CHAPTER III

CHAPTER - III

GAUSSIAN QUADRATURE

3.1 INTRODUCTION

All Newton-Cotes integration formulae developed in the previous chapter have the following form.

$$\int_a^b f(x)dx \doteq \sum_{i=0}^{n} w_i f(x_i) \qquad \ldots(3.1)$$

where, the (n+1) values w_i are known as the weights. These weights are associated with (n+1) function values $f(x_i)$. Here the values of base points x_i are equally spaced, and thus we have no choice in the selection of the base points.

The Gaussian quadrature formulae are also of the same type but in Gaussian quadrature, these base points x_i are not fixed and also there is no any other restriction on their selection. Thus, in this case, we have (2n+2) undetermined parameters, namely (n+1) values of w_i and (n+1) values of x_i. For these (2n+2) parameters we can define a polynomial of (2n+1) degree. Further, we select the values of base points x_i so that the sum of the (n+1) appropriately weighted functional values given in equation (3.1) gives the exact value of the integral when f(x) is a polynomial of degree (2n+1) or less. For such determination we require some background knowledge of orthogonal polynomials. For a given weight w(x), it is possible to define a sequence of polynomials $P_0(x).P_1(x),\ldots,$ which are orthogonal and in which $P_n(x)$ is of exact degree n; i.e

$$(P_m,P_n) = \int_a^b w(x)P_m(x)P_n(x) = 0 , \qquad m \neq n$$

Further, by multiplying each $P_n(x)$ by an appropriate constant we can produce a set of polynomials P_n^* which are orthonormal, i.e.

$$\langle P_m^*,P_n^*\rangle = 0, \qquad \text{if } m \neq n$$
$$= 1, \qquad \text{if } m = n$$

Orthogonal polynomials are popular because of the property, that the zeros of real orthogonal polynomials are real, simple and located in the interior of [a,b].

Here, we state the theorem. [4, pp 74] Let $w(x)\geq 0$ and is defined on [a,b], with corresponding orthonormal polynomials $P_n^*(x)$. Let the zeros of $P_n^*(x)$ be $x_1,x_2,\ldots,x_n$ such that $a<x_1<x_2<\ldots<x_n<b$.

Then we can find positive constants $w_1,w_2,\ldots,w_n$ such that

$$\int_a^b w(x)P(x)dx = \sum_{k=1}^{n} w_k P(x_k) \qquad \ldots(3.2)$$

whenever, $P(x)$ is a polynomial of class p_{2n-1}. The weights w_k have the explicit representation,

$$w_k = -\frac{k_{n+1}}{k_n} \cdot \frac{1}{P_{n+1}^*(x_k)\, P_n^{*\prime}(x_k)} . \qquad \ldots(3.3)$$

If abscissas and weights are determinded as above then the resulting integration rule is said to be of GAUSS type. Here k_n is the coefficient of the n^{th} degree term in the orthonormal polynomial P_n^* and we take k_n as positive.

3.2 SOME SPECIAL ORTHOGONAL POLYNOMIALS

Just, we have seen that the orthogonal polynomials play an important role in Gaussian quadrature. There are some well known and very common families of orthogonal polynomials. Their intervals and corresponding weight functions are as follows:

3.2.1 Legendre Polynomials

Let $P_n(x)$ denote the Legendre polynomial. These polynomials are orthogonal on the interval [-1,1] with respect to the weight function w(x) = 1. Therefore, we have

$$\int_{-1}^{+1} P_n(x)P_m(x)dx = \emptyset, \qquad \text{if } n \neq m$$

$$= c(n) \neq \emptyset, \text{ if } n = m \qquad \ldots(3.4)$$

These polynomials can be obtained from the recursive relation,

$$P_n(x) = \frac{2n-1}{n} xP_{n-1}(x) - \frac{n-1}{n} P_{n-2}(x) \qquad (n \geq 2)$$

$$\text{with } P_0(x) = 1, \; P_1(x) = x. \qquad \ldots(3.5)$$

3.2.2 Chebyshev Polynomials

Let us denote these polynomials by $T_n(x)$. These polynomials, are orthogonal on the interval [-1,1] with respect to weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$. Thus, we have

