CHAPTER-3

NEWMAN- PENROSE TYPE FORMALISM FOR RIEMANNIAN V_2

1. Introduction:

In the 4-dimensional space time of General Theory of Relativity amongst all the formalisms, Newman Penrose (1962) formalism has been proved to be the most powerful formalism. The formalism is widely used right from its inception in many applications. Especially in the study of

1. exact solutions of Einstein's field equations Kramer, stephani, Herlt, and MacCallum (1980)
2. electromagnetism, Tariq and Tupper (1975, 76), Debney and Zund (1981)
3. the black holes, Hawking and Ellis (1973), Chandrashekhar (1983).

It is the language of many working relativists. Its exposition is available in the following books. Flaherty (1976), Carmeli (1977), Kramer et.al (1980), Chandrashekhar (1983). Many authors have exploited this technique in their research work. To mention few of them are : Zafar etal. (2001) have used it to obtain the Lanczos potential for perfect fluid space times. Ng. Ibohal Singh (2002, 2005) has shown by using the technique of Newman-Penrose formalism that every electrical radiation of the non –rotating black hole leads to a reduction in its mass by some quantity. If such reduction takes place continuously for a long time in the black hole body the original mass of the black hole may be evaporated completely. Katkar and Khairmode (2005, 2007) had it to prove that not-every non-empty non flat space time can be embedded locally and isometrically in a five dimensional space of non-zero constant curvature and also to study the existence of second rank Killing tensor in non-empty space times.

In this chapter an attempt is made to develop the Newman-Penrose type formalism and applied to study the geometry of 2-dimensional Riemannian space V_2. It is interesting to note that it works beautifully. The components of connection 1-form and curvature 2-forms are expressed in this formalism. It is shown that the curvature 2-form is exact satisfying

$\Omega^{\alpha}_{\beta} = d\omega^{\alpha}_{\beta}$. The detail exposion of the Newman Penrose type formalism is given in the following section and is applied to show that the Riemannian space V_2 has constant curvature. The commulator relation and the field equation in V_2 are derived in the section 3 in the form

$$\left(\bar{\delta}\delta - \delta\bar{\delta}\right)\phi = \kappa\bar{\delta}\phi + \bar{\kappa}\delta\phi,$$

$$\bar{\delta}\kappa + \delta\bar{\kappa} - 2\kappa\bar{\kappa} + \psi + \phi_{12} - \frac{R}{6} = 0$$

2. Newman-Penrose type formalism for 2-dimensional Riemannian space:

It is well-known that the metric of V_2 is given by

$$ds^2 = r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \tag{2.1}$$

where

$$g_{ij} = \begin{pmatrix} r^2 & 0 \\ 0 & r^2\sin^2\theta \end{pmatrix} \tag{2.2}$$

$$g = \left|g_{ij}\right| = r^4\sin^2\theta$$

and

$$g^{ij} = \begin{pmatrix} \frac{1}{r^2} & 0 \\ 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix} \tag{2.3}$$

Let us define a diode $e_{(\alpha)i}$

$$e_{(\alpha)i} = \left(m_i,\ \overline{m}_i\right) \tag{2.4}$$

where m_i and $\overline{m}_i$ are the complex conjugate of each other such that

$$m_i m^i = \overline{m}_i \overline{m}^i = 0 \ \ and \ \ m_i \overline{m}^i = 1 \tag{2.5}$$

Then the equation

$$g_{\alpha\beta} = g_{ik}\, e^i_{(\alpha)}\, e^k_{(\beta)}$$

gives

$$g_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.6}$$

Hence the vector of the dual diode becomes

$$e_i^{(\alpha)} = \left(\overline{m}_i \quad m_i\right) \tag{2.7}$$

Thus the relation between the metric tensor of the space and the null complex vectors of the diode is given by

$$g_{ik} = g_{\alpha\beta}\, e_i^{(\alpha)}\, e_k^{(\beta)}$$

This becomes

$$g_{ik} = m_i \overline{m}_k + \overline{m}_i m_k \qquad (2.8)$$

This gives $\quad g_{ik}\, g^{ik} = 2$

For the metric (2.1), define the basis 1-form θ^α as

$$\theta^1 = \frac{1}{\sqrt{2}}\left(r\, d\theta + ir \sin\theta\, d\phi\right)$$

and

$$\theta^2 = \frac{1}{\sqrt{2}}\left(r\, d\theta - ir \sin\theta\, d\phi\right) \qquad (2.9)$$

