CHAPTER - FIVE

FUZZY IDEALS AND FUZZY DUAL IDEALS

Chapter 5

FUZZY IDEALS AND FUZZY DUAL IDEALS

§5.1 Fuzzy Ideals: -

Let $(X, \wedge, \vee)$ be a Fuzzy lattice. A nonempty nonfuzzy subset S of X is a fuzzy ideal of X if,

i) $(S, \wedge, \vee)$ is a fuzzy sub lattice.

ii) $x \in X, a \in S \Rightarrow x \wedge a \in S$.

➢ **Theorem 5.1: -**

Let (X, R) be a fuzzy lattice. Let S be nonempty crisp subset of X. Now S is a fuzzy ideal of X iff,

i) $x, y \in S \Rightarrow x \vee y \in S$

ii) $x \wedge a = x$ and $x \vee a = a, x \in X, a \in S \Rightarrow x \in S$

OR

$R(x, a) > 0, x \in X, a \in S \Rightarrow x \in S$

Proof: - **Necessary Part:**

Let S be a fuzzy ideal of X.

To Prove: i) $x, y \in S \Rightarrow x \vee y \in S$

ii) $x \wedge a = x$ and $x \vee a = a, x \in X, a \in S \Rightarrow x \in S$

OR

$R(x, a) > 0, x \in X, a \in S \Rightarrow x \in S$

Now, by definition of fuzzy ideal, S is a fuzzy sub lattice.

$\therefore x, y \in S \Rightarrow x \vee y \in S$ (i condition proved)

Let $x \wedge a = x$ and $x \vee a = a, x \in X, a \in S$.

To Prove: $x \in S$

Now, $x \in X, a \in S \Rightarrow x \wedge a \in S$. (As S is a fuzzy ideal)

$\Rightarrow x \in S$ (As $x \wedge a = x$)

OR equivalently, Let $R(x, a) > 0$, $x \in X$, $a \in S$.

To Prove: $x \in S$

Now, $x \in X, a \in S \Rightarrow x \wedge a \in S$. (As S is a fuzzy ideal)

$\therefore$ Fuzzy greatest lower bound of a subset $A = \{x, a\}$ of X is in S.

Let $x \wedge a = m \in S$.

Now by definition of Fuzzy Lower Bound, ___(*)

$L(R, A)(m) > 0$ and $R(z, m) > 0$ for all z that supports $L(R,A)$.

$\therefore (R \le [x] \cap R \le [a])(m) > 0$

$\therefore \min\{R \le [x](m), R \le [a](m)\} > 0$ (By fuzzy intersection)

$\therefore R \le [x](m) > 0$ and $R \le [a](m) > 0$

$\therefore R(m, x) > 0$ and $R(m, a) > 0$ ___(1) (By definition of $R \le$)

Now, $R(x, a) > 0$ (given) and $R(x, x) > 0$ (By fuzzy reflexivity)

$\therefore R \le [a](x) > 0$ and $R \le [x](x) > 0$ (By definition of $R \le$)

$\therefore \min\{R \le [a](x), R \le [x](x)\} > 0$

$\therefore (R \le [a] \cap R \le [x])(x) > 0$

$\therefore L(R, A)(x) > 0$

$\therefore$ x supports $L(R, A)$.

$\therefore R(x, m) > 0$. ___(2) (By *)

Let, if possible, $x \neq m$.

From (1) and (2) we get,

$R(m, x) > 0$ and $R(x, m) > 0$ which will give a contradiction to fuzzy perfectly antisymmetric property.

$\therefore x = m$.

$\therefore$ x is a Fuzzy greatest lower bound of a subset A that is in S.

$\therefore x \in S$.

Thus, $R(x, a) > 0$, $x \in X$, $a \in S \Rightarrow x \in S$

Sufficient Part:

Let, i) $x, y \in S \Rightarrow x \vee y \in S$

ii) $x \wedge a = x$ and $x \vee a = a$, $x \in X$, $a \in S \Rightarrow x \in S$

OR $R(x, a) > 0$, $x \in X$, $a \in S \Rightarrow x \in S$

To Prove: S is a fuzzy ideal of X.

First we will prove that S is a fuzzy sub lattice of X.

i.e., To Prove: 1) $x, y \in S \Rightarrow x \vee y \in S$

2) $x, y \in S \Rightarrow x \wedge y \in S$

Now, $x, y \in S \Rightarrow x \vee y \in S$ (by i)

Let $x, y \in S$. To prove: $x \wedge y \in S$

Now, $(x \wedge y) \wedge x = x \wedge (y \wedge x)$ (by Associativity)

$= x \wedge (x \wedge y)$ (by Commutativity)

$= (x \wedge x) \wedge y$ (by Associativity)

$= x \wedge y$ (by Idempotent Law)

Also, $(x \wedge y) \vee x = x$ (by Absorption Law)

Thus, $(x \wedge y) \wedge x = x \wedge y$, $(x \wedge y) \vee x = x$, $x \wedge y \in X$ and $x \in S$

$\Rightarrow x \wedge y \in S$ (by ii)

Thus $x, y \in S \Rightarrow x \wedge y \in S$

Thus S is a fuzzy sub lattice of X.

