CHAPTER - TWO

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§2.1 FUZZY PARTIAL ORDERINGS: - [1]

Definition: "A Fuzzy Relation R in X is a Fuzzy Partial Ordering iff it is Reflexive, Perfectly Antisymmetric and (Max-Min) Transitive."

When X is finite, it is possible to represent R as a triangular matrix or a Hasse diagram. A fuzzy Hasse diagram is a valued oriented graph whose nodes are the elements of X. The link $x \to y$ exists iff $R(x, y) > 0$. Each link is valued by $R(x, y)$. Owing to fuzzy perfect antisymmetrivity and fuzzy max-min transitivity the graph has no cycle.

§2.2 FUZZY PARTIAL ORDERED SET: -

Definition: "Let X be non-empty non-fuzzy set. If R is fuzzy partial order defined on X. Then ordered pair (X, R) is called as Fuzzy Partial Ordered Set."

Note: In Short, A Fuzzy Partial ordered set is known as Fuzzy poset.

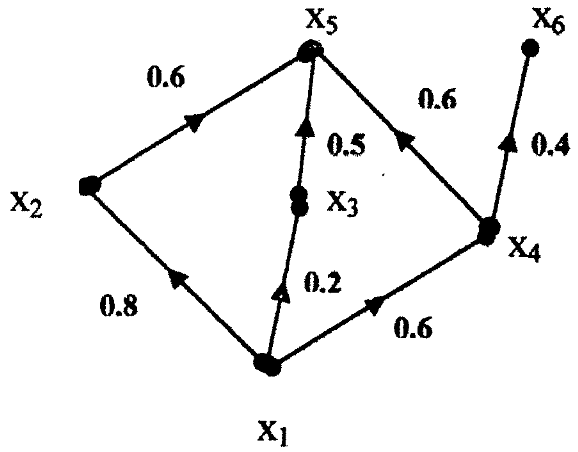
Examples: 1) $X=\{x_1, x_2, x_3, x_4, x_5, x_6\}$ and

$R(X, X)$ is given by grade membership matrix as,

	x_1	x_2	x_3	x_4	x_5	x_6
X_1	1	0.8	0.2	0.6	0.6	0.4
X_2	0	1	0	0	0.6	0
X_3	0	0	1	0	0.5	0
X_4	0	0	0	1	0.6	0.4
X_5	0	0	0	0	1	0
X_6	0	0	0	0	0	1

Here the ordered pair (X, R) is fuzzy partial ordered set.

The Hasse diagram is,



Note: The Hasse diagram is read as $R(x_1, x_2) = 0.8$ but $R(x_2, x_1) = 0$

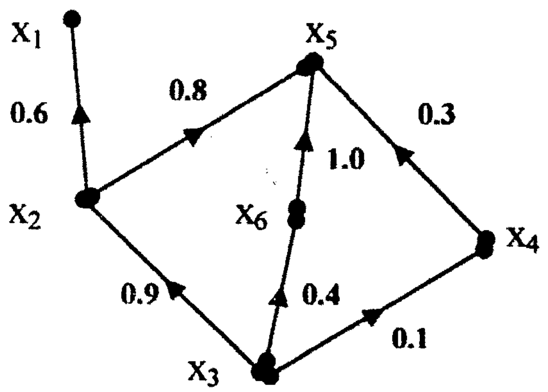
2) $X=\{x_1, x_2, x_3, x_4, x_5, x_6\}$ and

$R(X,X)$ is given by membership matrix as follows:

	x_1	x_2	x_3	x_4	x_5	x_6
x_1	1	0	0	0	0	0
x_2	0.6	1	0	0	0.8	0
x_3	0.6	0.9	1	0.1	0.8	0.4
x_4	0	0	0	1	0.3	0
x_5	0	0	0	0	1	0
x_6	0	0	0	0	1	1

Here the ordered pair (X, R) is fuzzy partial ordered set.

The Hasse diagram is,



3) Let $X = [a, a + n]$ where $a, n \in IR, n > 0$.