Hence the metric (2.1) reduces to the simple form

$$ds^2 = 2\theta^1 \theta^2 \qquad (2.10)$$

Now the equation

$$\theta^\alpha = e_i^{(\alpha)}\, dx^i \qquad \alpha = 1,2$$

gives

$$\theta^1 = \overline{m}_i\, dx^i \qquad (2.11)$$

and

$$\theta^2 = m_i\, dx^i \qquad (2.12)$$

Equations (2.9) to (2.12) give

$$m_i = \frac{1}{\sqrt{2}}\left(r, \quad -ir\sin\theta\right) \qquad (2.13)$$

$$\overline{m}_i = \frac{1}{\sqrt{2}}\left(r, \quad ir\sin\theta\right) \qquad (2.14)$$

Consequently the equation $m^i = g^{ik} m_k$ gives

$$m^i = \frac{1}{\sqrt{2}}\left(\frac{1}{r}, \quad \frac{-i}{r\sin\theta}\right) \qquad (2.15)$$

and

$$\overline{m}^i = \frac{1}{\sqrt{2}}\left(\frac{1}{r}, \quad \frac{i}{r\sin\theta}\right) \qquad (2.16)$$

From equations (2.13) to (2.16) we show that the null vectors satisfy the condition (2.5)

We start with Cartan's first equation of structure given by

$$d\theta^\alpha = -\omega^\alpha_\beta \wedge \theta^\beta \quad \alpha, \beta = 1,2.$$

Where'd' is the exterior derivative defined by

$$d = {}_{,i}\, dx^i$$
$$= {}_{,i}\, e^i_{(\alpha)}\, \theta^\alpha$$
$$= {}_{,i}\left(m^i\, \theta^1 + \overline{m}^i\, \theta^2 \right) \qquad (2.17)$$

or $$d = \delta\theta^1 + \overline{\delta}\,\theta^2 \qquad (2.18)$$

where $\delta = {}_{,i}\; m^i,\; \overline{\delta} = {}_{,i}\; \overline{m}^i$ *and* $\omega^\alpha_\beta = \gamma^\alpha_{\beta\gamma}\; \theta^\gamma$ (2.19)

In 2-dimensional space the connection 1-form has only one component and is either ω_{12} or ω_{21}.

Thus

$$\omega^1_{.1} = \eta^{1\alpha}\, \omega_{\alpha 1} \quad \alpha = 1,2.$$
$$\omega^1_1 = \omega_{21}$$
$$\Rightarrow \omega^1_1 = -\omega_{12}$$

and $$\omega^2_2 = \omega_{21}$$

Thus $$\omega^1_{.1} = -\omega_{12} = -\omega^2_2 \qquad (2.20)$$

and $$\omega^1_2 = \omega^2_1 = 0$$

Similarly, the Ricci's rotation coefficients $\gamma_{\alpha\beta\gamma}$ has only two independent components and are $\gamma_{121}\left(or\ \gamma_{211}\right)$ and $\gamma_{122}\left(or\ \gamma_{212}\right)$ and are given by

$$\gamma_{121} = -e_{(1)i,j}\; e^i_{(2)}\, e^j_{(1)}$$
$$\Rightarrow \gamma_{121} = -m_{i,j}\, \overline{m}^i\, m^j$$

Any tensor can be expressed as a linear combination of its basis vectors. Thus we express.

$$m_{i,j} = A m_i m_j + B\, m_i \overline{m}_j + C \overline{m}_i m_j + D\, \overline{m}_i\, \overline{m}_j$$

where the coefficients are defined as

$$A = m_{i;j}\,\overline{m}^{i}\,\overline{m}^{j} = \overline{\kappa}$$

$$B = m_{i,j}\,\overline{m}^{i} m^{j} = -\kappa \qquad (2.21)$$

$$C = m_{i,j}\, m^{i}\,\overline{m}^{j} = 0$$

$$D = m_{i,j}\, m^{i} m^{j} = 0$$

Thus

$$m_{i;j} = \overline{\kappa}\, m_i m_j - \kappa\, m_i \overline{m}_j \qquad (2.22)$$

