C H A P T E R – II

CHAPTER - II

LAPLACE TRANSFORM-COMPUTER IMPLEMENTATION

2.1 INTRODUCTION

The problem involving several variables can be solved by applying integral transformations successively with regard to several variables. In physical problems Laplace transformation is generally used first to remove the time variable and then other integral transformations on space variables are successively applied. Some examples of the repeated application of transformations are given by Carslaw, Ditkin, Sneddon [7] and Tranter [8].

The conventional Laplace transformation without S-multiplied form is defined by

$$f(s) = \int_0^{\infty} e^{-st} \cdot F(t)\, dt \qquad \ldots .(2.1.1)$$

has always been a subject of great interest because of its mathematical elegance and also because of its usefulness in solving certain types of boundary value problems. In certain cases it has already shown its distinct and superior mathematical character in comparison with the ordinary methods of solving such problems. As such many mathematicians have made a deep study of the Laplace transformation in its various aspects and some others have introduced more generalized integral transformations, of which the Laplace transformation becomes a perticular case.

There are many methods of finding Laplace transform. We enumerate below some of these methods.

1. Direct method.
2. Series method.
3. Method of differential equations.
4. Differentiation with respect to a parameter.
5. Miscellaneous methods.
6. Numerical approximation.

Also methods of solving inverse Laplace transformation are

1. Partial fractions method.
2. Series method.
3. Method of differential equations.
4. Differentiation with respect to a parameter.
5. Miscellaneous methods.
6. Numerical approximations.
7. Complex inversion formula.

Among these methods numerical approximation is a powerful method for finding transforms. Here we discuss the same method in detail.

2.2 DEFINITION

Laplace Transform

Let $F(t)$ be a function of t $(t > 0)$. Then the Laplace transform of $F(t)$, denoted by $L\{F(t)\}$, is defined by

$$L\{F(t)\} = f(s) = \int_0^\infty e^{-st} \cdot F(t)dt \qquad \ldots\ (2.2.1)$$

The Laplace transform of $F(t)$ is said to exist if (2.2.1) converges for some value of s; otherwise it does not exist. For sufficient conditions under which the Laplace transform does exist are $F(t)$ is sectionally continuous in every finite interval $0 \le t \le N$ in the range $t \ge 0$ and of exponential order r for $t > N$.

2.3 NUMERICAL COMPUTATION OF THE LAPLACE TRANSFORM

To compute the direct Laplace transform write

$$f(s) = \int_0^\infty e^{-st} \cdot F(t)dt$$

$$= \frac{1}{s} \int_0^\infty e^{-u} F\left(\frac{u}{s}\right)du \qquad \text{put } st = u$$

$$\cong \frac{1}{s} \sum_{k=1}^{n} w_k \cdot F\left(\frac{x_k}{s}\right) \qquad \ldots\ (2.3.1)$$

where x_k are the roots of the Laguerre polynomial. Laguerre polynomial $L_n(x)$ given by equation

$$L_n(x) = e^x \cdot \frac{d^n}{dx^n} (x^n \cdot e^{-x}) \qquad \ldots (2.3.2)$$

while the weights w_k are defined by

$$w_k = \int_0^{\infty} \frac{e^{-x} \cdot L_n(x)}{(x-x_k)\, L_n'(x_k)}\, dx$$

$$= \frac{(n!)^2}{x_k \cdot [L_n'(x_k)]^2} \qquad \ldots. (2.3.3)$$

For $n = 1$,

$$L_1(x) = 1-x$$

we have a zero at $x_1 = 1$.

The coefficient

$$w_1 = \frac{1}{[L_1'(1)]^2} = 1$$

For $n = 2$,

$$L_2(x) = x^2 - 4x + 2$$

we solve $L_2(x) \equiv x^2 - 4x + 2 = 0$

The zeros of $L_2(x)$ are

$x_1 = 0.58578644$ and $x_2 = 3.41421356$

and corresponding weight functions are obtain from (2.3.3) are

$w_1 = 0.85355339$ and $w_2 = 0.14644661$

Computing these and inserting them into the formula the x_k, W_k given by Francis Scheid [4]. Here the n point formula (2.3.2) is exact for polynomials of degree upto 2n-1.

