Chapter I

Chapter-I

Introduction

1.1 Preliminary Remark:-

The theory of Integral Transforms is a classical subject in mathematics whose literature can be traced back through at least 175 years. The classical theory of integral transforms of functions and their applications are too well known. Integral transforms of generalized functions are used in various problems of mathematical Physics and applied Mathematics. On the other hand, the theory of generalized functions is of recent origin, its advent being the publication of Laurent Schwartz's work, which appeared from 1944 onward, most notably of them is his two volume work, "Theorie des Distributions", published in 1950 and 1951. The concept of generalized integral transformation is confluence of these two mathematical streams. In this chapter we give a brief account of elementary concepts that are required for the development of the work of dissertation.

1.2 Integral transforms:-

Integral Transformation is widely used in pure and applied mathematics. They are used in solving some boundary value problems and integral equations. A function $F(s)$ is defined and denoted as in the form of integrals

$$F(s) = \int_{0 or -\infty}^{\infty} k(s,x) f(x) dx \text{-------}(1.2.1)$$

is called integral transform of function $f(x)$.

Where s-is real or complex.

$k(s,x)$ - Kernel of the transformation.

It is assumed that integral on R.H.S of (1.2.1) is convergent.

Different forms of kernel $k(s,x)$ and the range of integration, give rise to different integral transformations; such as Fourier, Laplace, Mellin, Hankel transformations.

The problems involving several variables can be solved by applying integral transformations successively with regards to several variables.

1.3 The Hankel Transformation:-

The conventional Hankel transformation is defined by

$$F(y) = h_\mu f = \int_0^\infty f(x)\sqrt{xy}\,J_\mu(xy)dx \text{ ---------(1.3.1)}$$

Where $0 < y < \infty$, μ is a real number, and J_μ is the Bessel function of first kind and order μ.

Inversion Theorem: (Watson [25]; p.456) If $f(x) \in L_1(0,\infty)$, if $f(x)$ is of bounded variation in a neighborhood of the point $x = x_0 > 0$, if $\mu \geq -\frac{1}{2}$, and if $F(y)$ is defined by (1.3.1),

then $\frac{1}{2}[f(x_0+0) + f(x_0-0)] = h_\mu^{-1}F = \int_0^\infty F(y)\sqrt{x_0 y}\,J_\mu(x_0 y)dy$ -------(1.3.2).

When $\mu \geq -\frac{1}{2}$, conventional inverse Hankel transformation h_μ^{-1} is defined precisely by the same formula as the direct Hankel transformation h_μ ; in symbols $h_\mu = h_\mu^{-1}$.

Theorem (1.3.2):- If $f(x)$ and G(y) are in $L_1(0,\infty)$, if $\mu \geq -\frac{1}{2}$, if $F(y) = h_\mu[f(x)]$, and if $g(x) = h_\mu^{-1}[G(y)]$,

Then $\int_0^\infty f(x)g(x)dx = \int_0^\infty F(y)G(y)dy$ ------------(1.3.3).

A series expansion for the Bessel function $J_\mu(z)$ of any order μ is

$$J_\mu(z) = \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{z}{2}\right)^{\mu+2k}}{k!\,\Gamma(\mu+k+1)}$$

1.4 Generalized functions and Distributions:-

P. Dirac [4] introduced delta function in 1947. The idea of specifying a function not by its value but by its behavior as a functional on some space of testing functions was a new concept. This new mode of thinking gave rise to the theory of generalized functions.

The impact of generalized functions on the integral transforms has recently revolutionized the theory of integral transformations. The foundations of the theory of generalized functions were laid by Bochner [1] and Sobolev [21]. But the work of Laurent Schwartz [20] was a systematic construction of theory of generalized functions.

Generalized functions:-

Let I be an open subset of R^n or C^n.

where R^n -Real n-dimensional Euclidean space.

C^n - Complex n-dimensional Euclidean space.

A set $V(I)$ is said to be a testing functions space on I if the following conditions are satisfied,

(i). $V(I)$ consists entirely of smooth complex valued functions defined on I.

(ii). $V(I)$ is either a complete countably multinormed space or a complete countable union space.

(iii). If sequence $\{\phi\}_{n=1}^{\infty}$ converges in $V(I)$ to zero then for then for every non negative integer k in R^n, $\{D^k\phi_n\}_{n=1}^{\infty}$ converges to the zero function uniformly on every compact subset of I.

A generalized function on I is any continuous linear functional on any testing function space on I.Thus f is called a generalized function, if it is a member of the dual space $V'(I)$ of some testing function space $V(I)$.