Now, To Prove: $x \in X$, $a \in S \Rightarrow x \wedge a \in S$.

Let $x \in X$ and $a \in S$

Now, $(x \wedge a) \wedge a = x \wedge (a \wedge a)$ (by Associativity)

$= x \wedge a$ (by Idempotent Law)

And $(x \wedge a) \vee a = a$ (by Absorption Law)

Thus, $(x \wedge a) \wedge a = x \wedge a$, $(x \wedge a) \vee a = a$,

$x \wedge a \in X$ and $a \in S \Rightarrow x \wedge a \in S$ (by ii)

Thus, $x \in X$, $a \in S \Rightarrow x \wedge a \in S$.

Thus, S is a fuzzy ideal of X. □

❖ **Examples of Fuzzy Ideals:**

1) Let $X = [a, a + n]$ where $a, n \in IR$ and $n > 0$. Let R = "**almost** less than or equal to" be a fuzzy relation defined on X as a function $R: X \times X \rightarrow [0,1]$ defined by,

$$R(x, y) = \begin{cases} 1 & \text{if } x = y \\ (y - x) / n & \text{if } x < y \\ 0 & \text{else.} \end{cases}$$

Here (X, R) is a fuzzy lattice.

Let $S = [a, p]$ where $p \in IR$ and $a < p \leq a + n$.

Then S is a fuzzy ideal of X.

Proof: - Let $S = [a, p]$ where $p \in IR$ and $a < p \leq a + n$.

Thus, S is a non-empty non-fuzzy subset of X.

Now, to prove:

i) $x, y \in S \Rightarrow x \vee y \in S$

ii) $R(x, y) > 0, x \in X, y \in S \Rightarrow x \in S$

So, Consider $x, y \in S$. To prove: $x \vee y \in S$

Now, $x \vee y$ exists and is unique (Since X is fuzzy lattice)

Let $x \vee y = m$ and $A = \{x, y\}$

i.e., To prove: $m \in S$

Now, $x \in S \Rightarrow x \leq p \Rightarrow R(x, p) > 0$ (By definition of R)

$y \in S \Rightarrow y \leq p \Rightarrow R(y, p) > 0$ (By definition of R)

$\therefore R(x \vee y, p) > 0$ (by theorem 2.2)

$\therefore R(m, p) > 0$ (Since $x \vee y = m$)

$\therefore m \leq p$ (By definition of R)

$\therefore m \in S$ (By definition of S)

$\therefore x \vee y \in S$ (Since $x \vee y = m$)

Thus $x, y \in S \Rightarrow x \vee y \in S$.

Let $R(x, y) > 0, x \in X, y \in S$. To Prove: $x \in S$

Now, $R(x, y) > 0 \Rightarrow x \le y$ and also $y \in S \Rightarrow y \le p$

$\therefore x \le p$ (Since $x \le y$ and $y \le p$)

$\therefore x \in S$

Thus, $R(x, y) > 0, x \in X, y \in S \Rightarrow x \in S$.

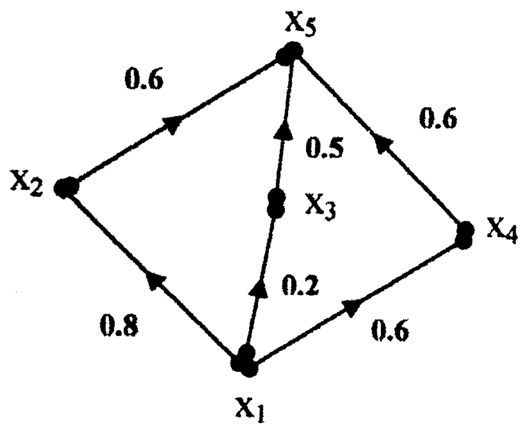
Thus, S is a fuzzy ideal.

2) $X=\{x_1, x_2, x_3, x_4, x_5\}$ and

R(X,X) is given by membership matrix as follows:

	x_1	x_2	x_3	x_4	x_5
x_1	1	0.8	0.2	0.6	0.6
x_2	0	1	0	0	0.6
x_3	0	0	1	0	0.5
x_4	0	0	0	1	0.6
x_5	0	0	0	0	1

The Hasse diagram is,



Here the ordered pair (X, R) is a fuzzy lattice.