We use the table of zeros and corresponding weight function given by Francis Scheid [4] for solving Laplace transform using formula (2.3.1).

2.4 COMPUTER IMPLEMENTATION

The method numerical approximation of Laplace transform can be implemented in computer as follows.

Step 1 : Read the value of n.

Step 2 : Read the values of $w_1, w_2, \ldots, w_n$

Step 3 : Read the values of $x_1, x_2, \ldots, x_n$

Step 4 : Read the value of s

Step 5 : For i = 1 to n do

$[x_i] = x_i/s$

Step 6 : sum = 0

Step 7 : for i = 1 to n do

sum = sum + w_i $F(x_i/s)$

Step 8 : Compute NA = Sum/S

Step 9 : Write NA

Step 10: Stop

2.5 PROGRAM LTAPPROXIMATION

```
Program : LTAPPROXIMATION (input, output);
          (* This program gives numerical approximation
             of Laplace transform *)
Var
          F,W,P,X: array [1..100] of real;
```

```
            N,I : 1..100;
            A,S,NA : real;
Function
            f_t(t:real):real;
            begin
                 f_t: = ......
            end;
Begin
            Write ('enter number of terms of approx. N');
            read (N);
            Writeln  ('enter values one after another W(1)  to
            W(N)');
            For I: = 1 to N do
            read (W[I]);
            Writeln ('enter the values one after another  x(1)
            to x(N)');
            for I: = 1 to N do
            read (x[I]);
            Writeln ('enter the value of S');
            read (S);
            While (S<>0) do
              begin
                A:= 0;
                for I:= 1 to N do
                  begin
```

```
        P[I]:= f_t(x[I]/S);
        F[I]:= W[I] * P[I];
        A:= A + F[I]
      end;
      NA:= A/S
      writeln ('the required approximate value
           :=', NA);
      writeln (' next value of S');
      read (S)
    end;
End.
```

2.6 EXAMPLES (For these examples we used Appendix - I)

(1) Given $F(t) = t\bar{e}^{\alpha t}$

Analytical formula

$$f(s) = L[F(t)] = \int_0^\infty \bar{e}^{st} . [t . \bar{e}^{\alpha t}] dt$$

$$= (s + \alpha)^{-2}; \quad ReS > -Re\alpha$$

Let $\alpha = 2$, Given $F(t) = t\bar{e}^{2t}$

$$f(s) = \int_0^\infty \bar{e}^{st} . (t\bar{e}^{2t}) = (S+2)^{-2}$$

S	Exact value of f(s)	Approximate value of f(s)
0.1	0.2268	
0.2	0.2066	
0.3	0.1890	
0.4	0.1736	0.2179
0.5	0.1600	0.1848
0.6	0.1479	0.1614
0.7	0.1372	0.1444
0.8	0.1276	0.1315
0.9	0.1189	0.1211
1.0	0.1111	0.1123
1.1	0.1041	0.1048
1.2	0.0977	0.0981
1.3	0.0918	0.0921
1.4	0.0865	0.0867
1.5	0.0816	0.0817

(2) Given $F(t) = t^{-1}(1 - \cos\alpha t)$

Analytical formula

$$f(s) = L[F(t)] = \int_0^\infty e^{-st} \cdot [t^{-1}.(1-\cos\alpha t)]dt$$

$$= \tfrac{1}{2} \log \left(1 + \frac{\alpha^2}{s^2}\right), \quad \mathrm{Re}S > |\mathrm{Im}\alpha|$$

