CHAPTER-1

INTRODUCTION AND MATHEMATICAL PRELIMINARIES

1. INTRODUCTION:

When it was realized that the Newton's mechanics is inadequate to explain the electromagnetic phenomenon, Albert Einstein published the theory of Special Relativity in 1905, which revolutionized in our understanding of the concept of space and time. This theory describes the motion of the fast moving particles and reduces to Newtonian mechanics when velocities involved are very small. However the theory is restricted in the sense that the gravitation is not incorporated in the theory of Special Relativity. The concept of gravitation was not compatible with the special theory of relativity; because in Special theory of Relativity no physical effects can propagate with velocity of light, where as in Newtonian Mechanics the gravitational effects throughout the space are function of its instantaneous position. In addition to this the Special theory of relativity does not resolve the conflict of gravitation with inertial observers which are so basic in the special theory of relativity. In order to resolve these conflicts Einstein proposed gravitation is universal, ever lasting and omnipresent. It can not be switched on and off at will and hence it is permanent to the space time region. This permanent characteristic of gravitation prompted Einstein to identify it with the non-Euclidean nature of space time geometry and proposed an astonishing result in 1915, Narlikar (1978).

Gravitation = space-time geometry

through the field equation $R_{ij} + \frac{1}{2} R g_{ij} = K T_{ij}$

The left hand side of this equation is referred as the geometry of the space time, while the right hand side is called as the dynamical part which represents the source of gravitation. Thus gravity is no more a force. It has been completely synthesized in to geometry of space time. In the history of mathematics this equation is described in the "Taj-Mahal of Science". Though the Einstein's theory of gravitation is referred as the most successful theory of gravitation, it however, does not explain the intrinsic spin of matter and cannot prevent singularities. However, these are different theories of gravitation besides Einstein theory of gravitation. Viz; Einstein-Cartan theory of gravitation also known as torsion theory of gravitation, Brans-Dicke theory of gravitation, Bergman-Wagoner theory of gravitation etc.

The natural generalizations of the Einstein theory of gravitation to space time with torsion is called the torsion theory of gravitation. The influence of the intrinsic spin of matter on the space time is considered in the Einstein –Cartan theory of gravitation. The theory is originated by Cartan (1923-24). He was the first man to introduce torsion in to gravitational theory. The theory is considered as a possible route to a unified theory of gravity and electromagnetism. The difference between the Einstein gravitation and torsion theory of gravitation is that the geometry of space time is not necessarily Riemannian and the mass and the spin are connected with geometry. As a matter of fact an affine connection compatible with the metric tensor is not necessarily symmetric in general. The torsion theory of gravitation reduces to the Einstein theory of gravitation when the torsion of the space time geometry vanishes.

Differential forms and exterior differential play a central role in the Einstein- Cartan theory and even in the modern literature. The use of differential forms can reduce the complexity of the computations. For example there are 40 Christoffel symbols to compute in tensor formulation, where as if we use Cartan's method of differential forms, there are only six connection forms to determine. In this chapter we briefly introduced the technique of differential forms and some mathematical preliminaries. No originality is claimed in this chapter. The material of this chapter can be obtained from the following books.

Choquet Bruhat (1982), B. F. Schutz (1980), F.de Felice and C.J.S. Clarke (1990) and S. Chandrashekhar (1983).

2. **TANGENT VECTOR**:

Let E^3 be a Euclidean space of 3-dimension. Let $f: E^3 \to \mathbb{R}$ be a real valued differentiable function on E^3. The set of all differentiable functions at P is denoted by $\mathcal{F}(p)$. Let $\gamma : I \to M \subset E^3$ be a smooth curve in E^3 and $p = \gamma(t_0)$ be a point on the curve γ. By a tangent vector to the curve γ at point P we mean a mapping V: $\mathcal{F}(p) \to \mathbb{R}$, defined by

$$V(f) = \frac{d}{dt} f(\gamma(t))|_{t=t_0} \tag{2.1}$$

This is the directional derivative of f at point P in the direction of the tangent to the curve γ at P. It can be shown that

$$V(f) = v^i \left(\frac{\partial}{\partial x^i} \right)_P (f)$$

As the function f is arbitrary, we write

$$V = v^i \left(\frac{\partial}{\partial x^i} \right)_P \tag{2.2}$$

where summation is defined over the repeated index i. The coefficient v^i is called 'tangent vector' to the curve γ at P. It has the following properties

$$(i) V(af + bg) = aV(f) + bV(g)$$
$$(ii) V(fg) = V(f) \cdot g(p) + f(p) \cdot V(g)$$

$$\forall a, b \in \mathbb{R} \ and \ f, g \in \mathcal{F}(p)$$

By virtue these properties a set of all vectors at $P \in M$ constitute on n-dimensional vector space called the tangent space of M at P and it is denoted by $T_P(M)$.

