CHAPTER I

PRELIMINARIES

1. Introduction:

Einstein's Special theory of relativity deals with only inertial observers and does not contain the gravitational force. When it was realized that gravitational phenomenon is incompatible with the concept of inertial frame, Einstein put forward his general theory of relativity in 1915. An inertial frame is specified as one in which a particle with no force on it appears to move with uniform velocity in a straight line. But in practice how can we achieve the state of no force? Since earth's gravitational force exists always and everywhere. It is possible to shield a piece of matter from all other interactions but not from gravitation. Einstein observed that the gravitation is an interaction which can not be switched on and off at will. It is present everywhere, everlasting and hence it is universal. If gravitation is universal, therefore it must act on both massive as well as mass less particles in the same manner. However, the mass less particle moves with velocity $C = 3\times 10^{10}\, cm/\sec$ which according to the Second postulate of special relativity is a universal constant. The key question is how to make gravity interact with mass less particle so that its velocity should not change? In the conventional framework, the only way force makes its presence felt is by changing the velocity of the particle on which it is acting. In an attempt to achieve gravitational force to act on mass less particle without changing its velocity, Einstein identifies gravitation with the intrinsic property of the space-time region. Thus in the space-time region of general relativity the particle feels gravity yet its velocity is constant. This characteristic property of gravitation, Einstein identified with the non-Euclidean nature of space-time geometry and proposed an astonishing result that

Gravitation = Space-time Geometry.

In 1915 he succeeded to formulate this law of nature mathematically after unremitting labor from the variational principle

$$\delta \int \left[R - \frac{16\pi G L}{C^4} \right] (-g)^{\frac{1}{2}} d^4x = 0$$

in the form

$$R_{ij} - \frac{1}{2} R g_{ij} = -K T_{ij}$$

where R_{ij} - the Ricci tensor ,

R- the Ricci curvature scalar ,

L- the matter Lagrangian ,

$K = \frac{8\pi G}{C^4}$ - a universal constant ,

$g = |g_{ik}|$,

$d^4x = dx^1dx^2dx^3dx^4$ - a 4-volume and

T_{ij} is the stress energy momentum tensor of matter which is the source of curvature of the space-time and hence mass directly influences the geometry of the space-time.

The space-time of general relativity is represented by the 4-dimensional manifold on which the metric $ds^2 = g_{ik}dx^i dx^k$ is defined, where the quantities g_{ik} are functions of coordinates and are called the components of the fundamental metric tensor. This theory deals with all types of motion and leads to successful relativistic generalization of Newton's theory of gravitation. However, in the neighbourhood of a point P of the manifold, there always exists inertial coordinates such that at the point P the components g_{ik} take up on their Minkoski space values and $\partial g_{ik}/\partial x^l = 0$. This equivalently means that locally space-time has the same structure as in special relativity and all experimentally observed special relativistic effects such as time dilation; length contraction and increase of mass with velocity are predicted. Further, though it is not apparent from Einstein's field equations of gravitation, but it is required that the equations should reduce to Newtonian gravitational theory in the limit of low masses and low velocities.

The Einstein's field equations of gravitation are ten in numbers. These are non-linear partial differential equations. The first solution of the Einstein's field equations for empty space-time was obtained by Schwarzschild in 1916 in the form

$$ds^2 = \left(1 - \frac{2m}{r}\right)dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) .$$

This metric describes the geometry in the neighbourhood of the Sun. Equivalently, it means that it determines the gravity of the Sun. Einstein derived the relativistic differential equations of the orbit of the mass particle in the gravitational field of the Sun in the form $\frac{d^2u}{d\phi^2} + u = \frac{m}{h^2} + 3mu^2$, where $h = r^2 \frac{d\phi}{ds}$

and for mass less particle as

$$\frac{d^2u}{d\phi^2} + u = 3mu^2 .$$

Solving these equations he has shown that

1. the perihelion of the Mercury shifts by an angle $43''$ per century.
2. the light deflects by an angle $1.75''$ of an arc.

It also governs

3. the collapse of a star to form black holes,
4. the expansion and recontraction of the universe.

The general theory of relativity enables us to understand the mysterious gravitational force through the geometry of the space-time structure. In spite of the fact that the Einstein's general relativity theory is satisfactorily supported by experimental tests on a macroscopic level it is incomplete because it

(i) does not incorporate Mach principle

(ii) does not explain the phenomenon of intrinsic spin of matter

(iii) can't prevent the singularity r = 0 in the solutions of the Einstein's field equations in empty space.

Hence a new theory of gravitation which includes above points is necessary.

Einstein-Cartan theory of gravitation:

A natural generalization of Einstein's general relativity to space-time with torsion is the Einstein-Cartan theory of gravitation. In the history, it is popularly known as U_4 theory of gravitation. The theory was originated by Cartan (1922) by considering the influence of intrinsic spin of matter on the space-time which was not included in the Einstein's general theory of relativity. Obviously, the geometry of the U_4 theory of gravitation is non-Riemannian, in which the christoffel symbols are not symmetric. The non-Riemannian part is described through the affine connections $\omega_{ij}{}^{l}$ and are defined as

$$\omega_{ij}{}^{l} = \Gamma_{ij}{}^{l} - K_{ij}{}^{l} \qquad \text{...(1.1)}$$

where $K_{ij}{}^{l}$ is the contortion tensor satisfying $K_{i(jl)} = 0$ ...(1.2) and $\Gamma_{ij}{}^{l} = \Gamma_{ji}{}^{l}$.

Thus the Einstein-Cartan theory of gravity is a modification of general relativity allowing space-time to have torsion.

In Einstein-Cartan theory the field equations are algebraic relation between the Ricci tensor and canonical energy momentum tensor as in general relativity and also between the torsion tensor and the dynamical spin angular momentum tensor. These are given by Helh et.al (1974, 76) as

$$R_{ij} - \frac{1}{2} R g_{ij} = -K t_{ij} \qquad \ldots(1.3)$$

and

$$Q_{ij}{}^{k} + \delta_i^k Q_{jl}{}^{l} - \delta_j^k Q_{il}{}^{l} = K S_{ij}{}^{k} \qquad \ldots(1.4)$$

where $K = \dfrac{8\pi G}{C^4}$,

$$Q_{ij}{}^{k} = -\frac{1}{2}(K_{ij}{}^{k} - K_{ji}{}^{k}) \text{ - the torsion tensor,} \qquad \ldots(1.5)$$

$S_{ij}{}^{k}$ - the Spin angular momentum tensor and

$$t_{ij} = T^{ij} + \left(\nabla_k + 2Q_{kl}{}^{l}\right)\left(S^{ijk} - S^{jki} + S^{kij}\right)$$

is the canonical energy momentum tensor and T^{ij} is the metric energy momentum tensor.

In recent years there has been a growing interest in the Einstein-Cartan theory of space-time [Trautman (2006), Pereira (2007), Galiakhmetov (2007), Kazmierczak (2008), Stelle (2008), SS Xue (2008, 09)] in which the intrinsic spin of matter is incorporated as the source of the torsion of the space-time manifold. The material of the thesis is organized as follows:

To make the thesis self explanatory we have discussed some basic mathematical concepts in the Chapter I. Hence the first chapter is introductory. The Chapter II deals with the electric part and magnetic part of Weyl tensor in Kerr-Newman space-time. In the last chapter we studied the spherically symmetric space-time field with dust in Einstein-Cartan theory of gravitation and the electric and magnetic parts of Weyl tensor is explored.