R = "**almost** less than or equal to" be fuzzy relation on X defined by the function $R : X \times X \to [0, 1]$ as, $\forall\, x, y \in X$

$$R(x, y) = \begin{cases} 1 & \text{if } x = y \\ (y-x)/n & \text{if } x < y \\ 0 & \text{else.} \end{cases}$$

Here the ordered pair (X, R) is a fuzzy partial ordered set.

§2.3 Dominating and Dominated class: - [2]

Let a fuzzy partial ordering R is defined on a non-empty non-fuzzy set X, Two fuzzy sets are associated with an element x in X.

1) Dominating class: -

Let $x \in X$. Then Dominating class of x is a fuzzy set, denoted by $R_{\geq}[x]$ and is defined by,

$$R_{\geq}[x](y) = R(x, y) \qquad \forall\, y \in X.$$

2) Dominated class: -

Let $x \in X$. Then Dominated class of x is a fuzzy set, denoted by $R_{\leq}[x]$ and is defined by,

$$R_{\leq}[x](y) = R(y, x) \qquad \forall\, y \in X.$$

§2.4.1 Fuzzy Upper Bound: [2]

For a crisp subset A of a set X on which a fuzzy partial ordering R is defined, the fuzzy upper bound for A is the fuzzy set, denoted by U(R, A), defined by,

$$U(R, A) = \bigcap_{x \in A} R_{\geq}[x]$$

Where $\cap$ denotes appropriate fuzzy intersection.

§2.4.2 Fuzzy Lower Bound:

For a crisp subset A of a set X on which a fuzzy partial ordering R is defined, the fuzzy lower bound for A is the fuzzy set, denoted by L(R, A), defined by,

$$L(R, A) = \bigcap_{x \in A} R_{\leq}[x]$$

Where $\cap$ denotes appropriate fuzzy intersection.

§2.5 Support of a Fuzzy set:

Definition: "The support of a fuzzy set A within a universal set X is the crisp set that contains all the elements of X that have nonzero membership grades in A." It is denoted as ***supp A*** = $\{x \in X \,/\, A(x) > 0\}$.

§2.6.1 Fuzzy Least Upper Bound: [2]

"For a nonempty nonfuzzy subset A of a set X on which a fuzzy partial ordering R is defined, the fuzzy least upper bound of A, if exists, is the unique element x in U(R, A) such that U(R, A) (x) > 0 and R(x, y) > 0 for all elements y in the support of U(R, A)."

It is denoted by, $\bigvee_{y \in A} y = x$

§2.6.2 Fuzzy Greatest Lower Bound:

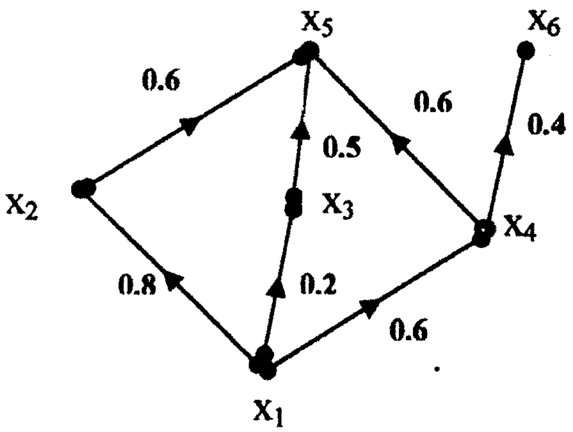
"For a nonempty nonfuzzy subset A of a set X on which a fuzzy partial ordering R is defined, the fuzzy greatest lower bound of A, if exists, is the unique element x in L(R, A) such that L(R, A) (x) > 0 and R(y, x) > 0 for all elements y in the support of L(R, A)."

It is denoted by, $\bigwedge_{y \in A} y = x$

Example: $X=\{x_1, x_2, x_3, x_4, x_5, x_6\}$ and

R(X,X) is given by membership matrix as follows:

	x_1	x_2	x_3	x_4	x_5	x_6
x_1	1	0.8	0.2	0.6	0.6	0.4
x_2	0	1	0	0	0.6	0
x_3	0	0	1	0	0.5	0
x_4	0	0	0	1	0.6	0.4
x_5	0	0	0	0	1	0
x_6	0	0	0	0	0	1



Here the ordered pair (X, R) is fuzzy partial ordered set.