3. **TENSORS AND 1-FORMS:**

Let T_P be a n-dimensional tangent space. Unlike the n-dimensional vector space $\mathbb{R}^n$, the general n-dimensional tangent space has no basis prescribed. Any collection of n linearly independent vectors $\{\bar{e}_i\}$ form a basis for T_P. Thus any vector $\overline{X}$ of T_P is uniquely expressed as a linear combination of the basis.

That is $\overline{X} = X^k e_k$,

where X^k are called the component of $\overline{X}$ with respect to the basis $\{\bar{e}_k\}$. It is wel-known that every vector space has infinite number of bases, and all bases have same number of elements, if that number is finite. Let therefore $\{\bar{e}_i\}$ and $\{\bar{e}_{i'}\}$ be two bases of T_P. Since any vector of T_P can be expressed uniquely as a linear combination of vectors of the basis. Hence each vector of the set $\{\bar{e}_i\}$ is a linear combination of the set $\{\bar{e}_{i'}\}$ and vice-versa. Thus we

$$\bar{e}_i = \wedge_i^{j'} \bar{e}_{j'} \quad \forall\ i \text{ and } j' \tag{3.1}$$

and also

$$\bar{e}_{i'} = \wedge_{i'}^{j} \bar{e}_j \quad \forall\ i' \text{ and } j \tag{3.2}$$

where $\wedge_i^{j'}$ and $\wedge_{i'}^{j}$ form an n x n non-singular matrices of reals and are called the matrices of transformation such that

$$\wedge_i^{k'} \ \wedge_{k'}^{j} = \delta_i^j \tag{3.4}$$

where $\delta_i^j = \begin{cases} 1 & when\ i=j \\ 0 & when\ i \neq j \end{cases}$

is called a Kronecker delta symbol.

Any vector $\overline{X}$ of T_P satisfying the equation

$$X^i = \wedge_{k'}^{i} X^{k'} \qquad (3.4)$$

$$or\ \ X^{i'} = \wedge_{k}^{i'} X^k$$

is called a contravariant vector or a contravariant tensor of rank one.

Vector space of 1-forms (Cotangent space):

Definition: A 1-form $\tilde{\omega}$ on T_P is a real valued linear function. That is a linear transformation $\tilde{\omega}: T_P \to \mathbb{R}$ from T_P to the 1-dimensional vector space $\mathbb{R}$ such that $\tilde{\omega}(\overline{X}) = \langle \tilde{\omega}, \overline{X} \rangle$ for every $\overline{X} \in T_P$, is called 1-form. The linearity of 1-form function is

$$\tilde{\omega}(f\overline{X} + g\overline{Y}) = f\tilde{\omega}(\overline{X}) + g\tilde{\omega}(\overline{Y}) \quad \forall\ f, g \in \mathbb{R}$$

A collection of all 1-forms $\tilde{\omega}$ under the operations

$$(i)(\tilde{\omega} + \tilde{\sigma})(\overline{X}) = \tilde{\omega}(\overline{X}) + \tilde{\sigma}(\overline{X}) \quad \forall\ \overline{X} \in T_P$$

and $$(ii)(f\tilde{\omega})(\overline{X}) = f\tilde{\omega}(\overline{X}) \quad \forall\ f \in \mathbb{R}$$

forms a new vector space $\overset{*}{T}_P$ is called a cotangent space or dual space.

