\infty} \frac{(-1)^n (1-\beta)_n}{n!\left(\frac{3}{2}\right)_n 2^{2n+1}} \frac{[(a)]_n S_{2n+1}(x;k,\ell) y^n}{[(e)]_n (1-\beta\ell)_{n\ell}} = \sum_{n=0}^{\infty} \frac{[(a)]_n x^{\ell} . Z_n^{-\beta\ell}\left(x^{2k};\ell\right) y^n}{[(e)]_n (1-\beta\ell)_{n\ell}}$$

$$= x^{\ell} \sum_{n=0}^{\infty} \frac{[(a)]_n . Z_n^{-\beta\ell}\left(x^{2k};\ell\right) y^n}{[(e)]_n (1-\beta\ell)_{n\ell}} \quad \text{---(3.4.9)}$$

By equations (3.4.8) and (3.4.9) we get,

$$\sum_{n=0}^{\infty} \frac{(-1)^n (1-\beta)_n}{n!\left(\frac{3}{2}\right)_n 2^{2n+1}} \frac{[(a)]_n S_{2n+1}(x;k,\ell) y^n}{[(e)]_n (1-\beta\ell)_{n\ell}} = x^{\ell} F^{(2)}\left[\begin{matrix} (a); \text{---}; \text{---}; \\ (e); \frac{(1-\beta\ell)}{\ell}, \ldots, \frac{(\ell-\beta\ell)}{\ell}; \text{---}; \end{matrix} \; y, -\left\{\left(\frac{x^{2k}}{\ell}\right)^{\ell} y\right\}\right] \quad \text{----(3.4.10)}$$

Particular cases:

(i) The equation (3.4.7) with values β= - ½, and ℓ=k=1 and the equation (3.2.8) with value k=1 are identical.

(ii) For β = - ½ , ℓ=k and k = 1 the equation (3.4.10) reduces to the equation (3.2.11)

By equation (3.2.5)

$$\sum_{n=0}^{\infty} \frac{[(a)]_n \left(1-\frac{\alpha+1}{k}\right)_n \text{---} \left(1-\frac{\alpha+k}{k}\right)_n Z_n^{\alpha-kn}(x;k) y^n}{[(e)]_n (\alpha+1-kn)_{kn}}$$

$$= F^{(2)}\left[\begin{matrix} (a); \text{---}; \left(1-\frac{\alpha+1}{k}\right), \ldots, \left(1-\frac{\alpha+k}{k}\right); \\ (e); \text{---}; \text{---}; \end{matrix} \; y, -\left\{\left(-\frac{x}{k}\right)^k y\right\}\right]$$

Replacing x by x^{2k} and k by ℓ and putting $\alpha=\ell n+\beta$

$$\sum_{n=0}^{\infty}\frac{[(a)]_n\left(1-\frac{n\ell+\beta+1}{\ell}\right)_n---\left(1-\frac{n\ell+\beta+\ell}{\ell}\right)_n Z_n^{\beta}(x^{2k};\ell)y^n}{[(e)]_n(1+\beta)_{n\ell}}$$

$$=F^{(2)}\left[\begin{matrix}(a);-;\left(1-\frac{n\ell+\beta+1}{\ell}\right)-,\ldots,\left(1-\frac{n\ell+\beta+\ell}{\ell}\right);\\(e);\text{———};\text{———};\end{matrix}\; y,-\left\{\left(-\frac{x^{2k}}{\ell}\right)^{\ell}y\right\}\right]---(3.4.11)$$

By equation (3.4.1)

$$S_{2n}(x;k,\ell)=\frac{(-1)^n n!(1+n)_n}{(1+\beta)_n}Z_n^{\beta}(x^{2k};\ell)$$

$$\frac{(-1)^n(1+\beta)_n S_{2n}(x;k,\ell)}{n!(1+n)_n}=Z_n^{\beta}(x^{2k};\ell)$$

Multiplying this equation by

$$\frac{[(a)]_n\left(1-\frac{n\ell+\beta+1}{\ell}\right)_n---\left(1-\frac{n\ell+\beta+\ell}{\ell}\right)_n y^n}{n!(1+n)_n[(e)]_n(1+\beta)_{n\ell}}.$$

and taking summation on both the sides over $n=0,1,2,\ldots.$

we obtain,

$$\sum_{n=0}^{\infty} \frac{(-1)^n (1+\beta)_n [(a)]_n \left(1-\frac{n\ell+\beta+1}{\ell}\right)_n ---\left(1-\frac{n\ell+\beta+\ell}{\ell}\right)_n S_{2n}(x;k,\ell) y^n}{n!(1+n)_n [(e)]_n (1+\beta)_{n\ell}}.$$

$$= \sum_{n=0}^{\infty} \frac{[(a)]_n \left(1-\frac{n\ell+\beta+1}{\ell}\right)_n ---\left(1-\frac{n\ell+\beta+\ell}{\ell}\right)_n}{[(e)]_n (1+\beta)_{n\ell}} . Z_n^{\beta}(x^{2k};\ell) y^n$$

$$= F^{(2)}\left[\begin{matrix}(a);\text{—};\left(1-\frac{n\ell+\beta+1}{\ell}\right),\ldots,\left(1-\frac{n\ell+\beta+\ell}{\ell}\right); \\ (e);\text{———};\text{———};\end{matrix} \; y, \left\{\left(\frac{-x^{2k}}{\ell}\right)^{\ell} y\right\}\right] \text{---}(3.4.12)$$