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} T_n(x)T_m(x)dx = \begin{cases} \emptyset, & n \neq m \\ c(n) \neq \emptyset, & n = m. \end{cases} \quad \ldots(3.6)$$

These polynomials have recursion relation given by

$$T_n(x) = 2x\, T_{n-1}(x) - T_{n-2}(x), \qquad (n \geq 2)$$

$$\text{and} \quad T_0(x) = 1, \quad T_1(x) = x\,. \qquad \ldots(3.7)$$

3.2.3 Hermite polynomials

Let us denote thses polynomials by $H_n(x)$. These are orthogonal on the interval $(-\infty,\infty)$ with respect to the weight function e^{-x^2}. Thus, we have

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x)H_m(x)dx = \begin{cases} \emptyset, & n \neq m \\ c_n(x) \neq \emptyset, & n = m. \end{cases} \qquad \ldots(3.8)$$

The recursion relation for these polynomials is given by

$$H_n(x) = 2x\, H_{n-1}(x) - 2(n-1)H_{n-2}(x), \qquad (n \geq 2).$$

$$\text{with} \quad H_0(x) = 1, \quad H_1(x) = 2x, \qquad \ldots(3.9)$$

3.2.4 Laguerre Polynomials

Let us denote these polynomials by $L_n(x)$.

These polynomials are orthogonal on the interval $[0,\infty)$ with weight function $w(x) = e^{-x}$. Thus, we have

$$\int_0^\infty e^{-x} L_n(x) L_m(x)\,dx = \begin{cases} 0 & n \neq m \\ c_n(x) \neq 0; & n = m. \end{cases} \qquad \ldots(3.10)$$

These polynomials are recursively obtained from the relations,

$$L_n(x) = (2n-x-1)\, L_{n-1}(x) - (n-1)^2 L_{n-2}(x), \quad (n \geq 2)$$

$$\text{with} \quad L_0(x) = 1, \qquad L_1(x) = -x + 1, \qquad \ldots(3.11)$$

Remark:

Any n^{th} degree arbitrary polynomial $P_n(x) = \sum_{i=0}^{n} \alpha_i x^i$ can be written in terms of a linear function of any of the above families of orthogonal polynomials. Further, any family of orthogonal polynomials, satisfying its respective recursion relation, is unique. In these cases, the Gauss formula is named after the name of the orthogonal polynomial.e.g Gauss-Legendre formula, Gauss-Laguerre formula, etc.

3.3 GAUSS-LEGENDRE QUADRATURE

In this case, we find the value of the integral $\int_a^b f(x)dx$ by approximating the function f(x) with n^{th} degree interpolating Lagrangian polynomial $P_n(x)$ with corresponding error term $R_n(x)$. Thus, we get

$$\int_a^b f(x)dx = \int_a^b P_n(x)dx + \int_a^b R_n(x)dx. \qquad \ldots(3.12)$$

As the base points x_i are unknown, we use the interpolating polynomial in that Lagrangian form where arbitrarily spaced base points are allowed, viz.

$$P_n(x) = \sum_{i=0}^{n} L_i(x)\ f(x_i), \qquad \ldots(3.13)$$

$$\text{where } L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{(x - x_j)}{(x_i - x_j)}, \qquad i = 0,1,2,\ldots,n.$$

with its corresponding error term given by,

$$R_n(x) = \left[\prod_{i=0}^{n} (x-x_i)\right] \frac{f^{(n+1)}(\xi)}{(n+1)!}, \qquad \text{where } a < \xi < b. \qquad \ldots(3.14)$$

Therefore, we can write,

$$f(x) = \sum_{i=0}^{n} L_i(x)f(x_i) + \left[\prod_{i=0}^{n} (x-x_i)\right]\frac{f^{(n+1)}(\xi)}{(n+1)!}, \qquad \text{where } a < \xi < b \qquad \ldots(3.15)$$

Also, for the sake of convenience, without loss of generality, we change the interval of integration to [-1.1] by a suitable transformation,

$$z = \frac{2x - (a+b)}{b-a} \qquad \ldots(3.16)$$

where, z is a new variable and $-1 \le z \le 1$.