The prime mathematical artifact namely the exterior differential forms plays a central role in simplifying the tensor equations considerably. A brief exposition of exterior differential forms is presented in the Section 2. Newman-Penrose (1962) tetrad formalism and its extension to space-times with torsion Jogia and Griffiths (1980) is the language of relativists and its exposition is available in many books and research articles. Some of them which are worth mentioning are:

Flaherty(1976), Carmeli(1977), Kramer Stephani, Herlt and McCallum(1980), S. Chandrasekhar(1983), Jogia and Griffiths(1980), McIntosh(1985), Singh and Griffiths(1990), Katkar(2008) and many others.

Cursory account of tetrad and basic notions is given in the Section 3. Throughout this thesis the use of NPJG formalism is demonstrated to study the electric part and magnetic part of Weyl tensor in General Relativity and Einstein-Cartan theory of gravitation. The thesis abounds in the transcription of tensor quantities into tetrad quantities. The efficiency of the NP tetrad formalism as an incisive tool is displayed in the Section 4. We portrayed the Jogia Griffiths formalism in the next Section 5. No originality is claimed in this chapter.

2. 1-Forms and Space of 1-Forms:

Let T_p be an n-dimensional tangent space. The elements of T_p are called vectors. The set of all vectors of T_p, with vector addition and scalar multiplication, T_p is a vector space.

Definition: A 1-form $\tilde{\omega}$ on T_p is a real valued linear function. That is $\tilde{\omega} : T_p \to R$ such that $\tilde{\omega}(\bar{X}) = \langle \tilde{\omega}, \bar{X} \rangle$ for every $\bar{X} \in T_p$, is called 1-form. The linearity of 1-form function is

$$\tilde{\omega}(f\bar{X} + g\bar{Y}) = f\,\tilde{\omega}(\bar{X}) + g\,\tilde{\omega}(\bar{Y}) \qquad \forall\ f,\ g \in R$$

A collection of all 1-forms $\tilde{\omega}$ under the operations

(i) $(\tilde{\omega} + \tilde{\sigma})(\bar{X}) = \tilde{\omega}(\bar{X}) + \tilde{\sigma}(\bar{X}) \qquad \forall\ \bar{X} \in T_p$

and (ii) $(f\,\tilde{\omega})(\bar{X}) = f\,\tilde{\omega}(\bar{X}) \qquad \forall\ f \in R$

forms a new vector space T_p^* is called a cotangent space or dual space.

Bases in the Cotangent space:

In the cotangent space T_p^* any linearly independent 1-forms constitute a basis. One can choose any arbitrary basis in T_p^*. However, once a basis $\{\bar{e}^i\}$ has been chosen for the vector of T_p at P, this induces a preferred basis for T_p^* called the dual basis $\{\tilde{\theta}^i\}$. The basis $\{\bar{e}_i\}$ determines 1-form basis $\{\tilde{\theta}^i\}$ by

$$\tilde{\theta}^i(\bar{e}_j) = \delta^i_j \qquad \ldots(2.1)$$

P-form and Exterior Differentiation:

A p-form is a completely skew-symmetric real valued function defined on $T_p \times T_p \times \cdots \times T_p$.

Let $\tilde{T}$ be a $p(\geq 2)$ form. It is defined as

$$\tilde{T} : T_p \times T_p \times \cdots \times T_p \to R \text{ such that}$$

$$\tilde{T}(\bar{X},\bar{Y},\cdots,\bar{Z}) = T_{ij\cdots k}(\tilde{\theta}^i \otimes \tilde{\theta}^j \otimes \cdots \otimes \tilde{\theta}^k)(\bar{X},\bar{Y},\cdots,\bar{Z})$$

That is $$\tilde{T} = T_{ij\cdots k}(\tilde{\theta}^i \otimes \tilde{\theta}^j \otimes \cdots \otimes \tilde{\theta}^k) \qquad \ldots(2.2)$$

where $\tilde{\theta}^i \otimes \tilde{\theta}^j \otimes \cdots \otimes \tilde{\theta}^k$ is n^p basis of $\otimes^p T_p^*$. Since p-form $p(\geq 2)$ is completely skew-symmetric covariant tensor in each pair of indices. This implies that all n^p components of $T_{ij\cdots k}$ are not linearly independent but has $\binom{n}{p}$ distinct components, where

$$\binom{n}{p} = \frac{n!}{p!(n-p)!}.$$

Thus we see form equation (2.2) that n^p basis $\tilde{\theta}^i \otimes \tilde{\theta}^j \otimes \cdots \otimes \tilde{\theta}^k$ of $\otimes^p T_p^*$ will no longer be the basis for a space of p-form. Thus the basis for a space of p-form is given by ${}^{[}\tilde{\theta}^i \otimes \tilde{\theta}^j \otimes \cdots \otimes \tilde{\theta}^{k]}$. Hence we have from equation (2.2) that

$$\tilde{T} = T_{ij\cdots k}{}^{[}\tilde{\theta}^i \otimes \tilde{\theta}^j \otimes \cdots \otimes \tilde{\theta}^{k]} \qquad \ldots(2.3)$$

where ${}^{[}\tilde{\theta}^i \otimes \tilde{\theta}^{j]} = \frac{1}{2}(\tilde{\theta}^i \otimes \tilde{\theta}^j - \tilde{\theta}^j \otimes \tilde{\theta}^i) = \tilde{\theta}^i \wedge \tilde{\theta}^j$.

Hence, ingeneral ${}^{[}\tilde{\theta}^i \otimes \tilde{\theta}^j \otimes \cdots \otimes \tilde{\theta}^{k]} = \tilde{\theta}^i \wedge \tilde{\theta}^j \wedge \cdots \wedge \tilde{\theta}^k$

where the wedge product $\wedge$ is defined in the next article.

Hence equation (2.3) can be written as

$$\widetilde{T} = T_{ij\cdots k}\widetilde{\theta}^i \wedge \widetilde{\theta}^j \wedge \cdots \wedge \widetilde{\theta}^k \qquad \ldots(2.4)$$

Which we call ingeneral p-form.

Here $T_{ij\cdots k}$ a completely anti-symmetric in each pair of indices are called the components of p-form or simply p-form or a form of degree p. Note that the set of all p-forms is a vector space and is denoted by $\wedge^p T_p^*$, where $\wedge^p T_p^* = T_p^* \wedge T_p^* \wedge \cdots \wedge T_p^* \subset \otimes^p T_p^*$.