Here $S = \{x_1, x_2\} \subseteq X$ is a fuzzy ideal by theorem 5.1.

Note: Every fuzzy ideal is a fuzzy sub lattices.

But converse need not be true.

Counter Example:

Consider above example

(i.e. example 2 of Fuzzy ideal)

In that consider $S = \{x_3, x_5\} \subseteq X$.

Clearly, S is a fuzzy sub lattice.

(Since $x_3 \vee x_5 = x_5 \in S$ and $x_3 \wedge x_5 = x_3 \in S$)

But S is not a fuzzy ideal as

$R(x_1, x_3) > 0, x_1 \in X, x_3 \in S \not\Rightarrow x_1 \in S$.

- **Theorem 5.2:**

Let S be any non-empty non fuzzy subset of fuzzy lattice X.

S be a fuzzy ideal of X iff $a \vee b \in S \Leftrightarrow a, b \in S$

Proof: **Necessary Part:**

Let S be a fuzzy ideal of X.

To prove: $a \vee b \in S \Leftrightarrow a, b \in S$

Let $a \vee b \in S$. To prove: $a, b \in S$

By definition of Fuzzy ideal,

$a \in X, a \vee b \in S \Rightarrow a \wedge (a \vee b) \in S$

$\Rightarrow a \in S$ (by Absorption Law)

Similarly,

$b \in X, a \vee b \in S \Rightarrow b \wedge (a \vee b) \in S$

$\Rightarrow b \wedge (b \vee a) \in S$ (Commutative Law)

$\Rightarrow b \in S$ (by Absorption Law)

Thus, $a \vee b \in S \Rightarrow a, b \in S$

Conversely, Let $a, b \in S$. To prove: $a \vee b \in S$

Now, S is a fuzzy ideal of X. Hence $a \vee b \in S$.

Thus, a, b $\in$ S $\Rightarrow$ a V b $\in$ S

Thus, a V b $\in$ S $\Leftrightarrow$ a, b $\in$ S

Sufficient Part:

Let a V b $\in$ S $\Leftrightarrow$ a, b $\in$ S

To Prove: S is a fuzzy ideal of X.

Now, a, b $\in$ S $\Rightarrow$ a V b $\in$ S (given data)

Let x Λ a = x and x V a = a, x $\in$ X, a $\in$ S

To prove: x $\in$ S

Now, x V a = a and a $\in$ S $\Rightarrow$ x V a $\in$ S

$\Rightarrow$ x, a $\in$ S (given data)

Thus, x $\in$ S:

Thus S is a fuzzy ideal of X. □

§5.2 Fuzzy Dual Ideal:

"Let (X, R) be a fuzzy lattice. Let D be any non-empty non-fuzzy subset of X. D is a fuzzy dual ideal of X if,

i) D is a fuzzy sub lattice of X.

ii) For x $\in$ X, a $\in$ D $\Rightarrow$ x V a $\in$ D.

➢ Theorem 5.3:

Let (X, R) be a fuzzy lattice. Let D be any non-empty non-fuzzy subset of X. D is a fuzzy dual ideal of X iff

i) a, b $\in$ D $\Rightarrow$ a Λ b $\in$ D.

ii) a Λ x = a and a V x = x, x $\in$ X, a $\in$ D $\Rightarrow$ x $\in$ D

OR

R(a, x) > 0, x $\in$ X, a $\in$ D $\Rightarrow$ x $\in$ D

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➢ **Theorem 5.4:**

Let D be any non-empty non-fuzzy subset of a fuzzy lattice (X, R). D is a fuzzy dual ideal of X iff $a \wedge b \in D \Leftrightarrow a, b \in D$.

❖ **Examples of Fuzzy Dual Ideal:**

1) Let $X = [a, a + n]$ where $a, n \in IR$ and $n > 0$.

Let R = "**almost** less than or equal to" be a fuzzy relation defined on X as a function $R: X \times X \rightarrow [0,1]$ defined by,

$$
\begin{aligned}
R(x, y) &= 1 && \text{if } x = y \\
&= (y - x) / n && \text{if } x < y \\
&= 0 && \text{else.}
\end{aligned}
$$

Here (X, R) is a fuzzy lattice.

Let $D = [m, a + n]$ where $m \in IR$ and $a \leq m < a + n$.

Then D is a fuzzy dual ideal of X.

2) $X=\{x_1, x_2, x_3, x_4, x_5\}$ and

R(X,X) is given by grade membership matrix as follows:

	x_1	x_2	x_3	x_4	x_5
x_1	1	0.8	0.2	0.6	0.6
x_2	0	1	0	0	0.6
x_3	0	0	1	0	0.5
x_4	0	0	0	1	0.6
x_5	0	0	0	0	1

Here the ordered pair (X, R) is a fuzzy lattice.