Therefore, $x = \dfrac{(b-a)z + (a+b)}{2}$ $\qquad \ldots(3.17)$

Let us assume that all the base points are in the interval [a,b]. i.e. $a \le x_0, x_1, \ldots, x_n \le b$. Also, we define a new function F(z) so that,

$$F(z) = f(x) = f\left(\frac{(b-a)z + (a+b)}{2}\right) \qquad \ldots(3.18)$$

With this convension, equation (3.15) becomes

$$F(z) = \sum_{i=0}^{n} L_i(z)F(z_i) + \left[\prod_{i=0}^{n}(z-z_i)\right]\frac{F^{(n+1)}(\xi)}{(n+1)!} \qquad \ldots(3.19)$$

where $L_i(z) = \prod_{\substack{j=0\\ j\neq i}}^{n} \dfrac{(z - z_j)}{(z_i - z_j)}$, and $\xi \in (-1, 1)$, $i = 0,1,2,\ldots,n$.

Here the base point value x_i is simply transformed to z_i. Obviously, $\sum_{i=0}^{n} L_i(z)F(z_i)$ is a polynomial of at most degree n, and $\prod_{i=0}^{n}(z-z_i)$ is a polynomial of degree (n+1). Therefore, if f(x) is assumed as a polynomial of degree (2n+1) then

$F^{(n+1)}(\xi)/(n+1)!$ must be a polynomial of degree n, say $q_n(z)$, i.e. let

$$q_n(z) = \frac{F^{(n+1)}(\xi)}{(n+1)!}$$

Then equation (3.19) becomes

$$F(z) = \sum_{i=0}^{n} L_i(z)F(z_i) + \left[\prod_{i=0}^{n}(z-z_i)\right]q_n(z) \qquad \ldots(3.20)$$

On integrating from -1 to 1, the above equation, we get

$$\int_{-1}^{1} F(z)dz = \int_{-1}^{1} \sum_{i=0}^{n} L_i(z)F(z_i)dz + \int_{-1}^{1} \left[\prod_{i=0}^{n}(z-z_i)\right]q_n(z)dz \qquad \ldots(3.21)$$

As $F(z_i)$ are fixed values, we can take summation operator outside the integral sign. Also, neglecting the rightmost integral, above equation becomes

$$\int_{-1}^{1} F(z)dz \doteq \sum_{i=0}^{n} F(z_i) \int_{-1}^{1} L_i(z)dz$$

$$\text{i.e} \quad \int_{-1}^{1} F(z)dz \doteq \sum_{i=0}^{n} w_i F(z_i), \qquad \ldots(3.22)$$

$$\text{where,} \quad w_i = \int_{-1}^{1} L_i(z)dz, \quad i = 0,1,\ldots,n. \qquad \ldots(3.23)$$

Actually, the equation (3.22) is in the required form (3.1) whereas the error term for this quadrature formula is the second integral on the right hand side of (3.21). Now, our goal is to select the values z_i in such a way that this error term,

$$\int_{-1}^{1}\left[\prod_{i=0}^{n}(z-z_i)\right] q_n(z)dz, \qquad \ldots(3.24)$$

vanishes. For this purpose, we make the use of the orthogonality property of the Legendre polynomials as follows:

First, we write the polynomials $q_n(z)$ and $\prod_{i=0}^{n}(z-z_i)$ in terms of the Legendre polynomials. Therefore, let

$$\prod_{i=0}^{n}(z-z_i) = b_0p_0(z) + b_1p_1(z) + \ldots + b_np_n(z) + b_{n+1}p_{n+1}(z)$$
$$= \sum_{i=0}^{n+1} b_ip_i(z). \qquad \ldots(3.25)$$

and $$q_n(z) = c_0p_0(z) + c_1p_1(z) + \ldots + c_np_n(z)$$
$$= \sum_{i=0}^{n} c_i p_i(z) \qquad \ldots(3.26)$$

