C H A P T E R - III

CHAPTER - III

MELLIN TRANSFORM - COMPUTER IMPLEMENTATION

3.1 INTRODUCTION : DEFINITION

Another pair of formulae embodying the same formal idea is given by

$$G(s) = \int_{0}^{\infty} t^{s-1} g(t)dt$$

and,

$$g(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{-s} G(s)ds$$

The idea of such a reciprocity occurs in Riemann's famous memoir on prime number. It was formulated explicitly by Cahen and the first accurate discussion was given by Mellin.

This transform is closely related to the Fourier transform and has its own peculiar uses. In perticular, it turns out to be a most convenient tool for deriving expansions, although it has many other application. Recall first the Fourier transform pair can be written as

$$A(iw) = \int_{-\infty}^{\infty} a(x)e^{iwx}\, dx, \quad \alpha < I_m(w) < \beta, \quad \ldots. (3.1.1)$$

and,

$$a(x) = \frac{1}{2\pi i} \int_{ir-\infty}^{ir+\infty} A(iw)e^{-iwx}\, dw, \ \alpha < r < \beta, \ \ldots (3.1.2)$$

If we introduced the variable change

$$s = iw$$

$$t = e^{x}$$

$$(g(t) = a(\ln t)$$

So that (3.1.1) and (3.1.2) becomes

$$G(s) = \int_{0}^{\infty} t^{s-1} \cdot g(t)\, dt, \quad \alpha < \operatorname{Re}(s) < \beta \quad \ldots (3.1.3)$$

and,

$$g(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{-s} \cdot G(s)\, ds \quad \ldots. (3.1.4)$$

Equation (3.1.3) is the Mellin transform and (3.1.4) is the Mellin inversion formula. The transforms normally exists only in the strip $\alpha < \operatorname{Re}(s) < \beta$ and the inversion contour must lie in this strip.

3.2 NUMERICAL COMPUTATION OF INVERSION OF THE MELLIN TRANSFORMS

Let $A = \{Z \varepsilon C : \theta_1 \le \arg z \equiv \theta \le \theta_2, Z \neq 0\}$

and $S = \{S \varepsilon C : a < \operatorname{Re} S < b\}$

are the subsets of the complex plane $\mathbb{C}$.

We defined two classes of functions :

1. The class M of functions g(z), which are analytic in A and satisfy the following conditions

   $|g(z)| \le 1\, \varrho^{-a}$ for $\varrho \le 1$

$$|g(z)| \le l\ \varrho^{-b} \quad \text{for} \quad \varrho > 1$$

where l is a constant.

2. The class m of functions G(s), which are analytic in s. and satisfy the conditions.

$$|G(s)| \le l\ e^{-\theta_2 y}, \qquad \text{for} \quad y \ge 0$$

$$|G(s)| \le l\ e^{-\theta_1 y}, \qquad \text{for} \quad y < 0$$

where $s = x+iy$

Then the Mellin transform, defined by formulae

$$G(s) = \int_0^\infty z^{s-1}\, g(z)dz, \qquad S \epsilon C \qquad \ldots. (3.2.1)$$

and,

$$g(z) = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} z^{-s}\, G(s)\, ds,\ a < x < b \ \ldots. (3.2.2)$$

Given the function G(s), to find a function g(z) such that

$$G(s) = M\,\{g(z)\}$$

analytically in most cases difficult. Here we find numerical inversion of the Mellin transform based on the expansion of the original function g(z) in a series of Laguerre orthogonal functions and determination of the coefficients of this expansion by means of a collocation on the real axis of the transformed plane.

Consider the expansion of g(t) :

$$g(t) = e^{ct} \sum_{n=0}^{\infty} a_n f_n \left(\frac{t}{T}\right) \qquad \ldots . \ (3.2.3)$$

$$(\Phi(t) = \Phi(t + io) = \Phi(z), \quad t > 0)$$

where c and T are constants and

$$f_n \left(\frac{t}{T}\right) = e^{-t/2T} L_n\left(\frac{t}{T}\right) \qquad \ldots . \ (3.2.4)$$

and the Laguerre orthogonal functions, $L_n(t/T)$ denoting the Laguerre polynomial.