Exterior Product (Wedge Product):

Let $\wedge^p T_p^*$ and $\wedge^q T_p^*$ be spaces of p-form and q-form respectively. If $\widetilde{\omega} \in \wedge^p T_p^*$ is any p-form and $\widetilde{\sigma} \in \wedge^q T_p^*$ is q-form, then their wedge product is denoted by $\wedge$ and it is defined as,

$$\wedge : \wedge^p T_p^* \times \wedge^q T_p^* \to \wedge^{p+q} T_p^*$$

satisfying the following basic properties :

(i) Associative property: $\widetilde{\omega} \wedge (\widetilde{\sigma} \wedge \widetilde{\rho}) = (\widetilde{\omega} \wedge \widetilde{\sigma}) \wedge \widetilde{\rho}$

(ii) Distributive property: $\widetilde{\omega} \wedge (f\,\widetilde{\sigma} + g\,\widetilde{\rho}) = f\,(\widetilde{\omega} \wedge \widetilde{\sigma}) + g\,(\widetilde{\omega} \wedge \widetilde{\rho})$

(iii) $\widetilde{\omega} \wedge \widetilde{\sigma} = (-1)^{pq}\,\widetilde{\sigma} \wedge \widetilde{\omega}$ $\qquad \ldots(2.5)$

(iv) $\widetilde{\omega} \wedge \widetilde{\omega} = 0$

(v) $f\,(\widetilde{\omega} \wedge \widetilde{\sigma}) = f\,\widetilde{\omega} \wedge \widetilde{\sigma} = \widetilde{\omega} \wedge f\,\widetilde{\sigma}$.

Exterior Differentiation:

Exterior differentiation is effected by an operator d applied to forms. It converts p-form to (p+1)-form obtained by taking either partial derivative or covariant derivative of the associated p^{th} order tensor.

Mathematically, $\quad d : \wedge^p T_p^* \to \wedge^{p+1} T_p^*$

with the following properties:

(i) The operator d applied to 0-form f, yields a 1-form df as

$$df = f_{,i}\,dx^i$$

(ii) If $\widetilde{\omega}$ and $\widetilde{\sigma}$ are two p-forms, then

$$d\,(a\widetilde{\omega} + b\widetilde{\sigma}) = a\,d\widetilde{\omega} + b\,d\widetilde{\sigma} \qquad a,b \in R$$

(iii) If $\widetilde{\omega}$ is a p-form and $\widetilde{\sigma}$ is a q-form, then

$$d(\widetilde{\omega} \wedge \widetilde{\sigma}) = d\widetilde{\omega} \wedge \widetilde{\sigma} + (-1)^{\deg \widetilde{\omega}}\, \widetilde{\omega} \wedge d\widetilde{\sigma} \qquad \ldots(2.6)$$

(iv) For any p-form $\widetilde{\omega}$, $d(d\widetilde{\omega}) = d^2\widetilde{\omega} = 0$.

$$d(f\,\widetilde{\omega}) = df \wedge \widetilde{\omega} + f\, d\widetilde{\omega} \quad \text{for } f \in R.$$

Note: The operator d subsumes the ordinary gradient, curl and divergence.

3. Newman-Penrose Formalism:

Let $x^i = x^i(s)$ be the world line of a particle in a given space-time of V_4. At each point of the world line we choose a tetrad consists of four covariant basis vector fields

$$e_{(\alpha)i} = \left(l_i, n_i, m_i, \overline{m}_i\right) \qquad \ldots(3.1)$$

where α is a tetrad index, l_i, n_i are real null vectors and m_i, $\overline{m}_i$ are two ccmplex null vectors. Associated with covariant vector fields, we have contravariant vector fields given by

$$e^i_{(\alpha)} = (l^i, n^i, m^i, \overline{m}^i) \qquad \ldots(3.2)$$

Then $$e^i_{(\alpha)} = g^{ik} e_{(\alpha)k} \qquad \ldots(3.3)$$

where g^{ik} are the contravariant components of the fundamental metric tensor. The null vector fields of the tetrad satisfy orthogonality conditions

$$l_i n^i = -m_i \overline{m}^i = 1,$$

$$l_i m^i = l_i \overline{m}^i = n_i m^i = n_i \overline{m}^i = 0,$$

$$l_i l^i = n_i n^i = m_i m^i = \overline{m}_i \overline{m}^i = 0 \qquad \ldots(3.4)$$

The tetrad of dual basis vectors is defined by $e^{(\alpha)}_i$, so that

$$e^i_{(\alpha)} e^{(\beta)}_i = \delta^\beta_\alpha \qquad \ldots(3.5)$$

The matrix of the scalar product of that basis vector fields is given by

$$\eta_{\alpha\beta} = g_{ik} e^i_{(\alpha)} e^k_{(\beta)} \qquad \ldots(3.6)$$

This gives

$$\eta_{\alpha\beta} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \qquad \ldots(3.7)$$

$\eta^{\alpha\beta}$ is the inverse of the matrix $\eta_{\alpha\beta}$, then

$$\eta_{\alpha\beta}\eta^{\beta\gamma} = \delta_{\alpha}^{\gamma} \qquad \ldots(3.8)$$

The matrix can be used to raise or to lower the tetrad indices.

Consequently, we have

$$e_i^{(\alpha)} = \eta^{\alpha\beta} e_{(\beta)i} \qquad \text{or} \qquad e_{(\alpha)i} = \eta_{\alpha\beta} e_i^{(\beta)} \qquad \ldots(3.9)$$

The tetrad of the dual basis vectors is given by

$$e_i^{(\alpha)} = (n_i,\ l_i,\ -\overline{m}_i,\ -m_i) \qquad \ldots(3.10)$$

Also the components of metric tensor in terms of the tetrad vectors are given by

$$g_{ik} = \eta_{\alpha\beta} e_i^{(\alpha)} e_k^{(\beta)} \qquad \ldots(3.11)$$

The relation between the geometry of space-time and the null basis is given by

$$g_{ik} = l_i n_k + n_i l_k - m_i \overline{m}_k - \overline{m}_i m_k \qquad \ldots(3.12)$$

Alternating Pseudo tensor:

The alternating pseudo-tensor η_{hijk} is defined as

$$\eta_{hijk} = \sqrt{-g}\,\varepsilon_{hijk} \qquad \ldots(3.13)$$

where $\varepsilon_{hijk} (= \varepsilon^{hijk})$ is the Levi-Civita permutation symbol, skew-symmetric in each pair of indices with

$$\varepsilon_{hijk} = \begin{cases} 1 & \text{if h, i, j, k is an even permutation of } 1, 2, 3, 4 \\ -1 & \text{if h, i, j, k is an odd permutation of } 1, 2, 3, 4 \\ 0 & \text{otherwise.} \end{cases} \qquad \ldots(3.14)$$

Raising the indices with the metric tensor g^{hi} yields

$$\eta^{hijk} = \frac{1}{\sqrt{-g}} \varepsilon^{hijk} \qquad \ldots(3.15)$$

Thus if $Q^{\cdots hi \cdots}$ is any tensor skew-symmetric in the pair hi, we define its dual with respect to this pair by

$$Q_{\cdots hi \cdots}^{\;\;*} = \frac{1}{2}\eta_{hijk} Q^{\cdots jk \cdots} \qquad \ldots(3.16)$$

Thus from equations (3.13) and (3.15) we have

$$\eta_{hijk}\eta^{jklm} = \varepsilon_{hijk}\varepsilon^{jklm} = 2\delta_{hi}^{lm}$$

where $\delta_{hi}^{lm} = 2(\delta_h^l \delta_i^m - \delta_i^l \delta_h^m)$ ...(3.17)

is the generalized Kronecker delta symbol.