Next, using equations (3.25) and (3.26) in (3.24), we get

$$\int_{-1}^{1}\left[\prod_{i=0}^{n}(z-z_i)\right]q_n(z)dz = \int_{-1}^{1}\left[\sum_{i=0}^{n}\sum_{j=0}^{n} b_ic_j \; p_i(z) \; p_j(z) + b_{n+1}\sum_{i=0}^{n} c_ip_i(z)p_{n+1}(z)\right]dz. \qquad \ldots(3.27)$$

Now, by orthogonality properties, all terms of this integral that are of the form

$$b_ic_j \int_{-1}^{1} p_i(z)p_j(z)dz, \qquad (i \neq j)$$

will vanish. Therefore, the equation (3.27) reduces to

$$\int_{-1}^{1}\left[\prod_{i=0}^{n}(z-z_i)\right]q_n(z)dz = \int_{-1}^{1}\sum_{i=0}^{n} b_i c_i[p_i(z)]^2 \, dz.$$

$$= \sum_{i=0}^{n} b_i c_i \int_{-1}^{1}[p_i(z)]^2 dz. \qquad \ldots(3.28)$$

To get this expression vanish, let us specify that the

$$b_i = 0, \qquad i = 0,1,\ldots,n.$$

However, the coefficient b_{n+1} of $p_{n+1}(z)$ remains still unspecified, but from equation (3.25), using $b_i = 0$ for $i = 0,1,\ldots,n$, we get

$$\prod_{i=0}^{n}(z-z_i) = b_{n+1}p_{n+1}(z). \qquad \ldots(3.29)$$

The important feature of above equation is that, the polynomial $\prod_{i=0}^{n}(z-z_i)$ is already in factored form, i.e. it has the (n+1) roots z_i, $i = 0,1,\ldots,n$. Further, these roots z_i must be the roots of $b_{n+1}p_{n+1}(z)$ because of equation (3.29). That is, these z_i are the roots of $p_{n+1}(z)$. Thus, the (n+1) base points, which are to be used so that the integration formula (3.22) becomes exact, are the (n+1) roots of the Legendre polynomial of degree (n+1). Also, the corresponding weights assigned to each function value $F(z_i)$ are given by (3.23). Base points and corresponding weights for 2,3,4,5,6,7,8,10,12 point formulae are given in the Appendix

Remark: In the evaluation of $\int_a^b f(x)dx$, where a and b are finite arbitrary, it is convenient to transform the Gauss-Legendre quadrature formula from the standard interval $-1 \le z \le 1$ to the desired interval $a \le x \le b$, using equation (3.17). Then, we get

$$\int_a^b f(x)dx = \frac{(b-a)}{2} \int_{-1}^{1} f\left(\frac{z(b-a)+b+a}{2}\right)dz. \qquad \ldots(3.30)$$

Since the Gauss-Legendre quadrature formula is given by

$$\int_{-1}^{1} F(z)dz \doteq \sum_{i=0}^{n} w_i F(z_i),$$

therefore, the equation (3.30) becomes

$$\int_a^b f(x)dx = \frac{(b-a)}{2} \sum_{i=0}^{n} w_i f\left(\frac{z_i(b-a)+b+a}{2}\right). \qquad \ldots(3.31)$$

The above equation gives the general formulation of the Gauss- Legendre quadrature. It is much suitable for computer because instead of symbolic transformation of f(x), base points z_i are transformed and weight factors w_i are modified by the constant (b-a)/2.