Bases in the Cotangent space:

In the vector space of 1-forms (cotangent space) $\overset{*}{T}_P$ any linearly independent 1-forms constitute a basis. One can choose any arbitrary basis in $\overset{*}{T}_P$. However, once a basis $\{\bar{e}_i\}$ has been chosen for the vector of T_P at P, this induces a preferred basis for $\overset{*}{T}_P$ called the dual basis $\{\tilde{\theta}^i\}$. The basis $\{\bar{e}_i\}$ determines 1-form basis $\{\tilde{\theta}^i\}$ by

$$\tilde{\theta}^i\left(\bar{e}_j\right)=\delta^i_j \tag{3.5}$$

Thus for any vector $\overline{X}\in T_P$, we have

$$\tilde{\theta}^i\left(\overline{X}\right)=X^i \tag{3.6}$$

This shows that the basis 1-forms $\tilde{\theta}^i$ map any vector $\overline{X}$ to its i^{th} component. Thus any arbitrary 1-form $\tilde{\omega}\in\overset{*}{T}_P$ is expressed as a linear combination of $\{\tilde{\theta}^i\}$ such that

$$\begin{aligned}\tilde{\omega}\left(\overline{X}\right)&=\tilde{\omega}\left(X^k\bar{e}_k\right)\\&=X^k\tilde{\omega}\left(\bar{e}_k\right)\end{aligned}$$

Define

$$\tilde{\omega}\left(\bar{e}_k\right)=\omega_k \tag{3.7}$$

Thus $\quad \tilde{\omega}\left(\overline{X}\right)=\omega_k\tilde{\theta}^k\left(\overline{X}\right)$ by using (3.6)

$$\Rightarrow\tilde{\omega}=\omega_k\tilde{\theta}^k \tag{3.8}$$

where ω_k are called the components of 1-form with respect to the basis $\left\{\tilde{\theta}^k\right\}$.

If ω_i and $\omega_{i'}$ are the components of 1-form $\tilde{\omega} \in \overset{*}{T}_P$ with respect to the basis $\left\{\tilde{\theta}^i\right\}$ and $\left\{\tilde{\theta}^{i'}\right\}$ respectively and satisfy the law

$$\omega_i = \wedge_i^{k'} \omega_{k'} \tag{3.9}$$

or $\omega_{i'} = \wedge_{i'}^{k} \omega_k$ is called covariant vector or a covariant tensor of rank one.

Tensor product of 1-forms:

Consider a space $\overset{*}{T}_P \otimes \overset{*}{T}_P = \otimes^2 \overset{*}{T}_P$ of n^2 dimensions. Let $\tilde{\omega}$ and $\tilde{\sigma}$ be 1-forms. Then the tensor product $\tilde{\omega} \otimes \tilde{\sigma}$ of 1-forms is defined as a bilinear real valued map on $T_P \times T_P$.

That is $\tilde{\omega} \otimes \tilde{\sigma} : T_P \times T_P \to \mathbb{R}$ defined by

$$\left(\tilde{\omega} \otimes \tilde{\sigma}\right)\left(\overline{X}, \overline{Y}\right) = \tilde{\omega}\left(\overline{X}\right) \cdot \tilde{\sigma}\left(\overline{Y}\right) \tag{3.10}$$

$$\forall \ \overline{X}, \overline{Y} \in T_P$$

Thus for $\overline{X} = X^i \ \overline{e}_i$ and $\overline{Y} = Y^k \ \overline{e}_k$, we have

$$\begin{aligned}\left(\tilde{\omega} \otimes \tilde{\sigma}\right)\left(\overline{X}, \overline{Y}\right) &= \tilde{\omega}\left(X^i \ \overline{e}_i\right) \cdot \tilde{\sigma}\left(Y^k \overline{e}_k\right) \\ &= X^i Y^k \tilde{\omega}\left(\overline{e}_i\right) \tilde{\sigma}\left(\overline{e}_k\right)\end{aligned}$$

On using equations (3.6) and (3.7), we obtain

$$\begin{aligned}\left(\tilde{\omega} \otimes \tilde{\sigma}\right)\left(\overline{X}, \overline{Y}\right) &= \omega_i \sigma_k \tilde{\theta}^i \left(\overline{X}\right) \tilde{\theta}^i \left(\overline{Y}\right) \\ &= \omega_i \sigma_k \left(\tilde{\theta}^i \otimes \tilde{\theta}^k\right)\left(\overline{X}, \ \overline{Y}\right) \qquad \textit{by } (3.10)\end{aligned}$$

Since $(\overline{X},\ \overline{Y})$ is an arbitrary pair of vectors of $T_P \times T_P$,

we have

$$\tilde{\omega}\otimes\tilde{\sigma} = \omega_i\sigma_k\ \tilde{\theta}^i\otimes\tilde{\theta}^k \tag{3.11}$$

$$or \quad \tilde{\omega}\otimes\tilde{\sigma} = T_{ik}\ \tilde{\theta}^i\otimes\tilde{\theta}^k \tag{3.12}$$

where $T_{ik} = \omega_i\sigma_k$ is a covariant tensor of rank two, and $\tilde{\theta}^i\otimes\tilde{\theta}^k$ is the basis for $\otimes^2 \overset{*}{T}_P$.