Distributions:

Let I be a non empty open set in R^n and K be a compact subset of I. $D_k(I)$ is the set of all complex valued smooth functions defined on I which vanish at those points of I, that are not in K. $D_k(I)$ is a linear space under the usual definitions of addition of functions and their multiplication by complex numbers. The Zero element in $D_k(I)$ is the identically zero function on I. For each non-negative integer k in R^n defined γ_k by

$$\gamma_k(\phi) = \sup_{t \in I} \left| D^k \phi(t) \right| \quad ; \phi \in D_k(I) \quad \text{-------------}(1.4.1)$$

then $\{\gamma_k\}$ is a countable multinorm on $D_k(I)$.

We assign to $D_k(I)$ the topology generated by $\{\gamma_k\}$ and thus $D_k(I)$ is a countably multinormed space. Moreover $D_k(I)$ is complete and hence a Frechet space.

Let $\{K_m\}_{m=1}^{\infty}$ be a sequence of compact subsets of I with properties,

i. $K_1 \subset K_2 \subset K_3 \subset -----$

ii. Each compact subsets of I is contained in one of the K_m,

then $I = \bigcup_{m=1}^{\infty} K_m$ and $D_{K_m}(I) \subset D_{K_{m+1}}(I)$ and topology of $D_{K_m}(I)$ is stronger than the topology induced on it by $D_{K_{m+1}}(I)$.

Now countable union space $D(I)$ is defined by

$$D(I) = \bigcup_{m=1}^{\infty} D_{K_m}(I) \text{--------}(1.4.2)$$

Its dual space is denoted by $D'(I)$. Members of $D'(I)$ are called distributions on I .Thus a distribution is a continuous linear functional on space $D(I)$.

Main advantage of generalized functions and distributions is that by widening the class of functions, many theorems and operations are freed from tedious restrictions. Generalized functions in mathematical physics are discussed by Vladimirov V. S. [24] .Gelfand I. M. and Shilov G. E. [7] is also available.

1.5 Generalized Integral Transformation:

The concept of the Generalized Integral transformation has been originated from the confluence of two mathematical disciplines, the theory of integral transformations and the theory of generalized functions. An important achievement of the theory of generalized functions published by Schwartz was the extension of Fourier transform to generalized functions, which led to a great process in the theory and applications of the generalized integral transformations in many directions and applied to various problems such as wave equation, electricity potential [26], heat condition, gas dynamics, heat diffusion and so on.

J. L. Lions [11] was the first, who extended Hankel transformation to generalized functions in such a way that an inversion formula could be stated for it. Recently Zemanian [26] extended various types of transforms to a certain class of generalized functions. Zemanian gave an alternative theory designed specifically for the Hankel transformation.

At present, the theory of generalized functions has numerous applications in physics, Mathematics and Engineering.

1.6 Generalization of Hankel Transformation:

A generalization of the Hankel transformation is different from that used for the Laplace and Mellin transformations. Here our definition is an indirect one based upon Parsevel's equation. On the other hand, in each of the previous cases constructed a testing function space, has been contain the kernel function, and then a transform is defined directly as the application of a generalized function to the kernel function. This latter approach doesn't always work now because the kernel function $\sqrt{xy}J_{\mu}(xy)$ as a function of x is not a member of H_{μ}, and therefore the equation

$$(h_{\mu}f)(y) = < f(x), \sqrt{xy}J_{\mu}(xy) >$$

does not possess a sense for every $f \in H_{\mu}^{'}$. However, under certain restriction on f, above equation will possess a sense and will agree with the definition

$$< h_{\mu}f, \phi > \triangleq < f, h_{\mu}\phi >$$

where $\mu \geq -\frac{1}{2}$, $f \in H_{\mu}^{'}$ and $\phi \in H_{\mu}$.

1.7 Some Definitions and Theorems:

Space of test functions:

The space of testing functions, which is denoted by $D(R)$, consists of all complex valued functions $\phi(t)$ that are infinitely smooth and zero out side some finite interval.

Theorem: A Local boundedness property of Distributions.

Every distribution f that is defined over some neighborhood of a fixed finite closed interval I in $R^{'}$ possesses the following boundedness property, there exist a nonnegative integer r and a constant C such that, for each ϕ in D whose support is in I,

$$\left| < f, \phi > \right| \leq C \sup_{t \in I} \left| \phi^{(r)}(t) \right|$$

C and r depend only on f and I.

Space S **:** The space of all complex valued function $\phi(t)$ that are infinitely smooth and are such that, as $|t| \to \infty$, They and all their partial derivatives decrease to zero faster than every power of $\frac{1}{|t|}$.

When t is one dimensional, every function $\phi(t)$ in S satisfies the infinite set of inequalities

$$\left|t^m \phi^{(k)}(t)\right| \leq C_{m,k} \quad , \qquad -\infty < t < \infty$$

Where m and k run through all nonnegative integers.

The elements of S are called testing functions of **rapid descent.**

Distribution of slow growth /Tempered distributions

A distributions f is said to be of slow growth if it is a continuous linear functional on the space S of testing functions of rapid descent.