Here $D = \{x_2, x_5\} \subseteq X$ is a fuzzy dual ideal of X.

Note: Every fuzzy dual ideal is a fuzzy sub lattice.

But converse need not be true.

Counter Example:

Consider above example (i.e. example 2 of fuzzy dual ideal)

In that consider $D = \{x_1, x_3\} \subseteq X$

Clearly, S is a fuzzy sub lattice.

But S is not a fuzzy dual ideal as

$R(x_1, x_5) > 0, x_1 \in D, x_5 \in X \not\Rightarrow x_5 \in D.$

§5.3 Definition: Fuzzy ideal generated by crisp subset

"Let (X, R) be a fuzzy lattice. Let H be any non-empty non-fuzzy subset of X. Then the smallest fuzzy ideal of X containing H is called fuzzy ideal generated by H and it is denoted by (H]."

i.e., $(H] = \cap \{I \subseteq X / I \text{ is a fuzzy ideal and } H \subseteq I\}$

§5.4 Definition: Fuzzy dual ideal generated by crisp subset

"Let (X, R) be a fuzzy lattice. Let H be any non-empty non-fuzzy subset of X. Then the smallest fuzzy dual ideal of X containing H is called fuzzy dual ideal generated by H and it is denoted by [H)."

i.e., $[H) = \cap \{D \subseteq X / D \text{ is a fuzzy dual ideal and } H \subseteq D\}$

➢ **Theorem 5.5:**

Let (X, R) be a fuzzy lattice. Let H be any non-empty non-fuzzy subset of X. Then,

$(H] = \{ x \in X / R(x, h_1 \vee h_2 \vee \ldots \vee h_n) > 0, h_i \in H, n \text{ is finite}\}$

Proof: Let $P = \{x \in X / R(x, h_1 \vee h_2 \vee \ldots \vee h_n) > 0, h_i \in H, n \text{ is finite}\}$

To Prove: $(H] = P$

Now, H is non-empty, Let $h \in H$.

Then $R(h, h \vee h_1) > 0, \forall h_1 \in H$ (by theorem 2.1)

$\therefore h \in P$ $\therefore H \subseteq P$ $\therefore P \neq \varnothing$

Thus, P is a nonempty nonfuzzy subset of X containing H.

Let $x, y \in P$.

Hence, $R(x, h_1 \vee h_2 \vee \ldots \vee h_n) > 0$, $h_i \in H$, n is finite and similarly, $R(y, k_1 \vee k_2 \vee \ldots \vee k_m) > 0$, $k_j \in H$, m is finite.

$\therefore R(x \vee y, h_1 \vee h_2 \vee \ldots \vee h_n \vee k_1 \vee k_2 \vee \ldots \vee k_m) > 0$,

where $h_i, k_j \in H$, $n + m$ is finite (by theorem 2.2)

$\therefore x \vee y \in P$. (By Definition of P)

Thus, $x, y \in P \Rightarrow x \vee y \in P$.

Let $R(x, a) > 0$, $x \in X$, $a \in P$ To Prove: $x \in P$

Now, $a \in P \Rightarrow R(a, h_1 \vee h_2 \vee \ldots \vee h_n) > 0$, $h_i \in H$, n is finite

$\therefore R(x, h_1 \vee h_2 \vee \ldots \vee h_n) > 0$, $h_i \in H$, n is finite

(by theorem 2.4)

$\therefore x \in P$ (by definition of P)

Thus, $R(x, a) > 0$, $x \in X$, $a \in P \Rightarrow x \in P$

Thus, P is a fuzzy ideal of X containing H.

Thus only to prove:

P is the smallest fuzzy ideal of X containing H.

Let, J be a fuzzy ideal of X containing H.

To Prove: $P \subseteq J$.

Let $x \in P$

$\therefore R(x, h_1 \vee h_2 \vee \ldots \vee h_n) > 0$, $h_i \in H$, n is finite

Now, $H \subseteq J$. Thus $h_i \in J$ and n is finite.

$\therefore h_1 \vee h_2 \vee \ldots \vee h_n \in J$ (Since J is a fuzzy ideal of X)

Now, J is a fuzzy ideal of X, $h_1 V h_2 V \dots V h_n \in J$ and

$R(x, h_1 V h_2 V \dots V h_n) > 0$,n is finite

$\therefore x \in J$ (by definition of fuzzy ideal)

Thus, $x \in P \Rightarrow x \in J$

Thus, $P \subseteq J$.

Thus, P is the smallest fuzzy ideal of X containing H.

