CHAPTER - THREE

FUZZY LATTICES

Chapter 3

⋆ Fuzzy Lattices ⋆

§3.1 Definition:

"A fuzzy partially ordered set X is called a Fuzzy Lattice if every two elements (i.e., every pair of elements) in X has a fuzzy least upper bound and a fuzzy greatest lower bound."

Let <X, R> be fuzzy partial ordered set. Let $x, y \in X$. If fuzzy least upper bound denoted by $x \vee y$ and fuzzy greatest lower bound denoted by $x \wedge y$ exists for all $x, y \in X$ then <X, R> is a fuzzy lattice.

Examples:

1) Let $X = [a, a + n]$ where $a, n \in IR$ and $n > 0$. Let R = "**almost** less than or equal to" be a fuzzy relation defined on X as a function $R: X \times X \rightarrow [0,1]$ defined by,

$$R(x, y) = \begin{cases} 1 & \text{if } x = y \\ (y - x)/n & \text{if } x < y \\ 0 & \text{else.} \end{cases}$$

Then (X, R) is a fuzzy lattice.

Proof: - First we will prove that (X, R) is a fuzzy poset.

i) **Fuzzy Reflexivity:**

Now, $R(x, x) = 1 \quad \forall x \in X.$ (by definition of R)

$\therefore$ R is a fuzzy reflexive relation.

ii) **Fuzzy Perfectly Antisymmetrivity:**

Let $x \neq y$ in X and $R(x, y) > 0$

Now, by definition of R, $R(x, y) = (y - x)/n$

$\therefore x < y$

$\therefore R(y, x) = 0$ (by definition of R)

$\therefore$ R is a fuzzy perfectly antisymmetric relation.

iii) Fuzzy max-min transitivity:

Let $(x, z) \in X^2$.

To prove: $R(x, z) \geq \max_{y \in X} \min (R(x, y), R(y, z))$ _____ (1)

case a) Let $x = z$,

$\therefore R(x, z) = 1$

$\therefore R(x, z) \geq \max_{y \in X} \min (R(x, y), R(y, z))$

case b) Let $x < z$

for all $y \in X$ such that $y < x$,

$R(x, y) = 0$ (by definition of R)

$\therefore \min (R(x, y), R(y, z)) = 0$

for all $y \in X$ such that $y = x$,

$R(x, y) = 1$ and $R(x, z) = R(y, z)$

$\therefore \min (R(x, y), R(y, z)) = R(y, z)$

$\therefore \min (R(x, y), R(y, z)) = R(x, z)$

for all $y \in X$ such that $x < y < z$,

$R(x, y) < R(x, z)$ and $R(y, z) < R(x, z)$

$\therefore \min (R(x, y), R(y, z)) < R(x, z)$

for all $y \in X$ such that $y = z$,

$R(y, z) = 1$ and $R(x, z) = R(x, y)$

$\therefore \min (R(x, y), R(y, z)) = R(x, y)$

$\therefore \min (R(x, y), R(y, z)) = R(x, z)$

for all $y \in X$ such that $z < y$,

$R(y, z) = 0$ (by definition of R)

$\therefore \min (R(x, y), R(y, z)) = 0$

Thus, $R(x, z) \geq \max_{y \in X} \min (R(x, y), R(y, z))$

case c) Let $x > z$

$\therefore R(x, z) = 0$

Thus to prove: $\max_{y \in X} \min (R(x, y), R(y, z)) = 0$

i.e., To prove, $\min(R(x,y), R(y,z)) = 0 \;\; \forall\, y \in X$

for all y such that $y \leq z$, Hence $y < x$

$\therefore R(x, y) = 0$

$\therefore \min (R(x, y), R(y, z)) = 0$

for all y such that $z < y < x$,

$\therefore R(x, y) = 0$ and $R(y, z) = 0$

$\therefore \min (R(x, y), R(y, z)) = 0$

for all y such that $z < x \leq y$,

$\therefore R(y, z) = 0$

$\therefore \min (R(x, y), R(y, z)) = 0$

$\therefore \min (R(x, y), R(y, z)) = 0 \;\; \forall\, y$

Thus from case (a), (b) and (c), equation in (1) holds true.

Thus R is fuzzy max – min transitive relation.

Hence (X, R) is fuzzy partial order set.

Now we will prove that any two-elements set in X has fuzzy least upper bound and fuzzy greatest lower bound.