Let $\tilde{T} = \tilde{\omega}\otimes\tilde{\sigma}$, then $\tilde{T}$ is bilinear, we mean

$$\tilde{T}\left(f\overline{X}+g\overline{Y},\ h\overline{Z}+k\overline{V}\right) = f\,h\tilde{T}\left(\overline{X},\overline{Z}\right)+f\,k\tilde{T}\left(\overline{X},\overline{V}\right)+$$
$$+gh\tilde{T}\left(\overline{Y},\overline{Z}\right)+gk\tilde{T}\left(\overline{Y},\overline{V}\right)$$

The space of all bilinear maps $\tilde{T}$ equipped with two operations defined by $(\tilde{T}+\tilde{S})(\overline{X},\ \overline{Y}) = \tilde{T}(\overline{X},\ \overline{Y})+\tilde{S}(\overline{X},\ \overline{Y})$ and

$$(f\tilde{T})(\overline{X},\ \overline{Y}) = f\tilde{T}(\overline{X},\ \overline{Y}) \quad \forall\ f\in\mathbb{R}$$

is a vector space defined by $\overset{*}{T}_P\otimes\overset{*}{T}_P$ or $\otimes^2\overset{*}{T}_P$ of dimension n^2.

The elements of $\otimes^2\overset{*}{T}_P$ are called the covariant tensor of rank two.

4. p-form (Differential form of degree $p\geq 2$) and exterior Differentiation:

A p-form $p\geq 2$ is a completely skew-symmetric mapping defined as follows.

Let $\tilde{T}$ be a $p(\geq 2)$ form. It is defined as a completely skew-symmetric mapping from $T_P\times T_P\times T_P\times\cdots\times T_P$ (p-times) onto $\mathbb{R}$.

That is $\tilde{T}: T_p \times T_p \times T_p \times \cdots \times T_p \to \mathbb{R}$ such that

$$\tilde{T}\left(\overline{X}, \overline{Y}, \cdots, \overline{Z}\right) = \tilde{T}\left(X^i \, \overline{e}_i, \, Y^j \, \overline{e}_j, \cdots, \, Z^k \, \overline{e}_k\right)$$
$$= X^i Y^j \cdots Z^k \tilde{T}\left(\overline{e}_i, \, \overline{e}_j, \cdots, \overline{e}_k\right)$$

Define

$$\tilde{T}\left(\overline{e}_i, \, \overline{e}_j, \cdots, \overline{e}_k\right) = T_{ij\cdots k} \tag{4.1}$$

are called the components of $\tilde{T}$ with respect to the basis $\left\{\overline{e}_i, \overline{e}_j, \cdots, \overline{e}_k\right\}$.

Using equation (3.7), we obtain

$$\tilde{T}\left(\overline{X}, \overline{Y}, \cdots, \overline{Z}\right) = T_{ij\cdots k} \, \tilde{\theta}^i\left(\overline{X}\right) \cdot \tilde{\theta}^j\left(\overline{Y}\right) \cdots \tilde{\theta}^k\left(\overline{Z}\right)$$

The definition of tensor product of 1-forms leads to

$$\tilde{T}\left(\overline{X}, \overline{Y}, \cdots, \overline{Z}\right) = T_{ij\cdots k} \left(\tilde{\theta}^i \otimes \tilde{\theta}^j \otimes \cdots \otimes \tilde{\theta}^k\right)\left(\overline{X}, \overline{Y}, \cdots, \overline{Z}\right)$$

Since $\left(\overline{X}, \overline{Y}, \cdots, \overline{Z}\right)$ is arbitrary, we have

$$\tilde{T} = T_{ij\cdots k} \left(\tilde{\theta}^i \otimes \tilde{\theta}^j \otimes \cdots \otimes \tilde{\theta}^k\right) \tag{4.2}$$