The tetrad components of the alternating pseudo tensor η_{hijk} are given by

$$\eta_{\alpha\beta\gamma\delta} = \eta_{hijk}\, e_{(\alpha)}^{h}\, e_{(\beta)}^{i}\, e_{(\gamma)}^{j}\, e_{(\delta)}^{k} \qquad \ldots(3.18)$$

Using equations (3.13) in (3.18) we get

$$\eta_{\alpha\beta\gamma\delta} = \sqrt{-g}\,\varepsilon_{hijk}\, e_{(\alpha)}^{h}\, e_{(\beta)}^{i}\, e_{(\gamma)}^{j}\, e_{(\delta)}^{k}$$

$$\eta_{\alpha\beta\gamma\delta} = \sqrt{-g}\,\varepsilon_{\alpha\beta\gamma\delta}\left|e_{(\lambda)}^{l}\right| \qquad \ldots(3.19)$$

We have from equation (3.6)

$$\eta_{\alpha\beta} = g_{ik} e_{(\alpha)}^{i} e_{(\beta)}^{k}$$

Taking determinants of this equation we get

$$\left|\eta_{\alpha\beta}\right| = \left|g_{ik}\right|\left|e_{(\alpha)}^{i}\right|\left|e_{(\beta)}^{k}\right|$$

$$1 = g\left|e_{(\alpha)}^{i}\right|^2$$

$$\sqrt{-g}\,\left|e_{(\alpha)}^{i}\right| = \pm\, i \qquad \ldots(3.20)$$

We make the convention that the tetrad be always to be so oriented with respect to the coordinates that the sign in (3.20) is positive.

Hence $\sqrt{-g}\,\left|e_{(\alpha)}^{i}\right| = i$ ...(3.21)

Substituting this value in equation (3.19) we get

$$\eta_{\alpha\beta\gamma\delta} = i\ \varepsilon_{\alpha\beta\gamma\delta} \qquad \ldots(3.22)$$

Similarly, we obtain

$$\eta^{\alpha\beta\gamma\delta} = i\ \varepsilon^{\alpha\beta\gamma\delta} \qquad \ldots(3.23)$$

We see from equations (3.13) and (3.22) that the advantage of null tetrad is that the inconvenient factor $\sqrt{-g}$ is replaced by i when components in tetrad system is used.

Bivectors and Dual Bivectors:

Bivectors are Skew-symmetric tensors of order two. Consider a skew-symmetric tensor F_{ij}. If $F_{\alpha\beta}$ are tetrad components of the bivector F_{ij} , then we have

$$F^{\alpha\beta} = \eta^{\alpha\gamma}\eta^{\beta\delta}F_{\gamma\delta}$$

$$F^{12} = \eta^{1\gamma}\eta^{2\delta}F_{\gamma\delta}$$

$$F^{12} = F_{21} ,$$

Similarly, we obtain

$$F^{13} = -F_{24} ; \qquad F^{14} = -F_{23} ;$$

$$F^{23} = -F_{14} ; \qquad F^{24} = -F_{13} ;$$

$$F^{34} = F_{43} . \qquad \ldots(3.24)$$

If $F^{*}_{\alpha\beta}$ is the dual tensor, then we have

$$F^{*}_{hi} = \frac{1}{2}\sqrt{-g}\,\varepsilon_{hijk}F^{jk}$$

If $F^{*}_{\alpha\beta}$ are the tetrad components of dual bivector, then we get

$$F^{*}_{\alpha\beta} = \frac{i}{2}\varepsilon_{\alpha\beta\gamma\delta}F^{\gamma\delta} \qquad \ldots(3.25)$$

where $\varepsilon_{\alpha\beta\gamma\delta}$ is the Levi-Civita permutation symbol and it is skew-symmetric in any pair of indices.

Hence

$$F^{*}_{12} = \frac{i}{2}\varepsilon_{12\gamma\delta}F^{\gamma\delta}$$

$$F^{*}_{12} = \frac{i}{2}\left(\varepsilon_{1234}F^{34} + \varepsilon_{1243}F^{43}\right)$$

$$= \frac{i}{2}\left(\varepsilon_{1234}F^{34} + \varepsilon_{1234}F^{34}\right)$$

$$= \frac{i}{2}\left(F^{34} + F^{34}\right)$$

$$F^{*}_{12} = iF^{34} = iF_{43}$$

Similarly,

$$F^{*}_{13} = -iF^{24} = iF_{13} ; \qquad F^{*}_{14} = iF^{23} = -iF_{14} ;$$

$$F^{*}_{23} = iF^{14} = -iF_{23} \ ; \qquad F^{*}_{24} = -iF^{13} = iF_{24} \ ;$$

$$F^{*}_{34} = iF^{12} = iF_{21} \ . \qquad \ldots(3.26)$$

We know a complex bivector F_{ij} is said to be self-dual if

$$F^{*}_{ij} = iF_{ij}$$

Thus, its tetrad components $F_{\alpha\beta}$ is self-dual if

$$F^{*}_{\alpha\beta} = iF_{\alpha\beta}$$

This implies

$$F_{14} = 0 = F_{23} \ , \qquad F_{21} = -F_{43} \ .$$

Thus, we see that, these are only three independent complex components for a self-dual tensor.

Thus, a self-dual complex bivector $F_{\alpha\beta}$ is characterized by

$$F_{14} = 0 \ , \qquad F_{23} = 0 \ , \qquad F_{21} = F_{34}$$

and has three independent complex components F_{13} , F_{24} , $F_{12} + F_{34}$.

Double Dual:

Consider any fourth order tensor Q_{hijk} skew-symmetric in each of the first and second pair of indices i.e.

$$Q_{hijk} = Q_{[hi]jk} = Q_{hi[jk]}$$

Then double dual can be evaluated as follows,

$$Q^{*\ jk}_{hi\ *} = \frac{1}{4}\varepsilon_{hilp}\varepsilon^{jkmn}Q^{lp}{}_{mn} \qquad \ldots(3.27)$$

where $\varepsilon_{hilp}\varepsilon^{jkmn} = \delta^{jkmn}_{hilp}$

where $\delta^{jkmn}_{hilp} = \begin{vmatrix} \delta^{j}_{h} & \delta^{j}_{i} & \delta^{j}_{l} & \delta^{j}_{p} \\ \delta^{k}_{h} & \delta^{k}_{i} & \delta^{k}_{l} & \delta^{k}_{p} \\ \delta^{m}_{h} & \delta^{m}_{i} & \delta^{m}_{l} & \delta^{m}_{p} \\ \delta^{n}_{h} & \delta^{n}_{i} & \delta^{n}_{l} & \delta^{n}_{p} \end{vmatrix}$

$$\delta^{jkmn}_{hilp} = \delta^{jk}_{hi}\delta^{mn}_{lp} + \delta^{jk}_{lp}\delta^{mn}_{hi} + \delta^{jk}_{lh}\delta^{mn}_{ip} + \delta^{jk}_{ip}\delta^{mn}_{lh} + \delta^{jk}_{hp}\delta^{mn}_{il} + \delta^{jk}_{il}\delta^{mn}_{hp}$$

Substituting this in equation (3.27) we get

$$Q_{hi}^{*}{}_{*}^{jk} = Q^{jk}{}_{hi} - \frac{1}{2}\delta_{hi}^{jk} Q^{mn}{}_{mn} + 4\delta_{[h}^{[j} Q_{i]l}^{k]l} \qquad \ldots(3.28)$$