So let $A = \{x, y\} \subseteq X$ where $x \neq y$

Now $x, y \in X \subseteq IR$

$\therefore$ Either $x < y$ or $y < x$

Let without loss of generality, $x < y$

$\therefore R(x, y) = (y - x) / n$ and

$\therefore R(y, x) = 0$ (By definition of R)

Now by Definition of $R_{\geq}$, We have

$$\therefore R_{\geq}[x] = \left\{ \frac{0}{u_1} + \frac{1}{u_2} + \frac{(u_3 - x)/n}{u_3} \middle/ \begin{matrix} u_1 < x,\ u_2 = x,\ u_3 > x \\ u_1, u_2, u_3 \in X \end{matrix} \right\}$$

$$\therefore R_{\geq}[y] = \left\{ \frac{0}{v_1} + \frac{1}{v_2} + \frac{(v_3 - y)/n}{v_3} \middle/ \begin{matrix} v_1 < y,\ v_2 = y,\ v_3 > y \\ v_1, v_2, v_3 \in X \end{matrix} \right\}$$

Now, $U(R, A) = R_{\geq}[x] \cap R_{\geq}[y]$

$$\therefore U(R, A) = \left\{ \frac{(y - x)/n}{y} + \frac{(m - y)/n}{m} \middle/ m > y \right\}$$

Now, Clearly ' y ' is a fuzzy least upper bound of a set A. Similarly, ' x ' is a fuzzy greatest lower bound of a set A.

Thus the set A has fuzzy least upper bound and fuzzy greatest lower bound.

Hence every two element non-fuzzy subset of X has a fuzzy least upper bound and fuzzy greatest lower bound.

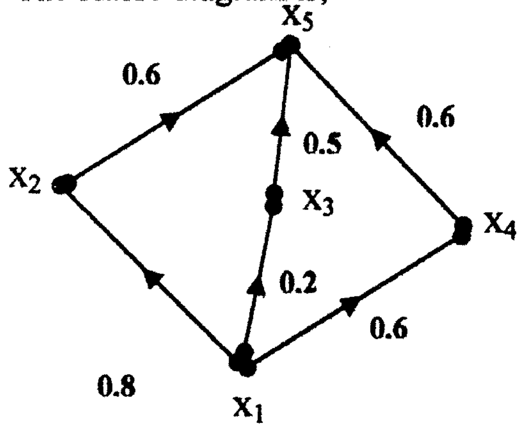
Hence, < X, R > is a fuzzy lattice. □

2) Consider $X=\{x_1, x_2, x_3, x_4, x_5\}$ and

R(X,X) is given by membership matrix as follows:

	x_1	X_2	X_3	x_4	x_5	x_6
x_1	1	0	0	0	0	0
x_2	0.6	1	0	0	0.8	0
x_3	0.6	0.9	1	0.1	0.8	0.4
x_4	0	0	0	1	0.3	0
x_5	0	0	0	0	1	0
x_6	0	0	0	0	1	1

The Hasse diagram is,



Here the ordered pair (X, R) is a fuzzy lattice. □

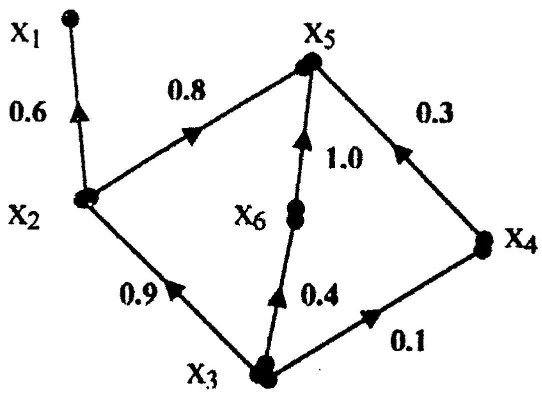
Note: Every fuzzy lattice is a fuzzy partial order set

But **converse need not be true.**

Counter Example: Let $X=\{x_1, x_2, x_3, x_4, x_5, x_6\}$ and $R(X,X)$ is given by membership matrix as follows:

	x_1	X_2	x_3	x_4	x_5
x_1	1	0.8	0.2	0.6	0.6
x_2	0	1	0	0	0.6
x_3	0	0	1	0	0.5
x_4	0	0	0	1	0.6
x_5	0	0	0	0	1

The Hasse diagram is,



Here (X, R) is a partially order set.