We write this as

$$\tilde{T} = T_{ij\cdots k} \tilde{\theta}^{ij\cdots k} \tag{4.3}$$

where $\tilde{\theta}^{ij\cdots k} = \tilde{\theta}^i \otimes \tilde{\theta}^j \otimes \cdots \otimes \tilde{\theta}^k$ is n^p basis of $\otimes^p \overset{*}{T}_p$. Since p-form ($p \geq 2$) is completely skew-symmetric covariant tensor in each pair of indices. This implies that all n^p components of $T_{ij\cdots k}$ are not linear independent but has $\binom{n}{p}$ distinct components, where

$$\binom{n}{p} = \frac{n!}{p!(n-p)!}$$

Thus we see from equation (4.2) that n^p basis $\tilde{\theta}^{ij\cdots k}$ of $\otimes^p \overset{*}{T}_p$ will no longer be the basis for a space of p-form. Thus the basis for a space of p-form is given by$[\tilde{\theta}^i \otimes \tilde{\theta}^j \otimes \cdots \otimes \tilde{\theta}^k]$. Hence we have from equation (4.2) that

$$\tilde{T} = T_{ij\cdots k} [\tilde{\theta}^i \otimes \tilde{\theta}^j \otimes \cdots \otimes \tilde{\theta}^k] \tag{4.4}$$

where $T_{ij\cdots k}$ a completely anti-symmetric in each pair of indices are called the components of p-form or simply p-form or a form of degree p. Note that the set of all p-forms is a vector space and is denoted by $\wedge^p \overset{*}{T}_p$, where $\wedge^p \overset{*}{T}_p = \overset{*}{T}_p \wedge \overset{*}{T}_p \wedge \cdots \wedge \overset{*}{T}_p \subset \otimes^p \overset{*}{T}_p$

The dimension of $\wedge^p \overset{*}{T}_p = \binom{n}{p}$.

Remarks: (i) It is convenient to denote $\wedge^0 \overset{*}{T}_p = \mathbb{R}$ a space of 0-form or simply scalars.

(ii) $\wedge' \overset{*}{T}_p = \overset{*}{T}_p$ is simply a space of 1-form.

It can be seen that $\tilde{\omega}$ is any p-form and $\tilde{\sigma}$ is a r-form then their tensor product $\tilde{\omega} \otimes \tilde{\sigma}$ is a tensor of rank p + r, that is skew-symmetric in the first p and the last r indices, but need not be skew-symmetric in all p + r entries. Hence the tensor product of p-form $\tilde{\omega}$ and r-form $\tilde{\sigma}$ is not a p + r –form. Hence Grassman defined new product called the wedge product or exterior product of forms which is indeed a form.

5. Exterior product (wedge product):

Let $\wedge^p \overset{*}{T}_P$ and $\wedge^q \overset{*}{T}_P$ be spaces of p-form and q-form respectively.

If $\tilde{\omega} \in \wedge^p \overset{*}{T}_P$ is any p-form and $\tilde{\sigma} \in \wedge^q \overset{*}{T}_P$ is q-form, then their wedge product $\tilde{\omega} \otimes \tilde{\sigma}$ is a p + q form and is defined by

$$\wedge : \wedge^p \overset{*}{T}_P \times \wedge^q \overset{*}{T}_P \rightarrow \wedge^{p+q} \overset{*}{T}_P .$$

In particular, if $\tilde{\omega}, \tilde{\sigma} \in \overset{*}{T}_P$ are two 1-forms then their wedge product is defined by

$$\tilde{\omega} \wedge \tilde{\sigma} = \frac{1}{2!}\left(\tilde{\omega} \otimes \tilde{\sigma} - \tilde{\sigma} \otimes \tilde{\omega}\right) \tag{5.1}$$

where $\tilde{\omega} \otimes \tilde{\sigma}$ is the tensor product of $\tilde{\omega}$ and $\tilde{\sigma}$.

On using equation (3.11), we get

$$\tilde{\omega} \wedge \tilde{\sigma} = \frac{1}{2!}\omega_l \sigma_k \left(\tilde{\theta}^l \otimes \tilde{\theta}^k - \tilde{\theta}^k \otimes \tilde{\theta}^l\right)$$

$$or \quad \tilde{\omega} \wedge \tilde{\sigma} = \omega_l \sigma_k \tilde{\theta}^l \wedge \tilde{\theta}^k \tag{5.2}$$

where, from equation (5.1)

$$\tilde{\theta}^l \wedge \tilde{\theta}^k = \frac{1}{2!}\left(\tilde{\theta}^l \otimes \tilde{\theta}^k - \tilde{\theta}^k \otimes \tilde{\theta}^l\right) = {}^{[}\tilde{\theta}^l \otimes \tilde{\theta}^{k\,]} \tag{5.3}$$