Thus we have

$$\overline{m}_{i,j} = -\overline{\kappa}\,\overline{m}_i m_j + \kappa\,\overline{m}_i\,\overline{m}_j \qquad (2.23)$$

The intrinsic derivative of the tetrad vector in the direction of m^i and $\overline{m}^j$ are given by

$$m_{i,j}\, m^{j} = -\kappa\, m_i$$

$$\overline{m}_{i,j}\, m^{j} = \kappa\,\overline{m}_i$$

$$\overline{m}_{i,j}\,\overline{m}^{j} = -\overline{\kappa}\,\overline{m}_i$$

$$m_{i,j}\,\overline{m}^{j} = \overline{\kappa}\, m_i \qquad (2.24)$$

Thus the non-vanishing Ricci's rotation coefficients are given by

$$\gamma_{121} = \kappa$$

and

$$\gamma_{122} = -\overline{\kappa} \qquad (2.25)$$

Now taking the exterior derivative of basis 1-form defined in (2.9) we get

$$d\theta^{1} = \frac{1}{\sqrt{2}}\, i\, r\cos\theta\, d\theta \wedge d\phi$$

where from equation (2.9), we have

$$d\theta = \frac{1}{\sqrt{2}}\frac{1}{r}\left(\theta^{1} + \theta^{2}\right)$$

and

$$d\phi = \frac{-i}{\sqrt{2}}\frac{1}{r\sin\theta}\left(\theta^{1} - \theta^{2}\right)$$

$$\Rightarrow \qquad d\theta \wedge d\phi = \frac{i}{r^2 \sin\theta} \theta^1 \wedge \theta^2 \tag{2.26}$$

Thus
$$d\theta^1 = -\frac{1}{\sqrt{2}} \frac{\cot\theta}{r} \theta^1 \wedge \theta^2 \tag{2.27}$$

Similarly, we obtain

$$d\theta^2 = \frac{1}{\sqrt{2}} \frac{\cot\theta}{r} \theta^1 \wedge \theta^2 \tag{2.28}$$

Also from Cartan's first equation of structure, we find

$$d\theta^1 = -\omega^1_1 \wedge \theta^1 - \omega^1_2 \wedge \theta^2 \tag{2.29}$$

where
$$\omega_{12} = \gamma_{12\alpha} \theta^\alpha$$

$$\Rightarrow \omega_{12} = \gamma_{121} \theta^1 + \gamma_{122} \theta^2$$

$$\Rightarrow \omega_{12} = \kappa \theta^1 - \bar{\kappa} \theta^2 \tag{2.30}$$

Thus due to equation (2.30), equation (2.29) becomes

$$d\theta^1 = \left(\kappa \theta^1 - \bar{\kappa} \theta^2\right) \wedge \theta^1$$

$$\Rightarrow d\theta^1 = \bar{\kappa}\, \theta^1 \wedge \theta^2 \tag{2.31}$$

Similarly we get

$$d\theta^2 = -\omega^2_2 \wedge \theta^2$$

$$\Rightarrow d\theta^2 = -\kappa \theta^1 \wedge \theta^2 \tag{2.32}$$

Comparing equations (2.27), (2.28) and (2.31), (2.32), we readily get

$$\kappa = \bar{\kappa} = -\frac{1}{\sqrt{2}} \frac{\cot\theta}{r} \tag{2.33}$$

Now from Cartan's second equation of structure, we have

$$\Omega^\alpha_\beta = d\omega^\alpha_\beta + \omega^\alpha_\sigma \wedge \omega^\sigma_{\cdot\beta}, \qquad \alpha, \beta, \sigma = 1, 2.$$

From this, we obtain the non-vanishing components of curvature 2-form as

$$\Omega^1_{\cdot 1} = d\omega^1_{\cdot 1},$$

and
$$\Omega^2_2 = d\omega^2_{\cdot 2}.$$

This show that $\quad \Omega^\alpha_\beta = d\omega^\alpha_{\cdot\beta}, \qquad \forall \alpha, \beta = 1, 2.$

This proves that the curvature 2-form is exact (Frankel (1997)).