1.8 Theory of Boehmians:

Zemanian A.H. [26] presents generalization of a number of commonly encountered integral transforms. His idea is to construct testing function spaces which contain appropriate kernels of integral transforms and to extend the classical theories of the corresponding testing function spaces. The theory thus developed is then applied to find generalized solution of partial differential equations.

The theory of Schwartz distributions was developed in order to give a solid mathematical foundation for generalizing the properties of Dirac δ

function introduced by British physicist P. A. M. Dirac in the late 1920's. Since Soboleff [21] and Schwartz [20] introduced the notation of distributions, there arise a number of theories of generalized functions. These theories differ from one another by generality, by application or by language, which is used to build them. One of the youngest generalizations of functions; and more particularly of Schwartz theory of distributions is Boehmians.

This idea of the construction of Boehmians was initiated by the concept of regular operators introduced by Boehme T.K [2]. Regular operators form a subalgebra of the field of Mikusinski operators and hence they include only such functions whose support is bounded from the left. Attempts were made to generalize the notion of regular operators in order to embrace all continuous functions and a general construction of Boehmians presented by Mikusinski [12]. In a concrete case the space of Boehmians contains all regular operators, all distributions and some objects which are neither operators nor distributions. An example of such a space is given in [12]. A concept of convergence of Boehmians was introduced and discussed in [13]. In the same paper the concrete space of Boehmians mentioned above is discussed in more detail. The space furnished with the introduced convergence appears to be a complete quasinormed space.

For every ring without zero divisors, there exists the corresponding field of quotients. The space C^{+} of all continuous functions on the real line R with supports bounded from the left forms a ring without zero divisors with respect to the convolution. The field of quotients for this space is known as the field of

Mikusinski operators [12].When replacing C^+ by the space C of all continuous functions, the construction of the field of quotients is impossible due to the presence of zero divisors in C. The construction of Boehmain is similar to the construction of the field of quotients and in some cases it gives just the field of quotients. On the other hand, the construction is possible where there are zero divisors, such as space C (with the operations of point wise addition and convolution).

In recent years the investigations in the Integral transforms for Boehmians have become an active and important part of the theory of generalized integral transforms. In the literature several integral transforms for various spaces of Boehmians are defined and their properties investigate [14], [16], [17].

The purpose of this talk is to give a general definition of Boehmians spaces and to explain how these spaces generalize the generalized functions spaces.

General Construction of Boehmian :

Let G be an additive commutative semigroup and S be a subset of G. We assume that to each pair of elements $\alpha \in G$ and $\delta \in S$ there is assigned an element of G denoted by $\alpha * \delta$. ($*$ is a map form $G \times S$ to G).

It is called the product of α and δ. We postulate that the product has the following properties:

I. If $\delta, \eta \in S$ then $\delta * \eta \in S$ and $\delta * \eta = \eta * \delta$.

II. If $\alpha \in G$ and $\delta, \eta \in S$ then $(\alpha * \delta) * \eta = \alpha * (\delta * \eta)$.

III. If $\alpha, \beta \in G$ and $\delta \in S$ then $((\alpha+\beta)*\delta = \alpha*\delta+\beta*\delta$.

Properties (I) and (II) imply that S is a multiplicative commutative semigroup.

Consider a family $\Delta \subset S^N$ satisfying the following conditions:

$\Delta 1$. If $\alpha, \beta \in G$, $\{\delta_n\} \in \Delta$ and $\alpha * \delta_n = \beta * \delta_n$, $\forall n \in N$ then $\alpha = \beta$ in G.

$\Delta 2$. If $\{\delta_n\}, \{\phi_n\} \in \Delta$ then $\{\delta_n * \phi_n\} \in \Delta$.

Elements of Δ are called **Delta Sequences.**

Consider the class "A" of pairs of sequences defined by

$$A = \{(\{f_n\}, \{\phi_n\}) : \{f_n\} \in G^n, \{\phi_n\} \in \Delta \}.$$

An element $(\{f_n\}, \{\phi_n\}) \in A$ is said to be quotient of sequences, denoted by f_n/ϕ_n if $f_i * \phi_j = f_j * \phi_i$, $\forall i, j \in N$.

We say that two quotients of sequences f_n/ϕ_n and g_n/φ_n are in relation if for each $n, m \in N$, we have $f_n * \varphi_m = g_m * \phi_n$. This relation is equivalence in A. Thus the family A splits into equivalence classes. The equivalence classes will be called Boehmians and the space of all Boehmians will be denoted by $B = B(G, \Delta)$.