Thus, P = (H]. □

➢ **Theorem 5.6:**

Let (X, R) be a fuzzy lattice. Let H be any non-empty non-fuzzy subset of X. Then,

$[H) = \{x \in X / R(h_1 V h_2 V \dots V h_n, x) > 0, h_i \in H, n \text{ is finite}\}$

§5.5 Definition: Let (X, R) be a fuzzy lattice. Let $a \in X$.

Define $(a] = \{x \in X / R(x, a) > 0\}$.

Then (a] is said to be **fuzzy principal ideal** generated by 'a'.

➢ **Theorem 5.7:**

If I is a fuzzy ideal of a fuzzy lattice (X, R). Let $a \in X$.

Then $I V (a] = \{x \in X / R(x, m V a) > 0, \text{ for some } m \in I\}$

is a fuzzy ideal of X.

Proof: Now, $I V (a] = \{x \in X / R(x, m V a) > 0, \text{ for some } m \in I\}$

As, $I \neq \varnothing$. Let $z \in I$.

Now, $R(z, z V a) > 0$. (by theorem 2.1)

Thus $z \in I V (a]$. (by definition of I V (a])

Hence $I V (a] \neq \varnothing$.

Thus, I V (a] is a non-empty non-fuzzy subset of X.

Let $x, y \in I \vee (a]$.

To prove: $x \vee y \in I \vee (a]$

Now, By definition of $I \vee (a]$

$x \in I \vee (a] \Rightarrow R(x, m_1 \vee a) > 0$, for some $m_1 \in I$.

$y \in I \vee (a] \Rightarrow R(y, m_2 \vee a) > 0$, for some $m_2 \in I$.

Now, $m_1, m_2 \in I \Rightarrow (m_1 \vee m_2) \in I$ (Since I is a fuzzy ideal)

$\therefore R(x \vee y, (m_1 \vee m_2) \vee a) > 0$ (by theorem 2.2)

$\therefore x \vee y \in I \vee (a]$. (by definition of $I \vee (a]$)

Thus, $x, y \in I \vee (a] \Rightarrow x \vee y \in I \vee (a]$

Let $R(x, p) > 0$, $x \in X$, $p \in I \vee (a]$ To prove: $x \in I \vee (a]$.

$p \in I \vee (a] \Rightarrow R(p, m \vee a) > 0$, for some $m \in I$.

$\therefore R(x, m \vee a) > 0$, where $m \in I$ (by theorem 2.4)

$\therefore x \in I \vee (a]$. (by definition of $I \vee (a]$)

Thus, $R(x, p) > 0$, $x \in X$, $p \in I \vee (a] \Rightarrow x \in I \vee (a]$.

Thus, $I \vee (a]$ is a fuzzy ideal of X. □

Note: Crisp union of two fuzzy ideals of a fuzzy lattice (X, R) need not be fuzzy ideal.

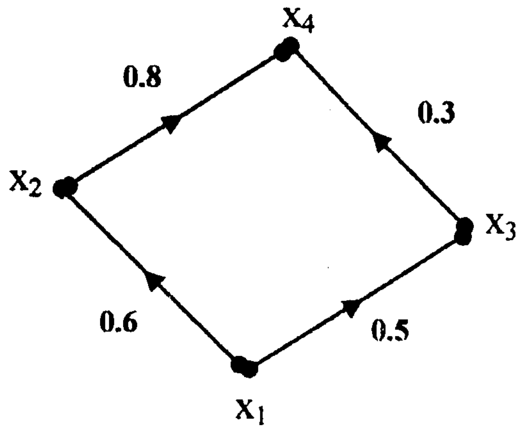
Counter Example:-

$X = \{x_1, x_2, x_3, x_4\}$

R(X,X) is given by grade membership matrix as follows:

	x_1	x_2	x_3	x_4
x_1	1	0.6	0.5	0.6
x_2	0	1	0	0.8
x_3	0	0	1	0.3
x_4	0	0	0	1

The Hasse diagram is,



Here the ordered pair (X, R) is a fuzzy lattice.

$\{x_1, x_3\}$ and $\{x_1, x_2\}$ are fuzzy ideals of X.

But $\{x_1, x_3\} \cup \{x_1, x_2\} = \{x_1, x_2, x_3\}$ is not a fuzzy ideal of X

(Since $x_2 \vee x_3 = x_4 \notin \{x_1, x_2, x_3\}$).

➢ **Theorem 5.8:**

Let (X, R) be a fuzzy lattice. Let $\mathcal{C}$ be a chain of fuzzy ideals of X. Then $P = \bigcup_{c \in \mathcal{C}} c$ is a fuzzy ideal of X.