In this case we have

$$\Omega_{12} = d\omega_{12} = \kappa\,\theta^1 \wedge \theta^2 \tag{2.34}$$

where κ is the curvature of the space V_2.

Now

$$\Omega^1_1 = d\omega^1_1$$
$$\Rightarrow \Omega^1_1 = -d\left(\kappa\,\theta^1 - \bar{\kappa}\,\theta^2\right)$$
$$\Rightarrow \Omega^1_1 = -d\kappa \wedge \theta^1 - \kappa\, d\theta^1 + d\,\bar{\kappa} \wedge \theta^2 + \bar{\kappa}\, d\theta^2$$

Using equations (2.18), (2.31) and (2.32) we obtain

$$\Omega^1_1 = -\left(\delta\kappa\,\theta^1 + \bar{\delta}\kappa\,\theta^2\right) \wedge \theta^1 - \kappa\left(\bar{\kappa}\,\theta^1 \wedge \theta^2\right) +$$
$$+\left(\delta\bar{\kappa}\,\theta^1 + \bar{\delta}\bar{\kappa}\,\theta^2\right) \wedge \theta^2 + \bar{\kappa}\left(-\kappa\,\theta^1 \wedge \theta^2\right)$$
$$\Rightarrow \Omega^1_{.1} = \left(\bar{\delta}\kappa + \delta\bar{\kappa} - 2\kappa\bar{\kappa}\right)\theta^1 \wedge \theta^2$$

Similarly on using $\Omega^2_{.2} = d\omega^2_{.2}$ we obtain

$$\Omega^2_2 = -\left(\bar{\delta}\kappa + \delta\bar{\kappa} - 2\kappa\bar{\kappa}\right)\theta^1 \wedge \theta^2$$

Thus we have

$$\Omega^1_1 = -\,\Omega^2_2 = \left(\bar{\delta}\kappa + \delta\bar{\kappa} - 2\kappa\bar{\kappa}\right)\theta^1 \wedge \theta^2$$

Using (2.33) we get

$$\Omega^1_1 = -\,\Omega^2_2 = \left(\bar{\delta}\kappa + \delta\bar{\kappa} - \frac{\cot^2\theta}{r^2}\right)\theta^1 \wedge \theta^2 \tag{2.35}$$

Solving the right hand side, we get

$$\Omega^1_{.1} = -\,\Omega^2_2 = \left[\left(-\frac{1}{\sqrt{2}\,r}\cot\theta\right)_{,i} \bar{m}^i + \left(-\frac{1}{\sqrt{2}\,r}\cot\theta\right)_{,i} m^i - \frac{\cot^2\theta}{r^2}\right]\theta^1 \wedge \theta^2\,, \quad i = 1,2.$$

Using equations (2.15) and (2.16) we obtain

$$\Omega^1_{\ 1} = -\ \Omega^2_{\ 2} = \left[\frac{\cos ec^2\theta}{\sqrt{2}\,r}\frac{1}{\sqrt{2}\,r} + \frac{1}{\sqrt{2}\,r}\cos ec^2\theta\frac{1}{\sqrt{2}\,r} - \frac{1}{r^2}\cot^2\theta\right]\theta^1 \wedge \theta^2$$

$$\Rightarrow \Omega^1_{\ 1} = -\ \Omega^2_{\ 2} = \frac{1}{r^2}\theta^1 \wedge \theta^2 \qquad (2.36)$$

Thus from equations (2.34) and (2.36) we have the curvature of V_2 is given by

$$K = \frac{1}{r^2} \qquad (2.37)$$

Now to find the non-vanishing components of the curvature tensor of V_2, we have from the definition

$$\Omega^\alpha_{.\beta} = \frac{1}{2}R^\alpha_{\ \beta\gamma\delta}\,\theta^\gamma \wedge \theta^\delta \qquad (2.38)$$

$$\Rightarrow\ \Omega^\alpha_{\ \beta} = R^\alpha_{\ \beta 12}\,\theta^1 \wedge \theta^2$$