Definition: The addition of two Boehmians is defined by

$$\left[f_n/\phi_n\right] + \left[g_n/\varphi_n\right] = \left[((f_n * \varphi_n) + (g_n * \phi_n))/(\phi_n * \varphi_n)\right]$$

and multiplication by a scalar can be defined in a natural way

$$\alpha\left[f_n/\phi_n\right] = \left[\alpha f_n/\phi_n\right] \quad, \alpha \in \mathbb{C}$$

The operation $*$ and the differentiation are defined by

$$\left[f_n \middle/ \phi_n \right] * \left[g_n \middle/ \varphi_n \right] = \left[(f_n * g_n) \middle/ (\phi_n * \varphi_n) \right]$$

and
$$D^{\alpha} \left[f_n \middle/ \phi_n \right] = \left[D^{\alpha} f_n \middle/ \phi_n \right]$$

In particular, if $\left[f_n \middle/ \phi_n \right] \in B$ and $\delta \in S$ is any fixed element then the product $*$ defined by $\left[f_n \middle/ \phi_n \right] * \delta = \left[(f_n * \delta) \middle/ \phi_n \right]$ is in $B(G, \Delta)$.

Remark:1 An element $f \in G$ can be identified as an element of B by the map

$$f \to \left[f * \delta_n \middle/ \delta_n \right]$$

here $\{\delta_n\} \in \Delta$. It is shown in [12] that the representation is independent of the delta sequences $\{\delta_n\}$ and the map gives an algebraic isomorphism of G into $B(G, \Delta)$.

Remark:2 G is also equipped with a notion of convergences. The intrinsic relation between the notion of convergence and the product $*$ is given by

i. If $f_n \to f$ as $n \to \infty$ in G and $\phi \in S$ is any fixed element then $f_n * \phi \to f * \phi$ as $n \to \infty$ in G.

ii. If $f_n \to f$ as $n \to \infty$ in G and $\{\delta_n\} \in \Delta$ then $f_n * \delta_n \to f$ as $n \to \infty$ in G.

(a). A sequence of Boehmians $\{x_n\}$ in B is said to be δ-convergence to a Boehmians $x \in B$, denoted by $x_n \xrightarrow{\delta} x$ if there exist a delta sequence $\{x_n\}$, such that , $\{x_n * \delta_k\}, \{x * \delta_k\} \in G \quad \forall k, n \in N$ and $\{x_n * \delta_k\} \to \{x * \delta_k\}$ as $n \to \infty$ in G, $\forall k, n \in N$.

From the condition (i) it is clear that $f_n \to f$ as $n \to \infty$ in G then $f_n \to f$ as $n \to \infty$ in B.

(b) A sequences of Boehmians $\{x_n\}$ in B is said to be Δ convergent to a Boehmians x in B, denoted by $x_n \xrightarrow{\Delta} x$, if there exist a delta sequences $\{\delta_n\} \in \Delta$, such that $(x_n - x) * \delta_n \in G$, $\forall n \in N$ and $(x_n - x) * \delta_n \to 0$ as $n \to \infty$ in G.

Support of a Boehmians:

Suppose U be an open set. A Boehmians $x \in B$ is said to vanish on U, if for each compact set $K \subseteq U$, there is a representative of f_n/ϕ_n of x, such that $f_n = 0$ on K for each $n \in N$.

Support of a Boehmians x is defined as the complement of the largest open set on which x vanishes.

Some Examples of Boehmians:

1. Let us now give an example of a Boehmian space in which the distributions D' can be imbedded. Take $G = C^\infty(R)$ equipped with the topology of uniform convergence on compact sets, $S = D(R)$ and Δ to be the class of sequences $\{\delta_n\}$ from D satisfying the conditions

(i) $\int\limits_{R^n} \phi_n(x)dx = 1$, $\forall n \in N$.

(ii) $\int\limits_{R} |\phi_n(x)| dx \leq M$ $\forall n$, for some $M > 0$.

(iii) $s(\phi_n) \to 0$ as $n \to \infty$ where $s(\phi) = \sup\{|x| : x \in R, \phi(x) \neq 0\}$

For $f \in G$, $\phi \in S$ we define the convolution $*$ as

$$(f * \phi)(x) = \int\limits_{R} f(x-t)\phi(t)dt.$$

Clearly $*$ defines an map from $G \times S$ to G and members of Δ satisfies the conditions ($\Delta 1$) and ($\Delta 2$). The Boehmians space thus obtained is denoted by $B = B(C^{\infty}(R), \Delta)$ and its members are called C^{∞}-Boehmians.

2. As a second example let us take G be the set of all locally integrable functions on R and identify two such functions whenever they are equal almost every where with respect with to the Lebesgue measure on R. The topology of this space is taken to be the semi norm topology generated by

$$P_n(f) = \int\limits_{-n}^{n} |f| d\lambda \qquad (\ n = 1,2,3,----)$$

Consider $S = D(R)$ and Δ to be class of sequences from D defined above. We get a corresponding Boehmians space $B = B(G, \Delta)$ called the space of Locally integrable Boehmians.