Let $\alpha = 1$, $F(t) = t^{-1}(1-\cos t)$

$$f(s) = \int_0^\infty e^{-st} \cdot F(t)dt = \tfrac{1}{2}\log \left(1 + \frac{1}{s^2}\right)$$

S	Exact value of f(s)	Approximate value of f(s)
0.1	2.3076	3.9087
0.2	1.6290	2.3255
0.3	1.2471	1.0180
0.4	0.9905	0.8560
0.5	0.8047	0.7895
0.6	0.6646	0.6740
0.7	0.5561	0.5632
0.8	0.4705	0.4738
0.9	0.4020	0.4033
1.0	0.3466	0.3470
1.1	0.3012	0.3013
1.2	0.2637	0.2637
1.3	0.2324	0.2324
1.4	0.2061	0.2061
1.5	0.1839	0.1838

(3) Given $F(t) = t^{-\frac{1}{2}} (1 + 2 \alpha t)$

Analytical formula

$$f(s) = L[F(t)] = \int_0^\infty \bar{e}^{st} \, . \, [t^{-\frac{1}{2}}.(1+2\alpha t)]dt$$

$$= \pi^{\frac{1}{2}} \, . \, s^{-3/2} \, . \, (s + \alpha), \ \text{Res} > 0$$

Let $\alpha = 3$, Given $F(t) = \bar{t}^{\frac{1}{2}}(1 + 6t)$

$$f(s) = \int_0^\infty \bar{e}^{s+} \, . \, F(t)dt = \frac{\sqrt{\pi} \, . \, (s + 3)}{s^{3/2}}$$

S	Exact value of f(s)	Approximate value of f(s)
0.1	173.7547	173.7020
0.2	63.4132	63.0560
0.3	35.5965	35.2178
0.4	23.8212	23.4555
0.5	17.5464	17.1991
0.6	13.7294	13.4000
0.7	11.1977	10.8847
0.8	09.4129	09.1143
0.9	08.0961	07.8104
1.0	07.0898	06.8156
1.1	06.2990	06.0350
1.2	05.6631	05.4084
1.3	05.1420	04.8957
1.4	04.7080	04.4693
1.5	04.3416	04.1099

(4) Given $F(t) = \sin \alpha t$

Analytical formula

$$f(s) = L[F(t)] = \int_0^\infty e^{-st} [\sin \alpha t]\, dt$$

$$= \alpha(s^2 + \alpha^2)^{-1}, \quad \text{Re } s > |I_m \alpha|$$

Let $\alpha = 3$, Given $F(t) = \sin 3t$

$$f(s) = \int_0^\infty e^{-st} \cdot F(t)\, dt = \frac{3}{s^2 + 9}$$

S	Exact value of f(s)	Approximate value of f(s)
0.5	0.3243	1.3415
1.0	0.3000	0.1508
1.5	0.2667	0.2962
2.0	0.2308	0.2275
2.5	0.1967	0.1963
3.0	0.1667	0.1667
3.5	0.1412	0.1412
4.0	0.1200	0.1200
4.5	0.1026	0.1026
5.0	0.0882	0.0882
5.5	0.0764	0.0764
6.0	0.0667	0.0667
6.5	0.0585	0.0585
7.0	0.0517	0.0517
7.5	0.0460	0.0460
8.0	0.0411	0.0411
8.5	0.0369	0.0369
9.0	0.0333	0.0333
9.5	0.0302	0.0302
10.0	0.0275	0.0275

2.7 DEFINITION

Inverse Laplace Transform

If $f(s) = L\{F(t)\}$, then $L^{-1}\{f(s)\}$ is given by,

$$F(t) = \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} e^{st} \cdot f(s)ds \quad , \quad t > 0 \quad \ldots(2.7.1)$$

and $F(t) = 0$ for $t < 0$. This result is called the inverse Laplace transform. The integral in (2.7.1) is exist if $f(s)$ be any function of the complex variable S that is analytic and of order $O(s^{-k})$ for all $s(s = x + iy)$, in a half plane $x \geq x_o$, where $k > 1$ also let $f(x)$ be real $(x \geq x_o)$.