Consider, $A = \{x_1, x_2\} \subset X$.

$$R_{\geq}[x_1] = \left\{\frac{1}{x_1} + \frac{0.8}{x_2} + \frac{0.2}{x_3} + \frac{0.6}{x_4} + \frac{0.6}{x_5} + \frac{0.4}{x_6}\right\}$$

$$R_{\geq}[x_2] = \left\{\frac{1}{x_2} + \frac{0.6}{x_5}\right\}$$

Now, $U(R,A) = R_{\geq}[x_1] \cap R_{\geq}[x_2]$

$$U(R, A) = \left\{\frac{0.8}{x_2} + \frac{0.6}{x_5}\right\}$$

Here, $U(R, A)(x_2) > 0$ and $R(x_2, y) > 0$ for all y in support of $U(R,A)$. Hence x_2 is the fuzzy least upper bound of A. Similarly, x_1 is the fuzzy greatest lower bound of A.

- **Theorem 2.1: -**

Let (X, R) be fuzzy partial order set. Let $a, b \in X$. If $a \vee b$ exists then $R(a, a \vee b) > 0$ and $R(b, a \vee b) > 0$. Also, if $a \wedge b$ exists then $R(a \wedge b, a) > 0$ and $R(a \wedge b, b) > 0$.

Proof:-

Let $a \vee b$ exists and let $a \vee b = m \in X$.

To prove: $R(a, m) > 0$ and $R(b, m) > 0$.

By definition of '$\vee$' i.e., Fuzzy least upper bound,

$U(R, \{a, b\})(m) > 0$ and **(Refer §2.6.1)**

$R(m, y) > 0$ for all y that supports $U(R, \{a, b\})$.

$\therefore (R_{\geq}[a] \cap R_{\geq}[b])(m) > 0.$ **(Refer §2.4.1)**

(By definition of fuzzy upper bound)

$\therefore \min\{R_{\geq}[a](m), R_{\geq}[b](m)\} > 0.$ **(Refer §1.2)**

$\therefore R_{\geq}[a](m) > 0$ and $R_{\geq}[b](m) > 0$.

$\therefore R(a, m) > 0$ and $R(b, m) > 0$. **(By definition of $R_{\geq}$ class)**

$\therefore R(a, a \vee b) > 0$ and $R(b, a \vee b) > 0$. **(Since $m = a \vee b$)**

Thus, if $a \vee b$ exists then $R(a, a \vee b) > 0$ and $R(b, a \vee b) > 0$.

Similarly, if $a \wedge b$ exists then $R(a \wedge b, a) > 0$ and $R(a \wedge b, b) > 0$.

□

- **Theorem 2.2: -**

Let (X, R) be fuzzy poset. Let $a, b, c, d \in X$. If $a \vee c$ and $b \vee d$ exists, $R(a, b) > 0$ and $R(c, d) > 0$ then $R(a \vee c, b \vee d) > 0$.

Proof: -

Let a V c and b V d exist, R(a, b) > 0 and R(c, d) > 0.

∴ R(a, a V c) > 0 and R(c, a V c) > 0.**(By theorem 2.1)**

Also, R(b, b V d) > 0 and R(d, b V d) > 0. **(By theorem 2.1)**

Let a V c = m and b V d = n.

∴ R(a, m) > 0, R(c, m) > 0, R(b, n) > 0 and R(d, n) > 0.

Thus R(a, b) > 0 (given data) and R(b, n) > 0

⇒ min{ R(a, b) , R(b, n) } > 0. ______ (1)

To prove:- R (m, n) > 0

Let, if possible, R (m, n) = 0

Now, m is fuzzy least upper bound of {a, c}

∴ U(R, {a, c}) (m) > 0 and **(Refer §2.6.1)**

R (m, y) > 0 for all y that supports U(R, {a, c}).

∴ n do not support U(R, {a, c}) as R (m, n) = 0.

∴ U(R, {a, c}) (n) = 0.