By truncating the series in (3.2.3) at n = N-1 and taking the Mellin transform of both sides,

$$G(s) \simeq \sum_{n=0}^{N-1} a_n F_n(s) \qquad \ldots . \ (3.2.5)$$

where

$$F_n(s) = M \{e^{-t(1/2T - c)} L_n (t/T)\}$$

$$= \frac{\Gamma(s+n)}{n!} \frac{(\alpha-\beta)^n}{\alpha^{s+n}} \, 2F_1 \left(-n, -s+1; -s-n+1; \frac{\alpha}{\alpha-\beta}\right)$$

where $\alpha = \frac{1}{2T} - c$, $\beta = \frac{1}{T}$, $2F_1$, denotes the hypergeometric function

If $x = \frac{\alpha}{\beta}$, then

$$\frac{\alpha}{\alpha-\beta} = \frac{x\beta}{x\beta-\beta} = \frac{x}{x-1}$$

and

$$F_n(s) = \frac{\Gamma(s+n)}{n!} \cdot \frac{1}{\alpha^s} \cdot \left(\frac{x-1}{x}\right)^n \cdot {}_2F_1\left(-n, -s+1; -s-n+1; \frac{x}{x-1}\right)$$

$$= \frac{\Gamma(s+n)}{n!} \cdot \frac{(-1)^n}{\alpha^s x^n} \cdot {}_2F_1(-n, -n; -s-n+1; x)$$

(According to the properties of the hypergeometric function and generalization of the Gamma function for negative argument)

For $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{4}$

$$\frac{1}{T} = \frac{1}{2} \quad \text{and} \quad c = \frac{1}{2T} - \frac{1}{2} = -\frac{1}{4}$$

$$F_n(s) = \frac{\Gamma(s+n)}{n!} \cdot (-1)^n 2^s \cdot {}_2F_1(-n, -n; -s-n+1; 1)$$

$$= (-1)^n \cdot 2^s \frac{\Gamma(s+n)}{n!} \frac{(-s+1)_n}{(-s-n+1)_n}$$

Hence

$$F_n(s) = (-1)^n 2^s \Gamma(s) \frac{(s-1)\ (s-2) \ldots (s-n)}{n!} \quad \ldots. \quad (3.2.6)$$

where $(a)_n = a(a+1) \ldots (a+n-1)$.

Now equations (3.2.5) and (3.2.6) gives

$$G(s) = 2^s \Gamma(s) \sum_{n=0}^{N-1} a_n (-1)^n \frac{(s-1) \ldots (s-n)}{n!}$$

$$= 2^s\Gamma(s) \sum_{n=1}^{N} a_{n-1}(-1)^{n-1} \frac{(s-1)\ldots[s-(n-1)]}{(n-1)!}$$

$$= 2^s\Gamma(s) \sum_{n=1}^{N} c_n(-1)^{n-1} \frac{(s-1)\ldots[s-(n-1)]}{(n-1)!} \quad ..(3.2.7)$$

where $c_n = a_{n-1}$

Introducing the notation

$$H_s \equiv H(s) = \frac{G(s)}{2^s\Gamma(s)} \quad .... (3.2.8)$$

Equation (3.2.7) becomes

$$H_s = c_1 + \sum_{n=2}^{N} c_n(-1)^{n-1} \frac{(s-1)\ldots[s-(n-1)]}{(n-1)!}$$

putting $s = 1,2, \ldots N$ successively gives

$$c_1 = H_1$$

$$c_2 = H_1 - H_2$$

$$c_s = (-1)^{s-1} \left[H_s - c_1 + \sum_{n=2}^{s-1} (-1)^n \binom{s-1}{n-1} c_n\right] \quad .... (3.2.9)$$

$$s = 3(1)N$$

Alternatively, from the recursive formula (3.2.9) one gives the formula

$$c_s = \sum_{n=1}^{s} (-1)^{n-1} \binom{s-1}{n-1} H_n, \quad s = 1(1)N \quad .... (3.2.10)$$

Hence (3.2.3) becomes

$$g(t) = \sum_{s=1}^{N} c_s e^{-t/2} L_{s-1}(t/2) \qquad \ldots\ldots (3.2.11)$$

the coefficients c_s given by either (3.2.9) or (3.2.10)

For computer implementation we use Laguerre recurrance relation

$$nL_n(x) = (2n-1-x) L_{n-1}(x) - (n-1) L_{n-2}(x)$$

$$L_o(x) = 1$$

and $L_1(x) = 1-x$

3.3 COMPUTER IMPLEMENTATION

The above method can be implemented in a computer as follows :

Step 1 : Input function G(n)

Step 2 : Calculate the constants C_s for s=1 to 30 by the formula,

$$c_s = \sum_{n=1}^{s} (-1)^{n-1} \binom{s-1}{n-1} \frac{G(n)}{2^n \Gamma(n)}$$

Step 3 : Input the values of T.