3.4 GAUSS-LAGUERRE QUADRATURE

This formula can be derived in the similar manner to that of Gauss Legendre quadrature. As, base points z_i are unknown, we use the interpolating polynomial (3.13) and error term given by (3.14). Thus, we have

$$F(z) = \sum_{i=0}^{n} L_i(z)\ F(z_i) + \left[\prod_{i=0}^{n}(z-z_i)\right]\frac{F^{(n+1)}(\xi)}{(n+1)!},$$

$$\emptyset < \xi < \infty \qquad \ldots(3.32)$$

where, $L_i(z) = \prod_{\substack{j=0 \\ j\neq i}}^{n}\left(\frac{z - z_j}{z_i - z_j}\right), \qquad i = \emptyset,1,\ldots,n.$

Obviously, if $F(z)$ is a polynomial of degree $(2n+1)$ then $F^{(n+1)}(\xi)/(n+1)!$ must be a polynomial of degree n, say $q_n(z)$. Thus, we can rewrite the above equation as

$$F(z) = \sum_{i=0}^{n} L_i(z)\ F(z_i) + \left[\prod_{i=0}^{n}(z-z_i)\right] q_n(z). \qquad \ldots(3.33)$$

As the weight function in this case is e^{-z}, we multiply the both sides of equation (3.33) by e^{-z} and then integrating within the limits $\emptyset$ to ∞, we get

$$\int_0^\infty e^{-z}F(z)dz = \sum_{i=0}^{n} F(z_i) \int_0^\infty e^{-z}L_i(z)dz$$
$$+ \int_0^\infty e^{-z}\left[\prod_{i=0}^{n}(z-z_i)\right] q_n(z)dz. \qquad \ldots(3.34)$$

or equivalently,

$$\int_0^\infty e^{-z}F(z)dz \doteq \sum_{i=0}^{n} w_i F(z_i) \qquad \ldots(3.35)$$

where $w_i = \int_0^\infty e^{-z}L_i(z)dz. \qquad \ldots(3.36)$

The equation (3.35) is in the desired form of equation (3.1) where the second term of right hand side of equation (3.34) gives the error term of the quadrature formula. We now select the values of z_i such that this error term get vanished by using orthogonality property of Laguerre polynomials. First we write $q_n(z)$ in terms of the Laguerre polynomials of degree n or less. Now if $\prod_{i=0}^{n}(z-z_i)$ is $(-1)^{n+1}L_{n+1}(z)$, the error term vanishes; and the roots of the Laguerre polynomial $L_{n+1}(z)$, will be the required base points z_i for this (n+1) point Gauss-Laguerre quadrature formula. Also from equation (3.36), we get the corresponding weight factors.

The error for this quadrature is given by

$$E_n = \frac{[(n+1)!]^2}{(2n+2)!} F^{(2n+2)}(\xi), \qquad \xi \in (0,\infty) \qquad \ldots(3.37)$$

<u>**Remark**</u>:

Using the linear transformation, x = z + a, where a is arbitrary but finite, we can use the Gauss-Laguerre quadrature to evaluate integrals of the type,

$$\int_a^\infty e^{-x} f(x)\,dx$$

Using above transformation, this integration becomes

$$\int_a^\infty e^{-x} f(x)\,dx = \int_0^\infty e^{-(z+a)} f(z+a)\,dz.$$

$$= e^{-a} \int_0^\infty e^{-z} f(z+a)\,dz.$$

Hence, the Gauss-Laguerre quadrature for arbitrary lower limit of integration a is written in general form as

$$\int_a^{\infty} e^{-x} f(x)dx \doteq e^{-a} \sum_{i=0}^{n} w_i f(z_i+a)$$

Here w_i and z_i are the same as that of previous case.

3.5 GAUSS-CHEBYSHEV QUADRATURE & GAUSS-HERMITE QUADRATURE

Using Chebyshev orthogonal polynomials and its properties, we can derive the quadrature formula known as Gauss-Chebyshev quadrature formula. The derivation of this formula is very similar to that of Gauss-Legendre or Gauss-Laguerre quadrature. Here we get

$$\int_{-1}^{1} \frac{1}{\sqrt{1-z^2}} F(z)dz \doteq \sum_{i=0}^{n} w_i F(z_i) \qquad \ldots(3.38)$$

If F(z) is a polynomial of degree (2n+1) or less then this integration is exact. Here the (n+1) base points z_i are the roots of the $(n+1)^{th}$ degree Chebyshev polynomial $T_{n+1}(z)$, so that

$$z_i = \cos \frac{(2i + 1)\pi}{(2n + 2)}, \qquad i = 0,1,\ldots,n.$$

Here the weights w_i are equal and have the vlaue $\frac{\pi}{n+1}$.