Thus, the Riemann tensor of rank 4 can also be written as

$$R_{hi}^{*}{}_{*}^{jk} = R_{hi}{}^{jk} - 4\delta_{[h}^{[j} S_{i]}^{k]}$$

where S_i^h is trace-free part of the Ricci tensor and defined as

$$S_i^h = R_i^h - \frac{1}{4}\delta_i^h R \ , \qquad\qquad R_{hi} = R^k{}_{hik} .$$

In case of the Weyl conformal tensor,

$$C_{hi}{}^{jk} = R_{hi}{}^{jk} - 2\delta_{[h}^{[j} R_{i]}^{k]} + \frac{1}{6}\delta_{hi}^{jk} R$$

$$C_{hi}{}^{jk} = R_{hi}{}^{jk} - \left(\delta_{[h}^{j} R_{i]}^{k} - \delta_{[h}^{k} R_{i]}^{j}\right) + \frac{1}{6}\left(\delta_h^j \delta_i^k - \delta_i^j \delta_h^k\right) R$$

$$C_{hijk} = R_{hijk} - \frac{1}{2}\left(g_{hi} R_{ik} - g_{ij} R_{hk} - g_{hk} R_{ij} + g_{ik} R_{hj}\right) + \frac{R}{6}\left(g_{hj} g_{ik} - g_{ij} g_{hk}\right)$$

$$C_{hijk}^{**} = C_{hijk} \qquad \ldots(3.29)$$

Also $\quad C_{hi\,jk}^{*} = C_{hi\,jk}^{\ \ \ \ *}$

Instead of Weyl tensor itself, it is equivalent and convenient to consider the complex combination,

$$C_{\alpha\beta\gamma\delta}^{(+)} = C_{\alpha\beta\gamma\delta} - iC_{\alpha\beta\gamma\delta}^{*} \qquad \ldots(3.30)$$

has all the algebraic symmetries of the Weyl tensor (including zero trace) and in addition it is self-dual in both index pairs i.e.

$$C_{\alpha\beta\ \gamma\delta}^{(+)\,*} = C^{(+)}{}_{\alpha\beta}{}^{*}{}_{\gamma\delta} = iC_{\alpha\beta\gamma\delta}^{(+)}$$

4. Newman-Penrose Spin Coefficients Formalism:

The Newman-Penrose spin coefficient formalism has been proved to be of great value in the context of general relativity. It has been particularly useful in the theoretical analysis of gravitational fields and in obtaining exact solutions of the Einstein field equations. We exploit in this thesis the Newman-Penrose formalism and its extension to spaces with torsion (Jogia and Griffiths 1980). A cursory account of the NP formalism is presented in this Section.

We recall here the tetrad consisting of four null vector fields

$$e_{(\alpha)i} = (l_i, n_i, m_i, \overline{m}_i),$$

where the null vector fields satisfy the orthonormality conditions(3.4). The computational advantage of differential equations becomes possible because of the fact that the covariant derivatives of null vector fields are expressed as linear combinations of the four null vector fields. For example

$$l_{i;j} = (\gamma+\overline{\gamma})l_i l_j - (\alpha+\overline{\beta})l_i m_j - (\overline{\alpha}+\beta)l_i\overline{m}_j + (\varepsilon+\overline{\varepsilon})l_i n_j - \overline{\tau} m_i l_j + \overline{\sigma} m_i m_j + \overline{\rho} m_i \overline{m}_j - \\ -\overline{\kappa} m_i n_j - \tau \overline{m}_i l_j + \rho \overline{m}_i m_j + \sigma \overline{m}_i \overline{m}_j - \kappa \overline{m}_i n_j \qquad \ldots(4.1)$$

Similarly, covariant derivatives of other vector fields can be expressed.

Here, the coefficients $\kappa, \nu, \rho, \ldots$ are called the Spin coefficients and are in fact the Ricci's coefficients of rotation, where the Ricci rotation coefficients are defined by

$$\gamma_{\alpha\beta\gamma} = -e_{(\alpha)i;j}\, e^{\,i}_{(\beta)}\, e^{\,j}_{(\gamma)} \qquad \ldots(4.2)$$

which are anti-symmetric in first two indices: $\gamma_{\alpha\beta\gamma} = -\gamma_{\beta\alpha\gamma}$ ...(4.3)

From which one can obtain

$$\kappa = -\gamma_{131} = l_{i;j} m^i l^j \;; \qquad \tau = -\gamma_{132} = l_{i;j} m^i n^j \;;$$

$$\sigma = -\gamma_{133} = l_{i;j} m^i m^j \;; \qquad \rho = -\gamma_{134} = l_{i;j} m^i \overline{m}^j \;;$$

$$\pi = \gamma_{241} = -n_{i;j} \overline{m}^i l^j \;; \qquad \nu = \gamma_{242} = -n_{i;j} \overline{m}^i n^j \;;$$

$$\mu = \gamma_{243} = -n_{i;j} \overline{m}^i m^j \;; \qquad \lambda = \gamma_{244} = -n_{i;j} \overline{m}^i \overline{m}^j \;;$$

$$\varepsilon = \frac{1}{2}\left(-\gamma_{121} + \gamma_{341}\right) = \frac{1}{2}\left(l_{i;j} n^i l^j - m_{i;j} \overline{m}^i l^j\right) ;$$

$$\gamma = \frac{1}{2}\left(-\gamma_{122} + \gamma_{342}\right) = \frac{1}{2}\left(l_{i;j} n^i n^j - m_{i;j} \overline{m}^i n^j\right) ; \qquad \ldots(4.4)$$

$$\beta = \frac{1}{2}\left(-\gamma_{123} + \gamma_{343}\right) = \frac{1}{2}\left(l_{i;j} n^i m^j - m_{i;j} \overline{m}^i m^j\right) ;$$

$$\alpha = \frac{1}{2}\left(-\gamma_{124} + \gamma_{344}\right) = \frac{1}{2}\left(l_{i;j} n^i \overline{m}^j - m_{i;j} \overline{m}^i \overline{m}^j\right).$$

It is clear that the complex conjugate of any quantity can be obtained by interchanging the indices 3 and 4 in the expression.

Representation of the Weyl tensor, the Ricci tensor, and Riemann tensor:

The Weyl tensor is the trace-free part of the Riemann tensor and its tetrad components are given by

$$C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} + \frac{1}{2}\left(\eta_{\alpha\gamma}R_{\beta\delta} + \eta_{\beta\delta}R_{\alpha\gamma} - \eta_{\alpha\delta}R_{\beta\gamma} - \eta_{\beta\gamma}R_{\alpha\delta}\right) + \frac{R}{6}\left(\eta_{\alpha\delta}\eta_{\beta\gamma} - \eta_{\alpha\gamma}\eta_{\beta\delta}\right) \qquad \ldots(4.5)$$

where the Ricci tensor and curvature scalar are defined by

$$R_{\alpha\gamma} = \eta^{\beta\delta}R_{\alpha\beta\gamma\delta} \qquad \text{and} \qquad R = \eta^{\alpha\beta}R_{\alpha\beta} \qquad \ldots(4.6)$$

and the trace-free part of the Weyl tensor is characterized by

$$\eta^{\alpha\delta}C_{\alpha\beta\gamma\delta} = 0 \qquad \ldots(4.7)$$

together with the cyclic property

$$C_{\alpha\beta\gamma\delta} + C_{\alpha\gamma\delta\beta} + C_{\alpha\delta\beta\gamma} = 0 \qquad \ldots(4.8)$$