Consider $A = \{x_1, x_5\}$

Now, $R_{\geq}[x_1] = \{1/x_1\}$ and $R_{\geq}[x_5] = \{1/x_5\}$

Now, $U(R, A) = R_{\geq}[x_1] \cap R_{\geq}[x_5]$ **(Refer §2.4.1)**

$\therefore U(R, A) = \varnothing$

i.e. fuzzy least upper bound of A does not exists.

$\therefore$ The ordered pair (X, R) is not a fuzzy lattice. □

§3.2 PROPERTIES OF FUZZY LATTICES:

Idempotent Law: $x \vee x = x$ and $x \wedge x = x$

Commutative Law: $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$

Associative Law: $x \vee (y \vee z) = (x \vee y) \vee z$

and $x \wedge (y \wedge z) = (x \wedge y) \wedge z$

Absorption Law: $x \vee (x \wedge y) = x$ and $x \wedge (x \vee y) = x$

➢ **Theorem 3.1:**

If X is a non-empty non-fuzzy set in which two fuzzy binary operations $\wedge$, $\vee$ are defined and satisfy the above stated properties, then a fuzzy partial order can be defined on set X such that X becomes a fuzzy lattice in which $x \vee y$ and $x \wedge y$ are fuzzy least upper bound and fuzzy greatest lower bound of non-fuzzy subset $\{x, y\}$ of X respectively.

Proof:

Let X be a non-empty and non-fuzzy set in which two fuzzy binary operations $\wedge$, $\vee$ are defined and satisfy the above stated properties.

Define a fuzzy relation $R : X \times X \to [0, 1]$ as,

$\forall\, x, y \in X$ and $n \in (0, 1)$,

$$
\begin{aligned}
R(x, y) &= 1 && \text{if } x = y \\
&= n && \text{if } x \vee y = y \text{ and } x \wedge y = x \\
&= 0 && \text{else}
\end{aligned}
$$

To prove: - (X, R) is a fuzzy partial order set.

i) **Fuzzy Reflexivity**:

Now, by definition of R,

$R(x, x) = 1 \qquad \forall\, x \in X.$

$\therefore$ R is a fuzzy reflexive relation.

ii) **Fuzzy Perfectly Antisymmetrivity**:

Let $x \neq y$ and $R(x, y) > 0$

$\therefore R(x, y) = n$ (by definition of R)

$\therefore x \vee y = y$ and $x \wedge y = x$ (by definition of R)

By commutative property,

$y \vee x = x \vee y = y \neq x$ and $y \wedge x = x \wedge y = x \neq y$

Thus $y \vee x \neq x$ and $y \wedge x \neq y$

$\therefore R(y, x) = 0$ (by definition of R)

Thus $x \neq y$ and $R(x, y) > 0 \Rightarrow R(y, x) = 0$

$\therefore$ R is a fuzzy perfectly antisymmetric relation.

iii) **Fuzzy Max-Min Transitivity**:

Let $(x, z) \in X^2$

To prove: $R(x, z) \geq \max_{y \in X} \min (R(x, y), R(y, z))$

If $x = z$ then $R(x, z) = 1$ (by definition of R)

$$\therefore R(x, z) \geq \max_{y \in X} \min (R(x, y), R(y, z))$$

If $x \neq z$ then either $R(x, z) = n$ or $R(x, z) = 0$

case a) Let $R(x, z) = n$

$\therefore$ $x \vee z = z$ and $x \wedge z = x$ (by definition of R)

Now, for all $y \in X$ such that $y = x$ and $y \neq z$

$R(x, y) = 1$ and $R(y, z) = n$ or 0

$\therefore$ $\min(R(x, y), R(y, z)) = R(y, z)$

$= R(x, z)$ (As, $y = x$)

Now, for all $y \in X$ such that $y = z$ and $y \neq x$

$R(x, y) = n$ or 0 and $R(y, z) = 1$

$\therefore$ $\min(R(x, y), R(y, z)) = R(x, y) = R(x, z)$

Now, for all $y \in X$ such that $y \neq x$ and $y \neq z$

$R(x, y) = n$ or 0 and $R(y, z) = n$ or 0

$\therefore$ $\min(R(x, y), R(y, z)) = n$ or 0

$\leq R(x, z)$

Thus, $R(x, z) \geq \max_{y \in X} \min(R(x, y), R(y, z))$

Case b) Let $R(x, z) = 0$

$\therefore$ $x \vee z \neq z$ or $x \wedge z \neq x$ (by definition of R)