Therefore, after simplification of (3.38), we get

$$\int_{-1}^{1} \frac{1}{\sqrt{1-z^2}} F(z)dz \doteq \left(\frac{\pi}{n+1}\right) \sum_{i=0}^{n} F(z_i) \qquad \ldots(3.39)$$

The error term [3,pp 116] in this case is given by

$$E_n = \frac{2\pi}{2^{2n+2}\,(2n+2)!}\, F^{(2n+2)}(\xi), \qquad \xi \in (-1,1) \qquad \ldots(3.40)$$

Similarly, using orthogonality property of the Hermite polynomials we can derive another Gaussian formula,

$$\int_{-\infty}^{\infty} e^{-x^2} f(x)\,dx = \sum_{i=0}^{n} w_i f(x_i) \qquad \ldots(3.41)$$

This is known as the Gauss-Hermite quadrature. The roots of Hermite polynomial of degree (n+1) gives the base points x_i. The error term [3,pp 116] in this case is given by

$$E_n = \frac{(n+1)!\sqrt{\pi}}{2^{n+1}(2n+2)!}\, f^{(2n+2)}(\xi), \qquad \xi \in (-1,1) \qquad \ldots(3.42)$$

<u>Remark</u>:

By suitable transformation, we can use the Gauss-Chebyshev quadrature to evaluate the integral of the form

$$\int_a^b \frac{1}{\sqrt{1-x^2}} f(x)dx \qquad \text{or} \qquad \int_a^b f(x)dx\ ,$$

where, a and b are finite.

3.6 OTHER GAUSSIAN QUADRATURE FORMULAE

The four Gaussian quadrature formulae discussed up to this time allow the numerical evaluation of many well behaved integrals over semi-infinite, finite or infinite intervals of integration by using a suitable transformation of the variable of the integration. Sometimes, it is possible to find the value of integration of the function which has a singularity in the interval of integration by adjusting the singular term in the weight function. e.g. If the function to be integrated contains the factor $\frac{1}{\sqrt{1-x^2}}$ over the integration interval [-1,1] then it can be evaluated by the help of Gauss-Chebyshev quadrature.

We can generate a variety of other Gaussian quadrature formulae for particular weight functions, limits of integration and sets of orthogonal polynomials. Two famous particular cases of the Gaussian type are developed in which either or both end points of the interval of integration are base points. These formulae are called the Lobatto quadrature and the Radau quadrature. In Lobatto quadrature, both the limit points are included in base points, the remaining (n-1) points are found as the roots of $L_n'(x)$, i.e. the roots the derivative of the Legendre polynomial of degree n. This quadrature produces the integral exactly provided f(x) is a polynomial of degree (2n-1) or less. In case of Radau quadrature, only one limit point is included in the set of base points. It produces the integral exactly when f(x) is a polynomial of degree 2n or less.

We can also use the Gaussian quadrature formulae repeatedly on the subintervals of the interval of the integration and create the corresponding composite formulae. As many Gaussian formulae do not include the end points of the interval of integration the number of functional evaluations per subinterval may not be reduced as in the case of Newton-Cotes closed formulae. On the other hand, due to the inconvenient form of values of the weight factors and base points it is impossible to use high order formulae for hand calculation. Also, since we require a large amount of tabular information, it is tedious to prepare computer programs to implement a quadrature for many values of n. In another way we may compute base points using well known root finding methods and then appropriate weights which will probably become too wasteful of computing time for a frequently used integration program.