Step 4 : Calculate the inverse Mellin transform of given G(n), i.e. g(t) as

$$g(t) \cong \sum_{s=1}^{N} c_s e^{-t/2} L_{s-1}(t/2)$$

where, $L_o(x) = 1$,

$$L_1(x) = 1-x$$

and $$nL_n(x) = (2n-1-x)L_{n-1}(x) - (n-1)L_{n-2} \quad (\text{for } n \geq 2)$$

3.4 PROGRAM

```
Program Imtapprox (Input, Output)
{$N+} (*This  program gives numerical approximation  of  in-
     verse mellin transform*)
Uses crt;
Const
     PI=3.142;
     N=30;
Var
     fact:Array[0..N] of double;
     lag:Array[0..N] of double;
     C:Array[1..N] of double;
     Prod, sum, term, term1, Gsum:double;
     T:real;
     I,J,K,P,Q, sign:integer;
Begin
     (*constants*)
     prod:=1;fact[0]:=1;
     for Q:=1 to N DO
          Begin
            Prod:=prod*Q;
            Fact[Q]:=Prod
          End;
```

```
    For K:=1 to N DO
      Begin
        Sum:=0; Term:=0;Sign:=1;
        For J:=1 to K Do
          Begin
            Term:=Sign*Fact[K-1]/(Fact[J-1]*Fact[K-J]
            *exp(J*Ln(2)));
            Sum:=Sum + Term;
            Sign:= -Sign
         End;
        C[K]:=sum
      End;
(*Final Inverse Mellin Value PHI (T)*)
Clrscr;
Writeln;Writeln( '     ','Value of T', '     '),
'Value of PHI (T)');
Writeln;Writeln;
T:=0.0;
For I:=1 to 30 do
Begin
  T:=T+0.1;
  Lag[0]:=1;
  Lag[1]:=1-T/2;
  for K:=2 to N do
```

```
  Begin
     Lag[K]:=((2*K-1-T/2)*Lag[K-1]-(K-1)*Lag[K-2])/K
  End;
GSUM:=0;
For P:=1 to N DO
  Begin
     Term1:=C[P]*exp(-0.5*T)*Lag[P-1];
     Gsum:=Gsum+Term1
  End;
  Writeln( '     ',T:10:5, '      ',Gsum:10:5);
 End;
END.
```

3.5 NUMERICAL EXAMPLES

This method has been tested on known four functions, choses from Erdelyi et al (1954) and its behaviour on some of them is good, although problems of stability comes when the strip of existance s does not contain the interval [1,N] (The problem's stability can be obtained by using linear transformation).

The approximate formula (3.2.11) has been used with 30 terms and corresponding numerical results of test functions are tabulated in tables.

(1) $G(s) = \cos(\alpha s)\Gamma(s)$, with Re $s > 0$, $-\frac{1}{2}\pi < \text{Re}\,\alpha < \frac{1}{2}\pi$