By giving different values to $\alpha, \beta, \gamma, \delta$ from 1 to 4 in equations (4.7) and (4.8) one can obtain the following relations,

$$\begin{array}{ll} C_{1212} = C_{1342} + C_{1432}\ ; & C_{1213} = C_{1343}\ ; \\ C_{1214} = C_{1434}\ ; & C_{1223} = C_{2334}\ ; \\ C_{1224} = -C_{2434}\ ; & C_{1234} = C_{1324} + C_{1432}\ ; \\ C_{1314} = 0\ ; & C_{1323} = 0\ ; \\ C_{1342} = C_{1423} + C_{3434}\ ; & C_{1424} = 0\ ; \\ C_{1432} = C_{1324} + C_{3434}\ ; & C_{2324} = 0\ . \end{array} \qquad \ldots(4.9)$$

Using these equations, we obtain from equation (4.5) the relations:

$$\begin{array}{ll} C_{1212} = R_{1212} - R_{12} + \dfrac{R}{6}\ ; & C_{1213} = R_{1213} - \dfrac{R_{13}}{2}\ ; \\ C_{1214} = R_{1214} - \dfrac{R_{14}}{2}\ ; & C_{1223} = R_{1223} + \dfrac{R_{23}}{2}\ ; \\ C_{1224} = R_{1224} + \dfrac{R_{24}}{2}\ ; & C_{1234} = R_{1234}\ ; \\ C_{1313} = R_{1313}\ ; & C_{1314} = R_{1314} - \dfrac{R_{11}}{2}\ ; \\ C_{1323} = R_{1323} + \dfrac{R_{33}}{2}\ ; & C_{1324} = R_{1324} + \dfrac{1}{2}\left(R_{34} - R_{12}\right) + \dfrac{R}{6}\ ; \end{array}$$

$$C_{1334} = R_{1334} - \frac{R_{13}}{2} \; ; \qquad C_{1423} = R_{1423} + \frac{1}{2}(R_{43} - R_{12}) + \frac{R}{6} \; ;$$

$$C_{1424} = R_{1424} + \frac{R_{44}}{2} \; ; \qquad C_{1434} = R_{1434} + \frac{R_{14}}{2} \; ;$$

$$C_{2323} = R_{2323} \; ; \qquad C_{2324} = R_{2324} - \frac{R_{22}}{2} \; ;$$

$$C_{2334} = R_{2334} - \frac{R_{23}}{2} \; ; \qquad C_{2424} = R_{2424} \; ; \qquad \ldots(4.10)$$

$$C_{2434} = R_{2434} + \frac{R_{24}}{2} \; ; \qquad C_{3434} = R_{3434} + R_{34} + \frac{R}{6}$$

Tetrad Components of Weyl Tensor and Ricci Tensor:

The tetrad components of Weyl tensor are labeled

$$\psi_{\circ} = -C_{1313} = -C_{hijk} l^h m^i l^j m^k \; ,$$

$$\psi_1 = -C_{1213} = -C_{hijk} l^h n^i l^j m^k \; ,$$

$$\psi_2 = -C_{4213} = -C_{hijk} \overline{m}^h n^i l^j m^k \; ,$$

$$\psi_3 = -C_{4212} = -C_{hijk} \overline{m}^h n^i l^j n^k \; , \qquad \ldots(4.11)$$

$$\psi_4 = -C_{4242} = -C_{hijk} \overline{m}^h n^i \overline{m}^j n^k$$

While, the tetrad components of R_{ij} and R are given by

$$\phi_{\circ\circ} = -\frac{1}{2} R_{11} = -\frac{1}{2} R_{ij} l^i l^j \; ,$$

$$\phi_{\circ 1} = -\frac{1}{2} R_{13} = -\frac{1}{2} R_{ij} l^i m^j \; ,$$

$$\phi_{\circ 2} = -\frac{1}{2} R_{33} = -\frac{1}{2} R_{ij} m^i m^j \; ,$$

$$\phi_{1\circ} = -\frac{1}{2} R_{14} = -\frac{1}{2} R_{ij} l^i \overline{m}^j \; ,$$

$$\phi_{11} = -\frac{1}{4}(R_{12} + R_{34}) = -\frac{1}{4} R_{ij} (l^i n^j + m^i \overline{m}^j) \; ,$$

$$\phi_{12} = -\frac{1}{2} R_{23} = -\frac{1}{2} R_{ij} n^i m^j \; ,$$

$$\phi_{20} = -\frac{1}{2} R_{44} = -\frac{1}{2} R_{ij} \overline{m}^i \overline{m}^j \; ,$$

$$\phi_{21} = -\frac{1}{2}R_{24} = -\frac{1}{2}R_{ij}n^i\overline{m}^j \ ,$$

$$\phi_{22} = -\frac{1}{2}R_{22} = -\frac{1}{2}R_{ij}n^in^j \ , \qquad \ldots(4.12)$$

$$\wedge = \frac{1}{24}R \, .$$

5. Jogia and Griffiths Formalism:

A familiar Newman-Penrose formalism which we discussed in earlier section is extended by Jogia and Griffiths (1980) to include torsion into the space-times. Henceforth we refer to it as NPJG formalism. It is widely used to study certain problems in Einstein-Cartan theory of gravitation, and also hoped that the Jogia-Griffiths formalism may provide the most convenient method for the analysis and generation of exact solutions. The space-time of Einstein-Cartan theory of gravitation is described by non-Riemannian geometry, in which the christoffel symbols are not symmetric. The geometry is described through the torsion tensor given by

$$Q_{ij}^{\ \ k} = -\frac{1}{2}(K_{ij}^{\ \ k} - K_{ji}^{\ \ k}) \qquad \ldots(5.1)$$

where $K_{ij}^{\ \ k}$ is the contortion tensor satisfying

$$K_{i(jk)} = 0 \qquad \ldots(5.2)$$

The tetrad components of contortion tensor are given by

$$K_{\alpha\beta\gamma} = K_{ijk}\, e^i_{(\alpha)}\, e^j_{(\beta)}\, e^k_{(\gamma)} \qquad \ldots(5.3)$$

where $\qquad K_{\alpha(\beta\gamma)} = 0 \, .$

The torsion can most conveniently be incorporated into NPJG formalism in terms of the components of contortion tensor. These are conveniently denoted by the Jogia and Griffiths spin coefficients defined by:

$$\kappa_1 = K_{131} = K_{ijk}l^im^jl^k \ ; \qquad \pi_1 = -K_{142} = -K_{ijk}l^i\overline{m}^jn^k \ ;$$

$$\tau_1 = K_{231} = K_{ijk}n^im^jl^k \ ; \qquad \nu_1 = -K_{242} = -K_{ijk}n^i\overline{m}^jn^k \ ;$$

$$\sigma_1 = K_{331} = K_{ijk}m^im^jl^k \ ; \qquad \mu_1 = -K_{342} = -K_{ijk}m^i\overline{m}^jn^k \ ;$$

$$\rho_1 = K_{431} = K_{ijk}\overline{m}^im^jl^k \ ; \qquad \lambda_1 = -K_{442} = -K_{ijk}\overline{m}^i\overline{m}^jn^k \ ;$$

$$\varepsilon_1 = \frac{1}{2}(K_{121} - K_{143}) = \frac{1}{2}K_{ijk} l^i (n^j l^k - \overline{m}^j m^k) \ ;$$

$$\gamma_1 = \frac{1}{2}(K_{221} - K_{243}) = \frac{1}{2}K_{ijk} n^i (n^j l^k - \overline{m}^j m^k) \ ;$$

$$\beta_1 = \frac{1}{2}(K_{321} - K_{343}) = \frac{1}{2}K_{ijk} m^i (n^j l^k - \overline{m}^j m^k) \ ; \qquad \ldots(5.4)$$

$$\alpha_1 = \frac{1}{2}(K_{421} - K_{443}) = \frac{1}{2}K_{ijk} \overline{m}^i (n^j l^k - \overline{m}^j m^k) \ ;$$

with this notation we have

$$\kappa = \kappa^\circ + \kappa_1, \qquad \varepsilon = \varepsilon^\circ + \varepsilon_1, \text{ etc.}$$

It is clear that the complex conjugate of any quantity can be obtained by replacing the index 3 by 4 and conversely.