3.7 GAUSSIAN QUADRATURES WITH PREASSIGNED ABSCISSAS

Now, we consider Gauss type integration formulae with a certain number of preassigned abscissas. Such formulae are of the type

$$\int_a^b w(x)f(x)dx \approx \sum_{k=1}^{m} a_k f(y_k) + \sum_{k=1}^{n} w_k f(x_k) \qquad \ldots(3.43)$$

Here y_k's and x_k's are abscissas and y_k's are preassigned abscissas. Further, there are totally (m+2n) constants: a_k, w_k and x_k. We have to determine these constants so that the rule is exact for polynomials of the highest possible degree (m+2n-1). For this purpose, let us introduce the polynomials

$$r(x) = (x-y_1)(x-y_2)\ \dots\ (x-y_m),$$
$$s(x) = (x-x_1)(x-x_2)\ \dots\ (x-x_n) \qquad \dots(3.44)$$

Davis and Rabinowitz [4,pp77] have shown that the formula (3.43) is exact for all polynomials of degree $(m + 2n - 1)$ or less if and only if

(i) it is exact for all polynomials of degree $(m + n - 1)$ or less,

(ii) every polynomial $P(x)$ of degree $(n-1)$ or less should satisfy, $\int_a^b w(x)r(x)s(x)p(x)dx = 0$.

In numerical point of view, clearly we must determine $s(x) = (x-x_1)\dots(x-x_n)$ as one of a family of orthogonal polynomials on $[a,b]$ with respect to the weight function $w(x)c(x)$. At this point, theorem of Christoffel is of much importance which states: "Let $P_n(x)$, $n = 0,1,\dots$ be orthonormal polynomials on $[a,b]$ with respect to the weight $w(x) \geq 0$. Let $r(x) \geq 0$ be, as defined in (3.44) on $[a,b]$ and suppose that the y_i's are distinct. Let $q_n(x)$, $n = 0,1,\dots$ be orthogonal polynomials over $[a,b]$ with respect to $w(x)r(x)$. Then

$$r(x)q_n(x) = \begin{vmatrix} P_n(x) & P_{n+1}(x) & \dots & P_{n+m}(x) \\ P_n(y_1) & P_{n+1}(y_1) & \dots & P_{n+m}(y_1) \\ \vdots & \vdots & & \vdots \\ P_n(y_m) & P_{n+1}(y_m) & \dots & P_{n+m}(y_m) \end{vmatrix}$$

The common examples of the rules of Gauss type with preassigned abscissas are Radau and Lobatto rules of integration, as disscussed in the previous section. Here abscissas at the end points of the interval of the integration are preassigned abscissas and weight $w(x) \equiv 1$ is used.

3.8 THE ALGEBRAIC APPROACH TO THE GAUSSIAN QUADRATURES

Here we shall see the algebraic approach for the Gaussian quadratures. As usual, take $w(x) \geq 0$ as a weight function. We want to find 2n values: $w_1, w_2, \dots, w_n$; $x_1, x_2, \dots, x_n$ so that for $f(x) = 1, x, x^2, \dots, x^{2n-1}$, we have

$$\int_a^b w(x)f(x)dx = \sum_{i=1}^{n} w_i f(x_i). \qquad \dots(3.45)$$

Using the abbreviation, $\int_a^b w(x)x^j dx = m_j$ and writing out equation (3.45) for $f(x) = 1, x, x^2, \dots, x^{2n-1}$ successively, we get the following system,

$$m_0 = w_1 + w_2 + w_3 + \dots + w_n ,$$

$$m_1 = w_1x_1 + w_2x_2 + w_3x_3 + \dots + w_nx_n ,$$

$$m_2 = w_1x_1^2 + w_2x_2^2 + w_3x_3^2 + \dots + w_nx_n^2 ,$$

$$\vdots$$

$$m_{2n-1} = w_1x_1^{2n-1} + w_2x_2^{2n-1} + w_3x_3^{2n-1} + \dots + w_nx_n^{2n-1} \qquad \dots(3.46)$$

Let us define

$$P(x) = (x-x_1)(x-x_2)\dots(x-x_n)$$

$$= \sum_{k=0}^{n} c_kx^k, \qquad c_n = 1. \qquad \dots(3.47)$$

Now multiplying first n equations of system (3.46) successively by $c_0, c_1, c_2, \dots, c_n$ we get the following system

$$m_0c_0 = w_1c_0 + w_2c_0 + w_3c_0 + \dots + w_nc_0 ,$$

$$m_1c_1 = w_1c_1x_1 + w_2c_1x_2 + w_3c_1x_3 + \dots + w_nc_1x_n ,$$

$$m_2c_2 = w_1c_2x_1^2 + w_2c_2x_2^2 + w_3c_2x_3^2 + \dots + w_nc_2x_n^2 ,$$

$$\vdots$$

$$m_nc_n = w_1c_nx_1^n + w_2c_nx_2^n + w_3c_nx_3^n + \dots + w_nc_nx_n^n . \qquad \dots(3.48)$$