Analytical formula

$$g(t) = \int_{c-i\infty}^{c+i\infty} t^{-s} \,.\, G(s)ds$$

$$= e^{-t\cos\alpha} \,.\, \cos\,(t.\sin\alpha)$$

for $\alpha = \dfrac{\pi}{4}$

t	Exact value of g(t)	Approximate value of g(t)
0.2	0.8595	0.85944
0.3	0.7907	0.79071
0.4	0.7237	0.72370
0.5	0.6588	0.65878
0.6	0.5962	0.59627
0.7	0.5364	0.53646
0.8	0.4795	0.47952
0.9	0.4256	0.42563
1.0	0.3749	0.37487
1.1	0.3273	0.32731
1.2	0.2830	0.28297
1.3	0.2419	0.24184
1.4	0.2039	0.20388
1.5	0.1691	0.16903
1.6	0.1372	0.13720
1.7	0.1083	0.10830
1.8	0.08223	0.08219
1.9	0.05880	0.05876
2.0	0.03791	0.03788
2.1	0.01943	0.01939
2.2	0.00320	0.00317
2.3	-0.01092	-0.01095
2.4	-0.02307	-0.02310
2.5	-0.03341	-0.03344
2.6	-0.04207	-0.04210
2.7	-0.04920	-0.04923
2.8	-0.05493	-0.05496
2.9	-0.05939	-0.05942
3.0	-0.06271	-0.06274

(2) $G(s) = \alpha^{-s}\Gamma(s)$, with Re $s > 0$, and $\alpha = 1$

Analytical formula

$$g(t) = \int_{c-i\infty}^{c+i\infty} t^{-s} \,.\, G(s)ds = e^{-\alpha t}, \quad \mathrm{Re}\,\alpha > 0$$

t	Exact value of g(t)	Approximate value of g(t)
0.10000	0.90484	0.90484
0.20000	0.81873	0.81873
0.30000	0.74082	0.74082
0.40000	0.67032	0.67032
0.50000	0.60653	0.60653
0.60000	0.54881	0.54881
0.70000	0.49659	0.49659
0.80000	0.44933	0.44933
0.90000	0.40657	0.40657
1.00000	0.36788	0.36788
1.10000	0.33287	0.33287
1.20000	0.30119	0.30119
1.30000	0.27253	0.27253
1.40000	0.24660	0.24660
1.50000	0.22313	0.22313
1.60000	0.20190	0.20190
1.70000	0.18268	0.18268
1.80000	0.16530	0.16530
1.90000	0.14957	0.14957
2.00000	0.13534	0.13534
2.10000	0.12246	0.12246
2.20000	0.11080	0.11080
2.30000	0.10026	0.10026
2.40000	0.09072	0.09072
2.50000	0.08208	0.08208
2.60000	0.07427	0.07427
2.70000	0.06721	0.06721
2.80000	0.06081	0.06081
2.90000	0.05502	0.05502
3.00000	0.04979	0.04979

(3) $G(s) = (s+\alpha)/[(s+\alpha)^2 + \beta^2]$ with $\mathrm{Re}\,(s+\alpha) > |I_{m\beta}|$

Analytical formula

$$g(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \bar{t}^{s}\, G(s)\, ds$$

$$= \begin{cases} t^{\alpha}\mathrm{Cos}(\beta \log t), & 0<t<1 \\ o, & 1<t<\infty \end{cases}$$

for $\alpha=3$, $\beta=2$

t	Exact value of g(t)	Approximate value of g(t)
0.1	-0.000107	-0.06012
0.2	-0.007976	-0.04279
0.3	-0.02005	0.02257
0.4	-0.01656	0.10041
0.5	+0.02293	0.17050
0.6	+0.11274	0.22288
0.7	+0.25937	0.25422
0.8	+0.46185	0.26525
0.9	+0.71288	0.26891
1.0		0.23916
1.1		0.21015
1.2		0.17576
1.3		0.13930
1.4		0.10343
1.5		0.07014
1.6		0.04075
1.7		0.01606
1.8		-0.00363
1.9		-0.01835
2.0		-0.02842
2.1		-0.03434
2.2		-0.03671
2.3		-0.03619
2.4		-0.03343
2.5		-0.02908
2.6		-0.02370
2.7		-0.01780
2.8		-0.01178
2.9		-0.00601
3.0		-0.00072