This shows that

$$\tilde{\theta}^l \wedge \tilde{\theta}^k = -\tilde{\theta}^k \wedge \tilde{\theta}^l \tag{5.4}$$

Since the components of any $p(\geq 2)$ form are completely anti-symmetric in every pair of indices, hence $\omega_l \sigma_k$ are not the components of 2-form. To find the components of 2-form, interchange the indices l and k in equation (5.2), we get

$$\tilde{\omega} \wedge \tilde{\sigma} = \omega_k \sigma_l \tilde{\theta}^k \wedge \tilde{\theta}^l \tag{5.5}$$

Adding equations (5.2) and (5.5), we get

$$\tilde{\omega} \wedge \tilde{\sigma} = \omega_{[l} \sigma_{k]} \tilde{\theta}^l \wedge \tilde{\theta}^k \tag{5.6}$$

where

$$\omega_{[l} \sigma_{k]} = \frac{1}{2}(\omega_l \sigma_k - \omega_k \sigma_l) \tag{5.7}$$

are called the components of 2-form.

It can be proved that the wedge product satisfies the following properties.

1. Associative property: $\tilde{\omega} \wedge (\tilde{\sigma} \wedge \tilde{\rho}) = (\tilde{\omega} \wedge \tilde{\sigma}) \wedge \tilde{\rho}$.

2. Distributive property:

$$\tilde{\omega} \wedge (f\tilde{\sigma} + g\tilde{\rho}) = f(\tilde{\omega} \wedge \tilde{\sigma}) + g(\tilde{\omega} \wedge \tilde{\rho})$$

3. $f(\tilde{\omega} \wedge \tilde{\sigma}) = f\tilde{\omega} \wedge \tilde{\sigma} = \tilde{\omega} \wedge f\tilde{\sigma}$

4. $\tilde{\omega} \wedge \tilde{\omega} = 0$

5. $\tilde{\omega} \wedge \tilde{\sigma} = -\tilde{\sigma} \wedge \tilde{\omega}$ for any 1-form

6. In general if $\tilde{\omega}$ is any p-form and $\tilde{\sigma}$ is a q-form, then

$$\tilde{\omega} \wedge \tilde{\sigma} = (-1)^{pq} \tilde{\sigma} \wedge \tilde{\omega}$$

Exterior Derivative of p-form:

Exterior differentiation is effected by an operator d applied to p + 1- form obtained by taking the partial derivative or covariant derivative (it is immaterial which) of the associated p^{th} order tensor.

Mathematically, we define $d : \wedge^{p} \overset{*}{T}_{p} \rightarrow \wedge^{p+1} \overset{*}{T}_{p}$

It satisfies the following properties

1. $df = f_{,i}\, dx^{i}$

This shows for a 0-form f, df is a ordinary differentiation.

2. $d\left(a\tilde{\omega}+b\tilde{\sigma}\right)=a\,d\tilde{\omega}+b\,d\tilde{\sigma}$ for any $a,b\in\mathbb{R}$

3. $d\left(\tilde{\omega}\wedge\tilde{\sigma}\right)=d\tilde{\omega}\wedge\tilde{\sigma}+\left(-1\right)^{\deg\tilde{\omega}}\tilde{\omega}\wedge d\tilde{\sigma}$

4. $d\left(f\tilde{\omega}\right)=df\wedge\tilde{\omega}+f\,d\tilde{\omega}$ for $f\in\mathbb{R}$

5. $d\left(d\tilde{\omega}\right)=d^{2}\left(\tilde{\omega}\right)=0$ for any form $\tilde{\omega}$.

The most importantly the single exterior derivative operator d subsumes ordinary gradient, curl and divergence.

In local coordinate basis 1-form we mean

$$\tilde{\omega}=\omega_{i}\,dx^{i}.$$

2-form $\tilde{\omega}$ we mean the expression

$$\tilde{\omega}=\omega_{ik}\,dx^{i}\wedge dx^{k}$$

and in general only p-form we mean

$$\tilde{\omega}=\omega_{i_1 i_2 \cdots i_p}\,dx^{i_1}\wedge dx^{i_2}\wedge\cdots\wedge dx^{i_p}\ .$$