Then the inversion integral of $f(s)$ along any line $x = r$ where $r \geq x_o$ converges to a real valued function $F(t)$ that is independent of r.

2.8 NUMERICAL COMPUTATION OF THE INVERSE LAPLACE TRANSFORM

The inverse Laplace transform is given by the approximate formula

$$\frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} e^{st} f(s)ds \cong \sum_{k=1}^{m} A_k^{(m)} (t) \cdot f(k) \quad \ldots(2.8.1)$$

where the m-point formula is employed whenever $f(s)$ is consider to be of the form $\sum_{r=1}^{m} \frac{B_r}{s^r}$. For the sake of brevity we write $A_k^{(m)}(t) \equiv A_k$; the m and t both being understood. The A_k is defined as the coefficients of $f(k)$ in the inverse Laplace transform of the (m+1)-point Lagrange interpolation polynomial of the m^{th} degree in $1/s$ which is equal to $f(s)$

for $s=k$ where $k = 1,2,..m,\infty$. Since we always have $f(\infty) = 0$, even though $A_\infty^{(m)}(t) \neq 0$, the $(m+1)^{th}$ term corresponding to $k=\infty$ is absent from the summation.

METHOD OF COMPUTATION

An expression suitable for the computation of $L_k^{(m)}(t)$ was obtained by writing the (m+1)-point Lagrange polynomial in 1/s for f(s) at s=1,2 ... m,∞ in the form

$$\sum_{k=1}^{m} \frac{\left(\frac{1}{s}-\frac{1}{1}\right)\left(\frac{1}{s}-\frac{1}{2}\right)\cdots\left(\frac{1}{s}-\frac{1}{k-1}\right)\left(\frac{1}{s}-\frac{1}{k+1}\right)\cdots\left(\frac{1}{s}-\frac{1}{m}\right)\cdot\frac{1}{s}}{\left(\frac{1}{k}-\frac{1}{1}\right)\left(\frac{1}{k}-\frac{1}{2}\right)\cdots\left(\frac{1}{k}-\frac{1}{k-1}\right)\left(\frac{1}{k}-\frac{1}{k+1}\right)\cdots\left(\frac{1}{k}-\frac{1}{m}\right)\cdot\frac{1}{k}}\cdot f(k)$$

The coefficient of f(k) is easily seen to be

$$\frac{k^{m+1}}{s^{m+1}}\cdot\frac{s(s-1)}{k(k-1)}\frac{(s-2)\dots(s-k+1)(s-k-1)\dots(s-m)}{(k-2)\dots(k-k+1)(k-k-1)\dots(k-m)}$$

where the multiplier of k^{m+1}/s^{m+1} is a Lagrange coefficient in the variable s, for the (m+1) point k=0,1,...m at the point s=k or $1/m!\ L_k^{(m+1)}(s)$

$$\text{i.e.}\ \frac{1}{m!}L_k^{(m+1)}(s) = \frac{s(s-1)(s-2)\dots(s-k+1)(s-k-1)\dots(s-m)}{k(k-1)(k-2)\dots(k-k+1)(k-k-1)\dots(k-m)}$$

The inverse Laplace transform of the (m+1)-point Lagrange polynomial in 1/s is given by

$$L^{-1}\left\{\sum_{k=1}^{m}\frac{k^{m+1}}{s^{m+1}}\cdots\frac{1}{m!}L_k^{(m+1)}(s)\ .\ f(k)\right\}$$

$$= \frac{1}{2\pi i}\sum_{k=1}^{m}\frac{k^{m+1}}{m!}f(k)\int_{r-i\infty}^{r+i\infty}\frac{e^{st}}{s^{m+1}}L_k^{(m+1)}(s)\,ds \quad .(2.8.2)$$