Hence from equation (5.3) one can readily obtain the expression for contortion tensor as

$$\begin{aligned} K_{ijk} = 2[&(\varepsilon_1 + \overline{\varepsilon}_1) n_i l_{[j} n_{k]} + (\gamma_1 + \overline{\gamma}_1) l_i l_{[j} n_{k]} - (\varepsilon_1 - \overline{\varepsilon}_1) n_i m_{[j} \overline{m}_{k]} - (\gamma_1 - \overline{\gamma}_1) l_i m_{[j} \overline{m}_{k]} + \\ &\{-\nu_1 l_i l_{[j} m_{k]} + \lambda_1 m_i l_{[j} m_{k]} + \mu_1 \overline{m}_i l_{[j} m_{k]} - \pi_1 n_i l_{[j} m_{k]} - (\alpha_1 + \overline{\beta}_1) m_i l_{[j} n_{k]} - \\ &-(\alpha_1 - \overline{\beta}_1) m_i \overline{m}_{[j} m_{k]} - \tau_1 l_i m_{[j} n_{k]} + \overline{\sigma}_1 m_i m_{[j} n_{k]} - \kappa_1 n_i \overline{m}_{[j} n_{k]} + \rho_1 m_i \overline{m}_{[j} n_{k]}\} + \{c.c.\}] \end{aligned} \qquad \ldots(5.5)$$

where c.c. indicates the complex conjugate of the previous term.

In the Einstein- Cartan theory of gravitation, the trace-free part of the Weyl tensor and the cyclic property described in equations (4.7) and (4.8) give the relations

$$C_{1212} = -(C_{1324} + C_{1423}) = -(C_{2314} + C_{2413});$$

$$C_{1213} = -C_{1334}; \qquad C_{1214} = C_{1434};$$

$$C_{1223} = C_{2334}; \qquad C_{1224} = -C_{2434};$$

$$C_{1312} = -C_{3413}; \qquad C_{1314} = -C_{1413};$$

$$C_{1412} = C_{3414}; \qquad C_{1323} = -C_{2313};$$

$$C_{1424} = -C_{2414}; \qquad C_{2312} = C_{3423}; \qquad \ldots(5.6)$$

$$C_{2412} = -C_{3424}; \qquad C_{2324} = -C_{2423};$$

$$C_{1324} = -(C_{2314} + C_{3434}); \qquad C_{1423} = -(C_{2413} + C_{3434}).$$

In Einstein-Cartan theory of gravitation, the Ricci tensor is not necessarily symmetric and so has 16 independent components. These can be expressed in terms of the nine components of a Hermitian 3×3 matrix ϕ_{AB} , the real parameter $\wedge$, and three new complex components ϕ_A (A, B = 0, 1, 2). These are defined by

$$\phi_{\circ\circ} = -\frac{1}{2} R_{ik} l^i l^k \ ;$$

$$\phi_{\circ 1} = -\frac{1}{4} R_{ik} \left(l^i m^k + m^i l^k \right) ;$$

$$\phi_{\circ 2} = -\frac{1}{2} R_{ik} m^i m^k \ ;$$

$$\phi_{11} = -\frac{1}{8} R_{ik} \left(l^i n^k + n^i l^k + m^i \overline{m}^k + \overline{m}^i m^k \right) ;$$

$$\phi_{12} = -\frac{1}{4} R_{ik} \left(n^i m^k + m^i n^k \right) ;$$

$$\phi_{22} = -\frac{1}{2} R_{ik} n^i n^k \ ; \qquad \ldots(5.7)$$

$$\wedge = \frac{R}{24} \ ;$$

$$\phi_{\circ} = -\frac{1}{4} R_{ik} \left(l^i m^k - m^i l^k \right) ;$$

$$\phi_1 = -\frac{1}{4} R_{ik} \left(l^i n^k - n^i l^k - m^i \overline{m}^k + \overline{m}^i m^k \right) ;$$

$$\phi_2 = -\frac{1}{4} R_{ik} \left(\overline{m}^i n^k - n^i \overline{m}^k \right) .$$

Also the trace-free part of the curvature tensor has 20 independent real components. These can be expressed in terms of the five complex components, nine components of a Hermitian 3×3 matrix Θ_{AB} (A, B =0, 1, 2) and a real parameter χ . These are defined by

$$\psi_{\circ} = -C_{1313} = -C_{hijk} l^h m^i l^j m^k \ ;$$

$$\psi_1 = -\frac{1}{2}\left(C_{1213} + C_{4313} \right) = -\frac{1}{2} C_{hijk} \left(l^h n^i + \overline{m}^h m^i \right) l^j m^k \ ;$$

$$\psi_2 = -C_{4213} = -C_{hijk} \overline{m}^h n^i l^j m^k \ ;$$

$$\psi_3 = -\frac{1}{2}\left(C_{1242} + C_{4342} \right) = -\frac{1}{2} C_{hijk} \left(l^h n^i + \overline{m}^h m^i \right) \overline{m}^j n^k \ ;$$

$$\psi_4 = -C_{4242} = -C_{hijk} \overline{m}^h n^i \overline{m}^j n^k \ ;$$

$$\Theta_{\circ\circ} = -iC_{1314} = -C_{hijk} l^h m^i l^j \overline{m}^k \ ;$$

$$\Theta_{\circ 1} = -\frac{i}{2}\left(C_{1312} + C_{1343}\right) = -\frac{i}{2} C_{hijk} l^h m^i \left(l^j n^k + \overline{m}^j m^k\right) ; \qquad \ldots(5.8)$$

$$\Theta_{\circ 2} = iC_{1323} = iC_{hijk} l^h m^i n^j m^k \ ;$$

$$\Theta_{11} = \frac{i}{4}\left(C_{1212} + C_{1243} - C_{4312} - C_{4343}\right) = \frac{i}{4} C_{hijk}\left(l^h n^i - \overline{m}^h m^i\right)\left(l^j n^k + \overline{m}^j m^k\right);$$

$$\Theta_{12} = -\frac{i}{2}\left(C_{2312} + C_{2343}\right) = -\frac{i}{2} C_{hijk} n^h m^i \left(l^j n^k + \overline{m}^j m^k\right) ;$$

$$\Theta_{22} = -iC_{2423} = -i\, C_{hijk} n^h \overline{m}^i n^j m^k \ .$$

We record the results obtained by Katkar (2008) for the use in the Chapter 3.