To prove: $R(x, z) \geq \max_{y \in X} \min(R(x, y), R(y, z))$

OR $0 \geq \max_{y \in X} \min(R(x, y), R(y, z))$ (since $R(x, z) = 0$)

OR $\max_{y \in X} \min(R(x, y), R(y, z)) = 0$

OR $\min(R(x, y), R(y, z)) = 0 \quad \forall y \in X$

OR $R(x, y) = 0$ or $R(y, z) = 0 \quad \forall y \in X$

OR equivalently for $y \in X$,

Prove: $R(x, y) = 0$ or $R(y, z) = 0$

Let, if possible, $R(x, y) > 0$, $R(y, z) > 0$

i.e., $R(x, y) \geq n$ and $R(y, z) \geq n$

$\therefore$ $x \vee y = y$, $x \wedge y = x$, $y \vee z = z$, $y \wedge z = y$

Now, $(x \vee y) \vee z = y \vee z = z$ and

$(x \vee y) \vee z = x \vee (y \vee z) = x \vee z$

$\therefore$ $x \vee z = z$

Also, $(x \wedge y) \wedge z = x \wedge z$ and

$(x \wedge y) \wedge z = x \wedge (y \wedge z) = x \wedge y = x$

$\therefore$ $x \wedge z = x$

Thus we get $x \vee z = z$ and $x \wedge z = x$

Which is a contradiction to given data.

$\therefore$ $R(x, y) = 0$ or $R(y, z) = 0$

Thus R is fuzzy max-min transitive relation.

Thus R is fuzzy partial order.

Thus (X, R) is a fuzzy partial order set.

Now, To prove : (X, R) is a fuzzy lattice.

Let, $A = \{x, y\} \subseteq X$

Thus, To prove: Fuzzy least upper bound and fuzzy greatest lower bound of a non-empty non-fuzzy subset A of X exists and $x \vee y$ and $x \wedge z$ are the fuzzy least upper bound and fuzzy greatest lower bound of A respectively.

Now, $\wedge$ and $\vee$ are binary operations defined on X.

Hence $x \vee y$ and $x \wedge y$ exists in X.

Let $x \vee y = z_1$ and $x \wedge y = z_2$

Thus, to prove: z_1 and z_2 are the fuzzy least upper bound and fuzzy greatest lower bound of A respectively.

Consider, $x \wedge z_1 = x \wedge (x \vee y) = x$ and

$x \vee z_1 = x \vee (x \vee y) = x \vee y = z_1$

Thus, $x \wedge z_1 = x$ and $x \vee z_1 = z_1$

Thus, $R(x, z_1) > 0$

Similarly, $R(y, z_1) > 0$

$\therefore$ $\min(R(x, z_1), R(y, z_1)) > 0$

$\therefore$ $\min(R_{\geq}[x](z_1), R_{\geq}[y](z_1)) > 0$ **(By definition of R≥ class)**

$\therefore$ $(R_{\geq}[x] \cap R_{\geq}[y])(z_1) > 0$ **(By definition of Fuzzy intersection.)**

$\therefore$ $U(R, A)(z_1) > 0$ ____(1)**(By definition of fuzzy upper bound)**

Let m supports $U(R, A)$. i.e., $U(R, A)(m) > 0$

$\therefore$ $(R_{\geq}[x] \cap R_{\geq}[y])(m) > 0$ **(By definition of fuzzy upper bound)**

$\therefore$ $\min(R_{\geq}[x](m), R_{\geq}[y](m)) > 0$ **(Refer §1.2)**

$\therefore$ $\min(R(x, m), R(y, m)) > 0$ **(By definition of R≥ class)**

$\therefore$ $R(x, m) > 0$ and $R(y, m) > 0$

$\therefore$ $R(x \vee y, m) > 0$ **(By theorem 2.2)**

$\therefore$ $R(z_1, m) > 0$

Thus, $R(z_1, m) > 0$ for all m that supports $U(R, A)$ _____ (2)

Thus, from (1) & (2), and by definition of fuzzy least upper bound z_1 is a fuzzy least upper bound of A **(Refer §2.5.1)**

Similarly we can show that z_2 is a fuzzy greatest lower bound of A

Thus, any two elements non-fuzzy subset A of X has fuzzy least upper bound and fuzzy greatest lower bound.

i.e., $x \vee y$ and $x \wedge y$ is a fuzzy least upper bound and fuzzy greatest lower bound for any $x, y \in X$.