This gives

$$\Omega^1_{\ 1} = -\Omega^2_{.2} = R^1_{\ 112}\,\theta^1 \wedge \theta^2$$

Comparing the coefficients from (2.36) we obtain

$$R^1_{\ 112} = \frac{1}{r^2}$$

Consequently we get

$$R_{1212} = -\frac{1}{r^2} \qquad (2.39)$$

However the tetrad components of curvature tensor are given by

$$R^\alpha_{\ \beta\gamma\delta} = R^h_{\ ijk}\, e^{(\alpha)}_h\, e^i_{(\beta)}\, e^j_{(\gamma)}\, e^k_{(\delta)}$$

$$\Rightarrow R^1_{\ 112} = R^h_{\ ijk}\, e^{(1)}_h\, e^i_{(1)}\, e^j_{(1)}\, e^k_{(2)}$$

$$\Rightarrow \frac{1}{r^2} = R^h_{\ ijk}\,\overline{m}_h\, m^i\, m^j\, \overline{m}^k$$

$$\Rightarrow \frac{1}{r^2} = R_{hijk} \overline{m}^h m^i m^j \overline{m}^k$$

$$\Rightarrow \frac{1}{r^2} = R_{1212} \begin{pmatrix} \overline{m}^1 m^2 m^1 \overline{m}^2 - \overline{m}^1 m^2 m^2 \overline{m}^1 - \\ -\overline{m}^2 m^1 m^1 \overline{m}^2 + \overline{m}^2 m^1 m^2 \overline{m}^1 \end{pmatrix}$$

$$\Rightarrow \frac{1}{r^2} = R_{1212} \left(\frac{1}{r^4 \sin^2 \theta} \right)$$

or $$R_{1212} = r^2 \sin^2 \theta \tag{2.40}$$

is the tensor component of the curvature tensor.

Commutator Relation and Field Equations:

Let ϕ be a scalar invariant, we express tetrad components of covariant derivative of ϕ as

$$\phi_{,\alpha} = \phi_{,i}\, e^i_{(\alpha)} \tag{3.1}$$

From this we obtain

$$\phi_{,\alpha;\beta} = \left(\phi_{;i}\, e^i_{(\alpha)} \right)_{,j} \; e^j_{(\beta)}$$

$$\Rightarrow \phi_{,\alpha,\beta} = \phi_{,ij}\, e^i_{(\alpha)}\, e^j_{(\beta)} + \phi_{,i}\; e^i_{(\alpha),j}\, e^j_{(\beta)} \tag{3.2}$$

where from definition of Ricci's rotation coefficients, we have

$$e^i_{(\alpha),j}\, e^j_{(\gamma)} = -\gamma_{\alpha\beta\gamma}\, e^{(\beta)i}$$

Thus the above equation (3.2) becomes

$$\phi_{,\alpha,\beta} = \phi_{,ij}\, e^i_{(\alpha)}\, e^j_{(\beta)} - \phi_{,i}\; \gamma_{\alpha\sigma\beta}\; e^{(\sigma)i} \tag{3.3}$$

Interchanging α and β in (3.3) we get

$$\phi_{,\beta,\alpha} = \phi_{,ij}\, e^i_{(\beta)}\, e^j_{(\alpha)} - \phi_{,i}\; \gamma_{\beta\sigma\alpha}\; e^{(\sigma)i} \tag{3.4}$$

Subtracting equation (3.4) from (3.3) we get

$$\phi_{,\alpha,\beta} - \phi_{,\beta,\alpha} = -\phi_{,i}\; e^{(\sigma)i} \left(\gamma_{\alpha\sigma\beta} - \gamma_{\beta\sigma\alpha} \right) \tag{3.5}$$

$$\alpha,\, \beta,\, \sigma = 1, 2.$$

This gives

$$\phi_{,\alpha;\beta}-\phi_{,\beta,\alpha}=-\phi_{,i}\, e^{(1)i}\left(\gamma_{\alpha1\beta}-\gamma_{\beta1\alpha}\right)-\phi_{,i}\, e^{(2)i}\left(\gamma_{\alpha2\beta}-\gamma_{\beta2\alpha}\right)$$

$$\phi_{,\alpha;\beta}-\phi_{,\beta,\alpha}=-\phi_{,i}\,\overline{m}^{i}\left(\gamma_{\alpha1\beta}-\gamma_{\beta1\alpha}\right)-\phi_{;i}\, m^{i}\left(\gamma_{\alpha2\beta}-\gamma_{\beta2\alpha}\right)$$

$$\phi_{,\alpha,\beta}-\phi_{,\beta,\alpha}=-\overline{\delta}\phi\left(\gamma_{\alpha1\beta}-\gamma_{\beta1\alpha}\right)-\delta\phi\left(\gamma_{\alpha2\beta}-\gamma_{\beta2\alpha}\right) \tag{3.6}$$