(4) $G(s) = \sin(\alpha s) \cdot \Gamma(s)$ with $\mathrm{Re}\, s > -1$

and $-\tfrac{1}{2}\pi < \mathrm{Re}\, \alpha < \tfrac{1}{2}\pi$

Analytical formula

$$g(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{-s} \cdot G(s)\, ds$$

$$= e^{-x\cos\alpha} \cdot \sin(x\sin\alpha)$$

for $\alpha = \dfrac{\pi}{4}$

t	Exact value of g(t)	Approximate value of g(t)
0.1	0.06583	0.06584
0.2	0.12236	0.12235
0.3	0.17030	0.17030
0.4	0.21033	0.21034
0.5	0.24312	0.24315
0.6	0.26932	0.26937
0.7	0.28956	0.28962
0.8	0.30443	0.30449
0.9	0.31450	0.31457
1.0	0.32032	0.32038
1.1	0.32238	0.32244
1.2	0.32117	0.32121
1.3	0.31712	0.31717
1.4	0.31065	0.31070
1.5	0.30214	0.30219
1.6	0.29194	0.29198
1.7	0.28037	0.28040
1.8	0.26771	0.26774
1.9	0.25422	0.25425
2.0	0.24014	0.24017
2.1	0.22568	0.22572
2.2	0.21103	0.21106
2.3	0.19634	0.19637
2.4	0.18176	0.18179
2.5	0.16741	0.16744
2.6	0.15339	0.15342
2.7	0.13980	0.13982
2.8	0.12669	0.12671
2.9	0.11413	0.11415
3.0	0.10216	0.10218

3.6 NUMERICAL COMPUTATION OF MELLIN TRANSFORM

While obtaining numerical computation of direct Mellin transform, many difficulties arise. After devoting long period of time we could not overcome these difficul ties. So we consider numerical computation of direct Mellin transform of functions of the following form.

$$\text{since} \quad G(s) = \int_0^\infty x^{s-1} \cdot F(x)dx$$

where $F(x)$ is of the form

$$F(x) = \begin{cases} g(x), & 0 < x < a \\ 0, & x > a \end{cases}$$

$$\therefore G(s) = \int_0^a x^{s-1} \cdot g(x)dx$$

$$= \int_0^\infty (ae^{-t})^{s-1} \cdot g(ae^{-t})(ae^{-t})dt, \quad x = ae^{-t}$$

$$= a^s \int_0^\infty e^{-st} \cdot g(ae^{-t})\, dt$$

$$\cong \frac{a^s}{s} \sum_{k=1}^{n} w_k \cdot g(ae^{-x_k/s}) \qquad \ldots. (3.6.1)$$

where x_k are the zeros of the n^{th} Laguerre polynomial and w_k are the corresponding weight function.

3.7 NUMERICAL EXAMPLES

The method has been tested on known functions, taken from Erdelyi et al. (1954). Using computer program (2.5) with two changes.

i. while entering the function replace $g(x)$ by $g(ae^{-x})$.

and ii. Lastly multiply the numerical approximation by a^s.

The numerical results of test functions so obtained are tabulated in tables 1,2,3 and 4.

(1) $g(x) = \begin{cases} x^v, & 0 < x < 1 \\ o, & x > 1 \end{cases}$

Analytical formula

$$G(s) = \int_0^{\infty} x^{s-1} . g(x)dx$$

$$= \int_0^{1} x^{s-1} . x^v dx = (s+v)^{-1}, \quad Re\ s > - Re\ v$$

for v = 3

s	Exact value of G(s)	Approximate value of G(s)
0.5	2.8571E-01	2.4172E-01
1.0	2.5000E-01	2.4699E-01
1.5	2.2222E-01	2.2192E-01
2.0	2.0000E-01	1.9996E-01
2.5	1.8182E-01	1.8181E-01
3.0	1.6667E-01	1.6667E-01
3.5	5.5385E-01	1.5385E-01
4.0	1.4286E-01	1.4286E-01
4.5	1.3333E-01	1.3333E-01
5.0	1.2500E-01	1.2500E-01
5.5	1.1765E-01	1.1765E-01
6.0	1.1111E-01	1.1111E-01
6.5	1.0526E-01	1.0526E-01
7.0	1.0000E-01	1.0000E-01
7.5	9.5238E-02	9.5238E-02
8.0	9.0909E-02	9.0909E-02
8.5	8.6957E-02	8.6957E-02
9.0	8.3333E-02	8.3333E-02
9.5	8.0000E-02	8.0000E-02
10.0	7.6923E-02	7.6923E-02

(2) $g(x) = \begin{cases} \cos(\alpha \log x), & 0 < x < 1 \\ 0, & 1 < x < \infty \end{cases}$