Proof:

Now, $P \neq \varnothing$ (Since $c \in \mathcal{C} \Rightarrow c \neq \varnothing$)

Let $x, y \in P$. To Prove: $x \vee y \in P$.

$x \in P \Rightarrow x \in c_1$, for some $c_1 \in \mathcal{C}$.

$y \in P \Rightarrow x \in c_2$, for some $c_2 \in \mathcal{C}$.

Now, either $c_1 \subseteq c_2$ or $c_2 \subseteq c_1$ (Since $\mathcal{C}$ is a chain)

Without loss of generality, let $c_1 \subseteq c_2$

$\therefore x, y \in c_2$.

$\therefore x \vee y \in c_2$. (Since c_2 is a fuzzy ideal)

$\therefore x \vee y \in P$ (Since $c_2 \subseteq P$)

Let $R(x, y) > 0$, $x \in X$, $y \in P$ To Prove: $x \in P$.

$y \in P \Rightarrow y \in c$, for some $c \in \mathcal{C}$.

Now $R(x, y) > 0$, $y \in c$ and $x \in X$

$\Rightarrow x \in c$ (Since c is a fuzzy ideal)

$\therefore x \in P$

Thus, $R(x, y) > 0$, $x \in X$, $y \in P \Rightarrow x \in P$.

Thus, P is a fuzzy ideal of X. □

Remark: Let (X, R) be a fuzzy lattice.

Let S be a fuzzy ideal.

Then $a \in S$ or $b \in S \Rightarrow a \wedge b \in S$

(by definition of fuzzy ideal)

But converse need not be true.

i.e., $a \wedge b \in S \nRightarrow a \in S$ or $b \in S$.

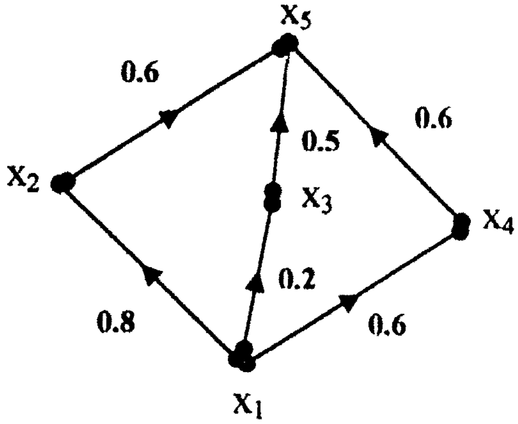
Consider the following example:

$X = \{x_1, x_2, x_3, x_4, x_5\}$ and

R(X,X) is given by grade membership matrix as follows:

	x_1	x_2	x_3	x_4	x_5
x_1	1	0.8	0.2	0.6	0.6
x_2	0	1	0	0	0.6
x_3	0	0	1	0	0.5
x_4	0	0	0	1	0.6
x_5	0	0	0	0	1

The Hasse diagram is,



Here the ordered pair (X, R) is a fuzzy lattice.

Consider, $S = \{x_1, x_2\}$.

Clearly S is a fuzzy ideal.

Consider $x_3 \wedge x_4 = x_1 \in S$. whereas, $x_3 \notin S$ and $x_4 \notin S$.

§5.6 Fuzzy Prime Ideals:

Definition "A proper fuzzy ideal S of a fuzzy lattice X is said to be fuzzy prime ideal of X, if $a \wedge b \in S \Rightarrow a \in S$ or $b \in S$."

❖ Examples of fuzzy prime ideals:

1) Let $X = [a, a + n]$ where $a, n \in \mathbb{R}$ and $n > 0$.

Let R = "**almost** less than or equal to" be a fuzzy relation defined on X as a function $R: X \times X \rightarrow [0,1]$ defined by,

$$
\begin{aligned}
R(x, y) &= 1 && \text{if } x = y \\
&= (y - x)/n && \text{if } x < y \\
&= 0 && \text{else.}
\end{aligned}
$$

Here (X, R) is a fuzzy lattice.

Let $S = [a, p]$ where $a < p < a + n$

Then S is a fuzzy prime ideal.

Solution: Now, S is a proper fuzzy ideal of X.

(See example 1of fuzzy ideal)

To Prove: S is a fuzzy prime ideal.

Let $x \wedge y \in S$.

To Prove: $x \in S$ or $y \in S$.

Let $m = x \wedge y$.

$\therefore m \in S$

Now $L(R, \{x, y\})(m) > 0$

and $R(q, m) > 0$ for all q that supports $L(R, \{x, y\})$.

Now, $m \in S \Rightarrow m \leq p$ (by definition of S)

$\therefore R(m, p) > 0$ (by definition of R)

By fuzzy perfectly antisymmetric property,

$R(p, m) = 0$

$\therefore$ p do not support $L(R, \{x, y\})$ __________ (1)

Let, if possible, $x \notin S$ and $y \notin S$.