Using (3.1) we have

$$\phi_{;1}=\phi_{,i}\, e^{i}_{(1)}$$
$$=\phi_{,i}\, m^{i}$$
$$\Rightarrow \phi_{,1}=\delta\phi \tag{3.7}$$

Similarly we have

$$\Rightarrow \phi_{,2}=\overline{\delta}\phi \tag{3.8}$$

Thus giving $\alpha,\ \beta=1,2.$ in (3.6) and using (3.7) and (3.8), we obtain

$$\left(\overline{\delta}\,\delta-\overline{\delta}\,\delta\right)\phi=-\overline{\delta}\phi\left(-\gamma_{211}\right)-\delta\phi\left(\gamma_{122}\right)$$

$$\Rightarrow\left(\overline{\delta}\,\delta-\overline{\delta}\,\delta\right)\phi=\kappa\,\overline{\delta}\phi+\overline{\kappa}\,\delta\phi \tag{3.9}$$

This is the commutator relation in V_2.

3. **Field equation:**

The tetrad components of the Weyl tensor are given by

$$R_{\alpha\beta\gamma\delta}=C_{\alpha\beta\gamma\delta}-\frac{1}{2}\left(g_{\alpha\gamma}R_{\beta\delta}+g_{\beta\delta}R_{\alpha\gamma}-g_{\beta\gamma}R_{\alpha\delta}-g_{\alpha\delta}R_{\beta\gamma}\right)+\frac{1}{6}\left(g_{\alpha\gamma}g_{\beta\delta}-g_{\beta\gamma}g_{\alpha\delta}\right) \tag{3.10}$$

where $R_{\alpha\beta}=R^{\gamma}_{\ \alpha\beta\gamma}$ denotes the tetrad components of the Ricci tensor and $R=g^{\alpha\beta}R_{\alpha\beta}$ the Ricci scalar curvature. However, in 2-dimensional Riemannian space V_2, equation (3.10) has only one component and is given by

$$R_{1212} = C_{1212} + R_{12} - \frac{R}{6} \tag{3.11}$$

where from equation (2.38), we have

$$\Omega_{\alpha\beta} = \frac{1}{2} R_{\alpha\beta\gamma\delta}\, \theta^{\gamma} \wedge \theta^{\delta}$$
$$\Rightarrow \Omega_{12} = R_{1212}\, \theta^{1} \wedge \theta^{2}$$

However, from equation (2.35), we have

$$-\left(\overline{\delta}\kappa + \delta\overline{\kappa} - 2\kappa\overline{\kappa}\right)\theta^{1} \wedge \theta^{2} = R_{1212}\, \theta^{1} \wedge \theta^{2}$$

This gives

$$R_{1212} = -\left(\overline{\delta}\kappa + \delta\overline{\kappa} - 2\kappa\overline{\kappa}\right) \tag{3.12}$$

Hence equation (3.11) becomes

$$-\left(\overline{\delta}\kappa + \delta\overline{\kappa} - 2\kappa\overline{\kappa}\right) = C_{1212} + R_{12} - \frac{R}{6}$$

Define

$$\begin{aligned}
\psi &= C_{1212} = C_{hijk} m^{h} \overline{m}^{i} m^{j} \overline{m}^{k} \\
\phi_{12} &= R_{12} = R_{ij}\, m^{i} \overline{m}^{j} \\
\phi_{11} &= R_{11} = R_{ij}\, m^{i} m^{j} \\
\phi_{22} &= R_{22} = R_{ij}\, \overline{m}^{i} \overline{m}^{j}
\end{aligned} \tag{3.13}$$

Hence we obtain the field equation

$$\overline{\delta}\kappa + \delta\overline{\kappa} - 2\kappa\overline{\kappa} + \psi + \phi_{12} - \frac{R}{6} = 0. \tag{3.14}$$