Analytical formula

$$G(s) = \int_0^\infty x^{s-1} \cdot g(x)dx$$

$$= \int_0^1 x^{s-1} \cdot (\cos(\alpha \log x))\, dx$$

$$= s(\alpha^2 + s^2)^{-1}, \quad \mathrm{Re}\ s > |I_m\ \alpha|$$

for $\alpha = 2$,

s	Exact value of G(s)	Approximate value of G(s)
1.0	0.2000	0.1922
1.5	0.2400	0.2402
2.0	0.2500	0.2500
2.5	0.2439	0.2439
3.0	0.2308	0.2308
3.5	0.2154	0.2154
4.0	0.2000	0.2000
4.5	0.1856	0.1856
5.0	0.1724	0.1724
5.5	0.1606	0.1606
6.0	0.1500	0.1500
6.5	0.1405	0.1405
7.0	0.1321	0.1321
7.5	0.1245	0.1245
8.0	0.1176	0.1176
8.5	0.1115	0.1115
9.0	0.1059	0.1059
9.5	0.1008	0.1008
10.0	0.0962	0.0962

(3) $g(x) = \begin{cases} \dfrac{x^{\alpha} - x^{\beta}}{\log x}, & 0 < x < 1 \\ 0, & 1 < x < \infty \end{cases}$

Analytical formula

$$G(s) = \int_0^{\infty} x^{s-1} \cdot g(x)dx = \int_0^{1} x^{s-1} \cdot \frac{x^{\alpha}-x^{\beta}}{\log x} dx$$

$$= \log \frac{s + \alpha}{s + \beta}, \quad \text{Re } s > -\text{Re } \alpha, -\text{Re } \beta.$$

for $\alpha = 2$, $\beta = 3$

s	Exact value of G(s)	Approximate value of G(s)
0.5	-3.3647E-01	-3.0647E-01
1.0	-2.8768E-01	-2.8618E-01
1.5	-2.5131E-01	-2.5119E-01
2.0	-2.2314E-01	-2.2313E-01
2.5	-2.0067E-01	-2.0067E-01
3.0	-1.8232E-01	-1.8232E-01
3.5	-1.6705E-01	-1.6705E-01
4.0	-1.5415E-01	-1.5415E-01
4.5	-1.4310E-01	-1.4310E-01
5.0	-1.3353E-01	-1.3353E-01
5.5	-1.2516E-01	-1.2516E-01
6.0	-1.1778E-01	-1.1778E-01
6.5	-1.1123E-01	-1.1123E-01
7.0	-1.0536E-01	-1.0536E-01
7.5	-1.0008E-01	-1.0008E-01
8.0	-9.5310E-02	-9.5310E-02
8.5	-9.0972E-02	-9.0972E-02
9.0	-8.7011E-02	-8.7011E-02
9.5	-8.3382E-02	-8.3382E-02
10.0	-8.0043E-02	-8.0043E-02

$$(4)\quad g(x) = \begin{cases} (1 + ax)^{-v}, & 0 < x < b \\ 0, & x > b \end{cases}$$

$$|\arg(1+ab)| < \pi$$

Analytical formula

$$G(s) = \int_0^\infty x^{s-1} \cdot g(x)dx = \int_0^b x^{s-1} \cdot (1+ax)^{-v}\, dx$$

$$= s^{-1} \cdot b^s {}_2F_1(v, s;\ 1+s;\ -ab)$$

$$\mathrm{Re}\ s > 0$$

for $b=2$, $a=1$, $v=2$

s	Approximate value of g(t)
0.5	1.43640
1.0	0.66621
1.5	0.48390
2.0	0.43195
2.5	0.43388
3.0	0.46944
3.5	0.53394
4.0	0.62917
4.5	0.76098
5.0	0.93888
5.5	1.17660
6.0	1.49306
6.5	1.91415
7.0	2.47499
7.5	3.22326
8.0	4.22362
8.5	5.56390
9.0	7.36348
9.5	9.78486
10.0	13.04942

REFERENCE

1. Erdelyi, A. et al.:"Tables of Integral Transform" , Vol. I, McGraw Hill Book Company, Inc., New York, 1954.
2. Francis Scheid : " Numerical Analysis", Mcgraw Hill International Book Company, Singapore.
3. P.S. Theocaris And A.C. Chrysakis : " Numerical Inversion of the Mellin Transform" J. Inst. Maths Applics 20 (1977) 73-83.