In equation (2.8.2) replace $L_k^{(m+1}(s)$ by its explicit expression $\sum_{j=1}^{m} a_j.s^j$, where the coefficients a_j which are understood of course to be functions of k are exact integers. Then the coefficients of f(k) namely $A_k^{(m)}(t)$ is given by

$$A_k^{(m)}(t) = \frac{k^{m+1}}{m!}\sum_{j=1}^{m}\frac{a_j}{2\pi i}\int_{r-i\infty}^{r+i\infty}\frac{e^{st}}{s^{m+1-j}}\,ds$$

$$= \frac{k^{m+1}}{m!}\sum_{j=1}^{m}a_j\frac{t^{m-j}}{(m-j)!} \qquad \ldots. (2.8.3)$$

using formula (2.8.3) H.E. Salzer [6] find the values of $A_k^{(m)}$ for different values of t. We use appropriate values to find inverse Laplace transform from (2.8.1).

2.9 COMPUTER IMPLEMENTATION

Now we implement this method numerical approximation for inverse Laplace transform in computer as follows.

Step 1 : Read the value of m

Step 2 : Read the values of $A_1, A_2, \ldots A_m$.

Step 3 : For i=1 to m

compute f(i)

Step 4 : A = 0

Step 5 : For i = 1 to m do

$A = A + A_i \cdot f(i)$

Step 6 : Write A

Step 7 : Stop

2.10 PROGRAM

```
          Program INVLAPTRANSAPP (input, output);
     (*  This program gives numerical approximation of in-
     verse Laplace transform*)
var
     F,W : array [1..100] of real;
     N,I : 1...100;
     A,T : real;
Function
     fx(x : real) : real;
     begin
          fx := ...
     end;
Begin
     write ('enter the number of terms of approximation N');
     read (N);
     Writeln ('enter the value of T ');
     read (T);
```

```
     Writeln ('enter values one after another w(1)to w(N)');
     for I : 1 to N do
     read (W[I]);
     for I := 1 to N do
     f(I) := f_x(I);
     A := 0;
     for I := 1 to N do
     begin
          F[I] := W[I] * f[I];
          A := A + F[I]
     end;
     writeln ('the required approximate value =',A)
End.
```

2.11 **EXAMPLES** (For these examples we used Appendix - II)

Evaluate the integral $\frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} e^{st} \cdot \tan^{-1} \left(\frac{2}{s}\right) ds$

Analytical formula

$$\frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} e^{st} \cdot \tan^{-1}(\alpha s^{-1}) ds = t^{-1} \cdot \sin(\alpha t)$$

for $\alpha = 2$,

using computer program we get approximate value of

$$F(t) = \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} e^{st} \cdot \tan^{-1}\left(\frac{2}{s}\right) ds$$

for different values of t. The exact and approximate values of F(t) for different t shown in following table.

t	Exact value of F(t)	Approximate value of F(t)
0.2	1.9471	1.9471
0.4	1.7934	1.7936
0.6	1.5534	1.5531
0.8	1.2495	1.2488
1.0	0.9093	0.9096
1.2	0.5629	0.5643
1.4	0.2393	0.2387
1.6	-0.0365	-0.0364
1.8	-0.2458	-0.2492
2.0	-0.3784	-0.3864
2.2	-0.4325	-0.4454
2.4	-0.4151	-0.4273
2.6	-0.3398	-0.3412
2.8	-0.2255	-0.2070
3.0	-0.0931	-0.0560
3.2	0.0364	+0.1491
3.4	0.1453	0.2959
3.6	0.2205	0.3744
3.8	0.2547	0.4468
4.0	0.2473	0.3787