Replace x by x^{2k}, k by ℓ and put $\alpha = n\ell-\beta\ell$ in equation (3.2.5)

$$\sum_{n=0}^{\infty} \frac{[(a)]_n \left(1-\frac{n\ell-\beta\ell+1}{\ell}\right)_n ---\left(1-\frac{n\ell-\beta\ell+\ell}{\ell}\right)_n Z_n^{-\beta\ell}(x^{2k};\ell) y^n}{[(e)]_n (1-\beta\ell)_{n\ell}}$$

$$= F^{(2)}\left[\begin{matrix}(a);\text{—};\left(1-\frac{n\ell-\beta\ell+1}{\ell}\right),\ldots,\left(1-\frac{n\ell-\beta\ell+\ell}{\ell}\right); \\ (e);\text{———};\text{———};\end{matrix} \; y, \left\{\left(-\frac{x^{2k}}{\ell}\right)^{\ell} y\right\}\right] \text{---}(3.413)$$

By equation (3.4.2)

$$S_{2n+1}(x;k,\ell) = \frac{(-1)^n n! \left(\frac{3}{2}\right)_n 2^{2n+1} x^{\ell} . Z_n^{-\beta\ell}\left(x^{2k};\ell\right)}{(1-\beta)_n}$$

$$\frac{(-1)^n(1-\beta)_n S_{2n+1}(x;k,\ell)}{n!\left(\frac{3}{2}\right)_n 2^{2n+1}} = x^{\ell}.Z_n^{-\beta\ell}\left(x^{2k};\ell\right)$$

Multiplying this equation by

$$\frac{[(a)]_n\left(1-\frac{n\ell-\beta\ell+1}{\ell}\right)_n --- \left(1-\frac{n\ell-\beta\ell+\ell}{\ell}\right)_n y^n}{[(e)]_n(1-\beta\ell)_{n\ell}}$$

and taking summation over n = 0 , 1, 2,.....

$$\sum_{n=0}^{\infty}\frac{(-1)^n(1-\beta)_n}{n!\left(\frac{3}{2}\right)_n 2^{2n+1}}\frac{[(a)]_n\left(1-\frac{n\ell-\beta\ell+1}{\ell}\right)_n --- \left(1-\frac{n\ell-\beta\ell+\ell}{\ell}\right)_n S_{2n+1}(x;k,\ell)y^n}{[(e)]_n(1-\beta\ell)_{n\ell}}$$

$$=\sum_{n=0}^{\infty}\frac{[(a)]_n\left(1-\frac{n\ell-\beta\ell+1}{\ell}\right)_n --- \left(1-\frac{n\ell-\beta\ell+\ell}{\ell}\right)_n y^n}{[(e)]_n(1-\beta\ell)_{n\ell}} x^{\ell}.Z_n^{-\beta\ell}\left(x^{2k};\ell\right) --- (3.4.14)$$

By equations (3.4.13) and (3.4.14) we obtain,

$$\sum_{n=0}^{\infty}\frac{(-1)^n(1-\beta)_n}{n!\left(\frac{3}{2}\right)_n 2^{2n+1}}\frac{[(a)]_n\left(1-\frac{n\ell-\beta\ell+1}{\ell}\right)_n --- \left(1-\frac{n\ell-\beta\ell+\ell}{\ell}\right)_n S_{2n+1}(x;k,\ell)y^n}{[(e)]_n(1-\beta\ell)_{n\ell}}$$

$$= x^{\ell}F^{(2)}\left[\begin{matrix}(a);\text{---};\left(1-\frac{n\ell-\beta\ell+1}{\ell}\right),\dots,\left(1-\frac{n\ell-\beta\ell+\ell}{\ell}\right); \\ (e);\text{---------};\text{------};\end{matrix}\; y, -\left\{\left[-\frac{x^{2k}}{\ell}\right]^{\ell} y\right\}\right] -- (3.4.15)$$

Particular cases:

i The equation (3.4.12) with values β= - ½, and ℓ=k=1 and the equation (3.2.8) with value k = 1 are identical.

ii For β = - ½ , ℓ = k and k = 1 the equation (3.4.15) reduces to the equation (3.2.17).