$$C_{1212} = R_{1212} - \frac{1}{2}\left(R_{12} + R_{21}\right) + \frac{R}{6} \ ; \qquad C_{1213} = R_{1213} - \frac{R_{13}}{2} \ ;$$

$$C_{1214} = R_{1214} - \frac{R_{14}}{2} \ ; \qquad C_{1223} = R_{1223} + \frac{R_{23}}{2} \ ;$$

$$C_{1224} = R_{1224} + \frac{R_{24}}{2} \ ; \qquad C_{1234} = R_{1234} \ ;$$

$$C_{1312} = R_{1312} - \frac{R_{31}}{2} \ ; \qquad C_{1313} = R_{1313} \ ;$$

$$C_{1314} = R_{1314} - \frac{R_{11}}{2} \ ; \qquad C_{1323} = R_{1323} + \frac{R_{33}}{2} \ ;$$

$$C_{1324} = R_{1324} + \frac{1}{2}\left(R_{34} - R_{12}\right) + \frac{R}{6} \ ; \qquad C_{1334} = R_{1334} - \frac{R_{13}}{2} \ ;$$

$$C_{1412} = R_{1412} - \frac{R_{41}}{2} \ ; \qquad C_{1413} = R_{1413} - \frac{R_{11}}{2} \ ;$$

$$C_{1414} = R_{1414} \ ; \qquad C_{1423} = R_{1423} + \frac{1}{2}\left(R_{43} - R_{12}\right) + \frac{R}{6} \ ;$$

$$C_{1424} = R_{1424} + \frac{R_{44}}{2} \ ; \qquad C_{1434} = R_{1434} + \frac{R_{14}}{2} \ ;$$

$$C_{2312} = R_{2312} + \frac{R_{32}}{2} \ ; \qquad C_{2313} = R_{2313} + \frac{R_{33}}{2} \ ;$$

$$C_{2314} = R_{2314} + \frac{1}{2}\left(R_{34} - R_{21}\right) + \frac{R}{6} \ ; \qquad C_{2323} = R_{2323} \ ;$$

$$C_{2324} = R_{2324} - \frac{R_{22}}{2} ; \qquad C_{2334} = R_{2334} - \frac{R_{23}}{2} ;$$

$$C_{2412} = R_{2412} + \frac{R_{42}}{2} ; \qquad C_{2413} = R_{2413} + \frac{1}{2}(R_{43} - R_{21}) + \frac{R}{6} ;$$

$$C_{2414} = R_{2414} + \frac{R_{44}}{2} ; \qquad C_{2423} = R_{2423} - \frac{R_{22}}{2} ; \qquad \ldots(5.9)$$

$$C_{2424} = R_{2424} ; \qquad C_{2434} = R_{2434} + \frac{R_{24}}{2} ;$$

$$C_{3412} = R_{3412} ; \qquad C_{3413} = R_{3413} - \frac{R_{31}}{2} ;$$

$$C_{3414} = R_{3414} + \frac{R_{41}}{2} ; \qquad C_{3423} = R_{3423} - \frac{R_{32}}{2} ;$$

$$C_{3424} = R_{3424} + \frac{R_{42}}{2} ; \qquad C_{3434} = R_{3434} + \frac{1}{2}(R_{43} + R_{34}) + \frac{R}{6} .$$

The trace-free part of Weyl tensor has components,

$$C_{1212} = C_{3434} = (-\psi_2 + i\chi - i\Theta_{11}) + (-\overline{\psi}_2 + \overline{i}\overline{\chi} - \overline{i}\overline{\Theta}_{11}) ;$$

$$C_{1213} = -C_{1334} = -\psi_1 - i\Theta_{01} ;$$

$$C_{1223} = C_{2334} = \overline{\psi}_3 - i\Theta_{12} ;$$

$$C_{1234} = \psi_2 - \overline{\psi}_2 + 2i\Theta_{11} - 2i\chi ;$$

$$C_{1312} = -C_{3413} = -\psi_1 + i\Theta_{01} ;$$

$$C_{1313} = -\psi_\circ ;$$

$$C_{1314} = -C_{1413} = i\Theta_{\circ\circ} ;$$

$$C_{1323} = -C_{2313} = -i\Theta_{\circ 2} ;$$

$$C_{1324} = C_{2413} = \psi_2 ;$$

$$C_{2312} = C_{3423} = \overline{\psi}_3 + i\Theta_{12} ;$$

$$C_{2323} = -\overline{\psi}_4 ; \qquad \ldots(5.10)$$

$$C_{2324} = -C_{2423} = -i\Theta_{22} ;$$

$$C_{3412} = \psi_2 - \overline{\psi}_2 - 2i\Theta_{11} - 2i\chi$$

and C_{1214}, C_{1224}, C_{1412}, C_{1414}, C_{1423}, C_{1424}, C_{1434}, C_{2314}, C_{2412}, C_{2414}, C_{2424}, C_{2434}, C_{3414}, C_{3424} are obtain from above equations by interchanging 3 and 4, in the components C_{1213}, C_{1223}, C_{1312}, $C_{13:3}$, C_{1324}, C_{1323}, C_{1334}, C_{2413}, C_{2312}, C_{2313}, C_{2323}, C_{2334}, C_{3413}, C_{3423}.

Consequently, Weyl tensor C_{hijk} in tetrad components is expressed as

$$\begin{aligned} C_{hijk} = [&-\psi_4 Z^1_{hi} Z^1_{jk} + \psi_3 \left(Z^1_{hi} Z^2_{jk} + Z^2_{hi} Z^1_{jk} \right) - \psi_2 \left(Z^1_{hi} Z^3_{jk} + Z^2_{hi} Z^2_{jk} + Z^3_{hi} Z^1_{jk} \right) + \\ &+ \psi_1 \left(Z^2_{hi} Z^3_{jk} + Z^3_{hi} Z^2_{jk} \right) - \psi_\circ Z^3_{hi} Z^3_{jk} + i\Theta_{\circ\circ} Z^3_{hi} \overline{Z}^3_{jk} + i\Theta_{\circ 1} \left(\overline{Z}^2_{hi} Z^3_{jk} - Z^3_{hi} \overline{Z}^2_{jk} \right) + \\ &+ i\Theta_{\circ 2} \left(\overline{Z}^1_{hi} Z^3_{jk} + Z^3_{hi} \overline{Z}^1_{jk} \right) + i\Theta_{11} Z^2_{hi} \overline{Z}^2_{jk} - i\Theta_{12} \left(Z^2_{hi} \overline{Z}^1_{jk} - \overline{Z}^1_{hi} Z^2_{jk} \right) + i\Theta_{22} \overline{Z}^1_{hi} Z^1_{jk} + \\ &+ 2i\chi Z^2_{hi} Z^2_{jk}] + [c.c.] \end{aligned} \qquad \ldots(5.11)$$

where

$$\begin{aligned} Z^1_{hi} &= -2 n_{[h} \overline{m}_{i]} ; \\ Z^2_{hi} &= -2 [l_{[h} n_{i]} - m_{[h} \overline{m}_{i]}] ; \qquad \ldots(5.12) \\ Z^3_{hi} &= 2 l_{[h} m_{i]} . \end{aligned}$$

It is hope that the equation (5.11) can be used for the analysis of Weyl tensor.