Hence (X, R) is a fuzzy lattice. □

§3.3 Fuzzy Lattice as Algebra:

"If X is a non-empty, non-fuzzy set in which two fuzzy binary operations Λ, V are defined and satisfy the four properties given below: Let x, y, z $\in$X

i) Idempotent Law: $x \vee x = x$ and $x \wedge x = x$

ii) Commutative Law: $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$

iii) Associative Law: $x \vee (y \vee z) = (x \vee y) \vee z$

and $x \wedge (y \wedge z) = (x \wedge y) \wedge z$

iv) Absorption Law: $x \vee (x \wedge y) = x$

and $x \wedge (x \vee y) = x$

Then < X, Λ, V > is called as Fuzzy lattice as Algebra."

§3.4 Fuzzy Complement of an element:

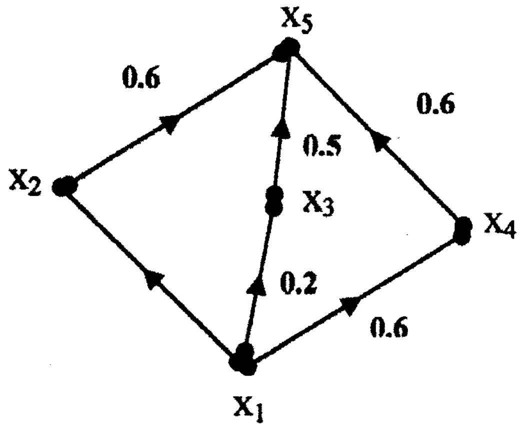
Definition: Let (X, R) be a fuzzy lattice with 0 and 1. An element a`$\in$X, if exists, is a fuzzy complement of an element a $\in$ X if a $\Lambda$ a` = **0** and a V a` = **1** (**0** is fuzzy zero and **1** is fuzzy unit element of X)

e.g. X={x_1, x_2, x_3, x_4, x_5} and

R(X,X) is given by membership matrix as follows:

	x_1	x_2	x_3	x_4	x_5
x_1	1	0.8	0.2	0.6	0.6
x_2	0	1	0	0	0.6
x_3	0	0	1	0	0.5
x_4	0	0	0	1	0.6
x_5	0	0	0	0	1

The Hasse diagram is,



Here the ordered pair (X, R) is a fuzzy lattice.

Also, x_3 is a fuzzy complement of x_2 and vice versa.

x_4 is a fuzzy complement of x_2 and vice versa.

Note: Fuzzy complement of an element need not be unique.

Properties :

1) Fuzzy complement of fuzzy zero is fuzzy unit i.e., **0`** = **1**
2) Fuzzy complement of fuzzy unit is fuzzy zero i.e., **1`** = **0**

§3.5 Fuzzy Complemented Lattice: -

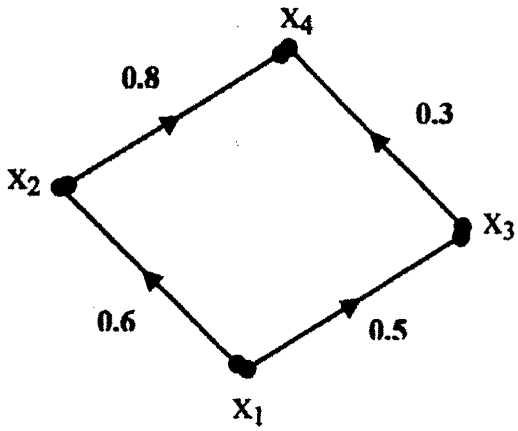
Definition: Let (X, R) be a fuzzy lattice with **0** and **1**. If a\` i.e. fuzzy complement exists for all element $a \in X$ then X is said to be Fuzzy Complemented Lattice.

e.g.: $X = \{x_1, x_2, x_3, x_4\}$

R(X,X) is given by membership matrix as follows:

	x_1	x_2	x_3	x_4
x_1	1	0.6	0.5	0.6
x_2	0	1	0	0.8
x_3	0	0	1	0.3
x_4	0	0	0	1

The Hasse diagram is,



Here the ordered pair (X, R) is a fuzzy lattice.

Now, $x_1' = x_4$, $x_2' = x_3$, $x_3' = x_2$, $x_4' = x_1$

Thus fuzzy complement exists for all elements in X.

Hence X is said to be Fuzzy Complemented Lattice.