$\therefore x > p$ and $y > p$. (by definition of S)

$\therefore R(p, x) > 0$ and $R(p, y) > 0$ (by definition of R)

$\therefore R_{\leq}[x](p) > 0$ and $R_{\leq}[y](p) > 0$ (by definition of $R_{\leq}$)

$\therefore \min\{R_{\leq}[x](p), R_{\leq}[y](p)\} > 0$

$\therefore (R_{\leq}[x] \cap R_{\leq}[y])(p) > 0$

$\therefore L(R, \{x, y\})(p) > 0$

$\therefore$ p supports $L(R, \{x, y\})$

which is contradiction to statement (1).

Hence our assumption $x \notin S$ and $y \notin S$ is wrong.

$\therefore x \in S$ or $y \in S$.

Thus, $x \wedge y \in S \Rightarrow x \in S$ or $y \in S$

Thus, S is a fuzzy prime ideal.

§5.7 Fuzzy Distributive Lattice:

Definition: "Let X be a fuzzy lattice. If $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$, $\forall\, a, b, c \in X$ then X is a fuzzy distributive lattice."

- **Theorem 5.9: (Stone's Theorem)**

Let X be a fuzzy distributive lattice. I be a fuzzy ideal and D be a fuzzy dual ideal of X such that $I \cap D = \varnothing$ then there exists a fuzzy prime ideal P of X such that $I \subseteq P$ and $P \cap D = \varnothing$.

Proof:

Let $\mathcal{F} = \{J \subseteq X / J$ is a fuzzy ideal of X containing I and $J \cap D = \varnothing\}$

Now, $I \subseteq X$, I is a fuzzy ideal of X, $I \subseteq I$ and $I \cap D = \varnothing$.

$\therefore I \in \mathcal{F} \quad \therefore \mathcal{F} \neq \varnothing$

Let $\mathcal{C}$ be chain in $\mathcal{F}$ and $\mathcal{M} = \bigcup_{c \in \mathcal{C}} c$

To prove: $\mathcal{M} \in \mathcal{F}$

Now, $\mathcal{M}$ is a fuzzy ideal of X (By theorem 5.8)

$c \in \mathcal{C} \Rightarrow c \in \mathcal{F} \Rightarrow I \subseteq c$ (by definition of $\mathcal{F}$)

Thus, $\forall\, c \in \mathcal{C}$, $I \subseteq c$

$\therefore I \subseteq \bigcup_{c \in \mathcal{C}} c \Rightarrow I \subseteq \mathcal{M}$

$c \in \mathcal{C} \Rightarrow c \in \mathcal{F} \Rightarrow c \cap D = \varnothing \qquad \forall\, c \in \mathcal{C}$

$\therefore \mathcal{M} \cap D = \left(\bigcup_{c \in \mathcal{C}} c\right) \cap D = \bigcup_{c \in \mathcal{C}} (c \cap D) = \bigcup_{c \in \mathcal{C}} \varnothing = \varnothing$

Thus, $\mathcal{M}$ is a fuzzy ideal of X containing I such that $\mathcal{M} \cap D = \varnothing$.

$\therefore \mathcal{M} \in \mathcal{F}$

We have, Zorn's Lemma as,

Let $X \neq \varnothing$, $\mathcal{F} \neq \varnothing$, $\mathcal{F} \subseteq \mathcal{P}(X)$, if for any chain C of $\mathcal{F}$, $\bigcup_{c \in C} c \in \mathcal{F}$

then there exists a maximal element in $\mathcal{F}$.

Thus, by Zorn's lemma, there exists a maximal element, say P, in $\mathcal{F}$

To prove: P is a fuzzy prime ideal of X.

Now, $P \in \mathcal{F}$

$\therefore$ $P \subseteq X$, P is a fuzzy ideal in X containing I and $P \cap D = \varnothing$.

As $D \neq \varnothing$, P is a proper subset of X.

Thus P is a proper fuzzy ideal of X.

Let $a \wedge b \in P$. To prove: $a \in P$ or $b \in P$.

Let, if possible, $a \notin P$ and $b \notin P$.

As, P is a maximal element in $\mathcal{F}$ and by theorem 5.7, we get,

$P \vee (a]$, $P \vee (b]$ are fuzzy ideals of X.

Also, $I \subseteq P \subseteq P \vee (a]$ and $I \subseteq P \subseteq P \vee (b]$.

Hence, $(P \vee (a]) \cap D \neq \varnothing$ and $(P \vee (b]) \cap D \neq \varnothing$.