∴ (R ≥[a] ∩ R ≥[c]) (n) = 0. . **(Refer §2.4.1)**

(By definition of fuzzy upper bound)

∴ min { R ≥[a] (n), R ≥[c] (n) } = 0. **(Refer §1.2)**

∴ min { R(a, n), R(c, n) } = 0 **(By definition of R≥ class)**

∴ R(a, n) = 0 or R(c, n) = 0.

Let without loss of generality, R(a, n) = 0.

By fuzzy max-min transitivity,

$$R(a, n) \geq \max_{y \in X} \min (R(a, y), R(y, n))$$

$$\therefore 0 \geq \max_{y \in X} \min (R(a, y), R(y, n))$$

$$\therefore \max_{y \in X} \min (R(a, y), R(y, n)) = 0$$

$\therefore \min (R(a, y), R(y, n)) = 0 \qquad \forall y \in X.$

Now, $b \in X$

$\therefore \min (R(a, b), R(b, n)) = 0$ ________ (2)

Thus, from (1) and (2), we get a contradiction.

Thus, our assumption that $R(m, n) = 0$ is wrong.

Thus, $R(m, n) > 0$.

Thus, $R(a \vee c, b \vee d) > 0$. (Since $m = a \vee c$, $n = b \vee d$)

Thus if $R(a, b) > 0$ and $R(c, d) > 0$ then $R(a\vee c, b\vee d) > 0$. □

➢ **Theorem 2.3: -**

Let (X, R) be fuzzy poset. Let $a, b, c, d \in X$.

If $a \wedge c$ and $b \wedge d$ exists, $R(a, b) > 0$ and $R(c, d) > 0$

Then $R(a \wedge c, b \wedge d) > 0$.

➢ **Theorem 2.4: -**

Let (X, R) be fuzzy poset. Let $a, b, c \in X$.

If $R(a, b) > 0$ and $R(b, c) > 0$ then $R(a, c) > 0$.

Proof: -

Let $R(a, b) > 0$ and $R(b, c) > 0$ To prove: $R(a, c) > 0$.

Let, if possible, $R(a, c) = 0$

By fuzzy max-min transitivity,

$$R(a, c) \geq \max_{y \in X} \min (R(a, y), R(y, c))$$

$$\therefore 0 \geq \max_{y \in X} \min (R(a, y), R(y, c))$$

$$\therefore \max_{y \in X} \min (R(a, y), R(y, c)) = 0$$

$\therefore \min (R(a, y), R(y, c)) = 0 \qquad \forall y \in X$

Now, $b \in X$

$\therefore$ min $(R(a, b), R(b, c)) = 0$

$\therefore$ $R(a, b) = 0$ or $R(b, c)) = 0$

which is contradiction to given data.

Hence our assumption that $R(a, c) = 0$ is wrong.

Hence $R(a, c) > 0$

If $R(a, b) > 0$ and $R(b, c) > 0$ then $R(a, c) > 0$. □

- **Theorem 2.5: -**

Let (X, R) be fuzzy poset. Let $a, b \in X$.

$a \wedge b$ exists and $R(a, b) > 0$ iff $a \wedge b = a$.

Proof: -

Let $a \wedge b$ exists and $R(a, b) > 0$.

Let, if possible, $a \wedge b \neq a$.

Now, $R(a \wedge b, a) > 0$. **(By theorem 2.1)**

By fuzzy perfectly antisymmetric property,

$R(a, a \wedge b) = 0$.

Now, $R(a, a) > 0$ **(Fuzzy reflexive property)**

and $R(a, b) > 0$ **(given data)**

$\therefore$ $R(a \wedge a, a \wedge b) > 0$ **(by theorem 2.3)**

$\therefore$ $R(a, a \wedge b) > 0$.

Which is a contradiction.

$\therefore$ Our assumption that $a \wedge b \neq a$ is wrong.

$\therefore$ $a \wedge b = a$.