2) Here we check complicated definite integral formula from Bieren's de Haan's compendium. It was a direct Laplace transform. The formula is,

$$\int_0^\infty e^{-st}.\sqrt{t}.\sin(\alpha t)dt=\tfrac{1}{2}\sqrt{\{-s^3+3\alpha^2 s+(\alpha^2+s^2)^{3/2}\}}.\sqrt{\frac{2\pi}{(\alpha^2+s^2)^3}}$$

choose $\alpha = 0.2$, the computer program gives approximate value of F(t), for different values of t. The following table shows comparision of exact and approximate values for different t. This formula also checked by numerical approximation of Laplace transform.

t	Exact value of F(t)	Approximate value of F(t)
0.0	0.0000	-0.0007
0.2	0.0179	0.0179
0.4	0.0505	0.0503
0.6	0.0927	0.0928
0.8	0.1425	0.1430
1.0	0.1987	0.1991
1.2	0.2604	0.2602
1.4	0.3270	0.3259
1.6	0.3979	0.3960
1.8	0.4726	0.4705
2.0	0.5507	0.5495
2.2	0.6318	0.6328
2.4	0.7154	0.7201
2.6	0.8012	0.8108
2.8	0.8888	0.9040
3.0	0.9780	-1.9485

3) Evaluate the integral $\frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} e^{st} \{ \frac{2s+3}{(s+1)^2} \} ds$

Analytical formula

$$\frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} e^{st} \cdot \frac{(\lambda s+\mu)}{(s+\alpha)^2} ds = [\lambda + (\mu-\alpha\lambda)t]\, e^{-\alpha t}$$

$$\frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} \frac{(2s+3)}{(s+1)^2} e^{st} ds = (2 + t)\, e^{-t}$$

for different values of t, the numerical approximation of given integral shown by following table.

t	Exact value of F(t)	Approximate value of F(t)
0.0	2.0000	2.0000
0.2	1.8012	1.8012
0.4	1.6088	1.6088
0.6	1.4269	1.4269
0.8	1.2581	1.2579
1.0	1.1036	1.1036
1.2	0.9638	0.9663
1.4	0.8384	0.8388
1.6	0.7268	0.7266
1.8	0.6281	0.6285
2.0	0.5413	0.5418
2.2	0.4654	0.4648
2.4	0.3991	0.3982
2.6	0.3416	0.3421
2.8	0.2919	0.2917
3.0	0.2489	0.2364

4) Evaluate the integral $\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{st} \cdot \frac{e^{-2s}}{s}\, ds$

Analytical formula

$$\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{st} \cdot \frac{e^{-\alpha s}}{s}\, ds = \begin{cases} 0, & 0 < t < \alpha \\ 1, & t > \alpha \end{cases}$$

$(\alpha > 0)$

for $\alpha = 2$

$$\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{st} \cdot \frac{e^{-2s}}{s}\, ds = \begin{cases} 0, & 0 < t < 2 \\ 1, & t > 2 \end{cases}$$

t	Exact value of F(t)	Approximate value of F(t)
0.0	0	0.0004
0.2	0	-0.0002
0.4	0	0.0007
0.6	0	0.0005
0.8	0	-0.0021
1.0	0	-0.0053
1.2	0	-0.0035
1.4	0	0.0127
1.6	0	0.0555
1.8	0	0.1386
2.0	0	0.2753
2.2	1	0.4758
2.4	1	0.7453
2.6	1	1.0810
2.8	1	1.4697
3.0	1	1.8853

2.12 APPLICATIONS

(1) This method is useful for any definite integral that is expressible as

$$\frac{1}{2\pi i}\int_{c-i\infty}^{r+i\infty} e^{st} \,.\, f(s)\,ds \quad \text{or} \quad \int_{0}^{\infty} e^{-st} \,.\, F(t)dt$$

(2) When f(s) (or F(t)) has complicated expression whose poles and residues are too difficult to obtain, then f(s) (or F(t)) might be itself known only numerically.

(3) We can use this method as a checkar for any definite integral that converted into Laplace or inverse Laplace transforms.

(4) By using computer program we can avoid laborious work of calculations and save the time.

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