Let, $d_1 \in P \vee (a]$, $d_2 \in P \vee (b]$ and $d_1, d_2 \in D$.

$d_1 \in P \vee (a] \Rightarrow R(d_1, m_1 \vee a) > 0$ for some $m_1 \in P$

$d_2 \in P \vee (b] \Rightarrow R(d_2, m_2 \vee b) > 0$ for some $m_2 \in P$

$R(d_1 \wedge d_2, (m_1 \vee a) \wedge (m_2 \vee b)) > 0$ (by theorem 2.2)

$\therefore R(d_1 \wedge d_2, (m_1 \wedge m_2) \vee (m_1 \wedge b) \vee (a \wedge m_2) \vee (a \wedge b)) > 0$

(Since X is a fuzzy distributive lattice)

Now, $d_1, d_2 \in D \Rightarrow d_1 \wedge d_2 \in D$. (Since D is fuzzy dual ideal)

$\therefore (m_1 \wedge m_2) \vee (m_1 \wedge b) \vee (a \wedge m_2) \vee (a \wedge b) \in D$. ______ (1)

(Since D is fuzzy dual ideal)

Now, $m_1, m_2 \in P$ and P is a fuzzy ideal

$\Rightarrow m_1 \wedge m_2 \in P$

Also, $R(m_1 \wedge b, m_1) > 0$, $m_1 \in P$ and P is a fuzzy ideal

$\Rightarrow m_1 \wedge b \in P$

Also, $R(a \wedge m_2, m_2) > 0$, $m_2 \in P$ and P is a fuzzy ideal

$\Rightarrow a \wedge m_2 \in P$

Also, $a \wedge b \in P$ (Given data)

$\therefore (m_1 \wedge m_2) \vee (m_1 \wedge b) \vee (a \wedge m_2) \vee (a \wedge b) \in P.$ _____ (2)

From (1) and (2), we get $P \cap D \neq \varnothing$ which is a contradiction.

Hence our assumption that $a \notin P$ and $b \notin P$ is wrong.

Thus, $a \in P$ or $b \in P$ whenever $a \wedge b \in P$.

$\therefore$ P is a fuzzy prime ideal.

Thus, if $I \cap D = \varnothing$ for any fuzzy ideal I and fuzzy dual ideal D then there exists a fuzzy prime ideal P of X such that $I \subseteq P$ and $P \cap D = \varnothing$. □

- **Theorem 5.10 :**

Let X be a fuzzy distributive lattice and $a \neq b$ in X. Then there exists a fuzzy prime ideal P in X containing exactly one of them.

Proof: Let $a \neq b$ in X.

case i) $R(a, b) \not> 0$ and $R(b, a) \not> 0$

$\therefore (a] \cap [b) = \varnothing$

By Stone's theorem, there exist a fuzzy prime ideal P such that $(a] \subseteq P$ and $P \cap [b) = \varnothing$.

$\therefore a \in P$ and $b \notin P$.

case ii) $R(a, b) > 0 \Rightarrow R(b, a) = 0$

(by fuzzy perfectly antisymmetric property)

$\therefore (a] \cap [b) = \varnothing$

By Stone's theorem, there exist a fuzzy prime ideal P such that $(a] \subseteq P$ and $P \cap [b) = \emptyset$.

$\therefore$ $a \in P$ and $b \notin P$.

case iii) $R(b, a) > 0 \Rightarrow R(a, b) = 0$

(by fuzzy perfectly antisymmetric property)

$\therefore$ $(b] \cap [a) = \emptyset$

By Stone's theorem, there exist a fuzzy prime ideal P such that $(b] \subseteq P$ and $P \cap [a) = \emptyset$.

$\therefore$ $b \in P$ and $a \notin P$.

From the above three cases, we get, whenever $a \neq b$ there exists a prime ideal P containing exactly one of them. □

- **Theorem 5.11**

Let P be fuzzy prime ideal of fuzzy lattice X then $X - P$ is a fuzzy dual ideal of fuzzy lattice X.

Proof: Let $x, y \in X - P$

$\Rightarrow x, y \notin P$

$\Rightarrow x \wedge y \notin P$ (Since P is fuzzy prime ideal)

$\Rightarrow x \wedge y \in X - P$

Let $x \wedge a = a$, $x \vee a = x$, $x \in X$, $a \in X - P$ To prove: $x \in X - P$

Now, $a \in X - P \Rightarrow a \notin P$

$\Rightarrow x \wedge a \notin P$ (Since $x \wedge a = a$, given)

$\Rightarrow x \notin P$ (Since P is fuzzy ideal)

$\Rightarrow x \in X - P$

Thus, $X - P$ is a fuzzy dual ideal of fuzzy lattice X. □