$\therefore R(a, b) > 0 \Rightarrow a \wedge b = a$ _____ (1)

Conversely, Let $a \wedge b = a$

Now, $R(a \wedge b, b) > 0$ **(by theorem 2.1)**

$\therefore$ R (a, b) > 0. **(Since a Λ b = a)**

$\therefore$ a Λ b = a $\Rightarrow$ R(a, b) > 0 _____ (2)

Thus, from (1) and (2), we get

R(a, b) > 0 iff a Λ b = a □

- **Theorem 2.6: -**

Let (X, R) be fuzzy poset. Let a, b $\in$ X.

a V b exists and R(a, b) > 0 iff a V b = b.

- **Theorem 2.7: -**

Let (X, R) be fuzzy poset. Let a, b $\in$ X.

R(a, b) > 0 and R(b, a) > 0 iff a = b.

Proof: -

Let R(a, b) > 0 and R(b, a) > 0

Let, if possible, a $\neq$ b

By fuzzy perfectly antisymmetric property,

R(a, b) > 0 $\Rightarrow$ R(b, a) = 0

which is a contradiction.

$\therefore$ our assumption that a $\neq$ b is wrong.

$\therefore$ a = b.

$\therefore$ R(a, b) > 0 and R(b, a) > 0 $\Rightarrow$ a = b _____ (1)

Conversely, let a = b

R(a, b) = R(a, a) = 1 > 0 **(Fuzzy reflexive property)**

R(b, a) = R(b, b) = 1 > 0 **(Fuzzy reflexive property)**

$\therefore$ a = b $\Rightarrow$ R(a, b) > 0 and R(b, a) > 0 _____ (2)

Thus, from (1) and (2), we get

R(a, b) > 0 and R(b, a) > 0 iff a = b □

§2.7.1 Fuzzy Zero Element: -

Let (X, R) be any fuzzy partial ordered set.

An element **0** $\in$ X, if exists, is called the fuzzy zero element of X

if R(**0**, x) > 0 $\forall$ x $\in$ X.

Here '**0**' is called the smallest element of X.

§2.7.2 Fuzzy Unit Element: -

Let (X, R) be any fuzzy partial ordered set.

An element **1** $\in$ X, if exists, is called the fuzzy unit element of X

if R(x, **1**) > 0 $\forall$ x $\in$ X

Here '**1**' is called the largest element of X.

§2.8 Fuzzy Bounded Posets: -

A fuzzy poset (X, R) is said to be fuzzy bounded poset

if **0** and **1** exists in X.

§2.9 Fuzzy Chain: -

A fuzzy poset (X, R) is a fuzzy chain

if for any two elements x, y $\in$ X, either R(x, y) > 0 or R(y, x) > 0.

Remark: -

Every fuzzy chain is fuzzy partial ordered set.

But converse need not be true.

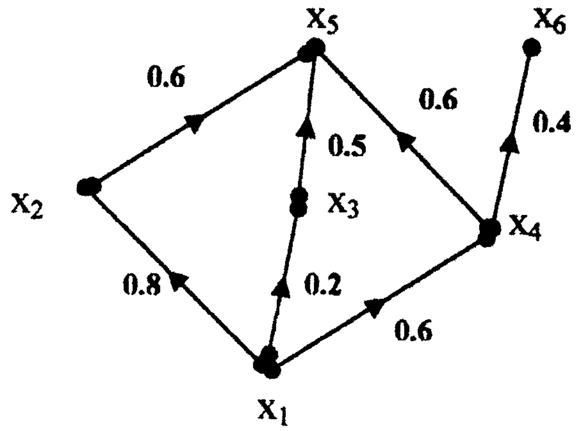
Counter example:

Let X={x_1, x_2, x_3, x_4, x_5, x_6}

R(X,X) is given by membership matrix as follows:

	x_1	x_2	x_3	x_4	x_5	x_6
x_1	1	0.8	0.2	0.6	0.6	0.4
x_2	0	1	0	0	0.6	0
x_3	0	0	1	0	0.5	0
x_4	0	0	0	1	0.6	0.4
x_5	0	0	0	0	1	0
x_6	0	0	0	0	0	1

The Hasse Diagram is,



Here the ordered pair (X, R) is fuzzy partial ordered set.

Consider, $x_2, x_3 \in X$.

Here, neither $R(x_2, x_3) > 0$ nor $R(x_3, x_2) > 0$

Hence, X is not a fuzzy chain.