Adding all these equations in the above system (3.48) we get

$$m_0c_0 + m_1c_1 + m_2c_2 + m_3c_3 + \dots + m_nc_n$$

$$= w_1(c_0 + c_1x_1 + c_2x_1^2 + \dots + c_nx_1^n)$$

$$+ w_2(c_0 + c_1x_2 + c_2x_2^2 + \dots + c_nx_2^n)$$

$$+ w_3(c_0 + c_1x_3 + c_2x_3^2 + \dots + c_nx_3^n) + \dots$$

$$+ w_n(c_0 + c_1x_n + c_2x_n^2 + \dots + c_nx_n^n).$$

The terms in parenthesis above have typical form $\sum_{k=0}^{n} c_kx_i^k$, $i = 0,1,\dots,n$. and from equation (3.47), its value is $P(x_i)$ which in turn becomes zero. Hence above equation reduces to

$$m_0c_0 + m_1c_1 + m_2c_2 + \dots + m_nc_n = 0.$$

Once again we do the same procedure for second, third, $\dots,(n+1)^{th}$ equations of the system (3.46), i.e. multiplying second, third, ..., $(n+1)^{th}$ equations of the system (3.46) by $c_0, c_1, \dots, c_n$ successively and then adding and using equation (3.47), we get

$$m_1c_0 + m_2c_1 + \dots + m_{n+1}c_n = 0.$$

In this way, since $c_n = 1$ we obtain the system of n equations in n quantities $c_0, c_1, c_2, \dots, c_{n-1}$; which is

$$m_0c_0 + m_1c_1 + m_2c_2 + \dots + m_{n-1}c_{n-1} = -m_n ,$$

$$m_1c_0 + m_2c_1 + m_3c_2 + \dots + m_nc_{n-1} = -m_{n+1} ,$$

$$m_2c_0 + m_3c_1 + m_4c_2 + \dots + m_{n+1}\,c_{n-1} = -m_{n+2} ,$$

$$\vdots$$

$$m_{n-1}c_0 + m_nc_1 + m_{n+1}c_2 + \dots + m_{2n-2}c_{n-1} = -\, m_{2n-1}. \qquad \dots(3.49)$$

The determinant D of this system is given by

$$D = |\, m_{i+j} | = \left| \int_a^b w(x)x^{i+j}dx \right| = \left| \int_a^b w(x)x^ix^jdx \right|.$$

This is the Gram determinant of the functions $1, x, x^2, \dots, x^{n-1}$. In fact, these functions are linearly independent and hence it follows that $D \neq \emptyset$. Hence the system (3.49) have a unique solution for the constants $c_0, c_1, c_2, \dots, c_{n-1}$. From the values of $c_0, c_1, \dots, c_{n-1}$, we can get the abscissas $x_1, x_2, \dots, x_n$ from equation (3.47). Clearly these abscissas will be real, simple and located in (a,b). After determined the abscissas, we can find the weights using system (3.46).

Remarks - (1) To obtain formulae of Gauss type with one or more fixed points, we can use the modification of this device.

(2). Very often, the relevant matrices are ill-conditioned. Hence, from numerical point of view this method of determining Gauss integration formulae is not mostly recommended. Also the evaluation of a polynomial in the form $\sum_{k=0}^{n} c_k x^k$ may be an ill- conditioned process for large n so that accurate determination of its zeros becomes impossible.

3.0 CONVERGENCE OF GUASSIAN RULES

To study the behavior of a sequence of approximations generated by a specific Gaussian quadrature as the number n, of ordinates is increased, we express the error for the n-point Guassian rule in the form,

$$E_n = R_n[f(x)] = \int_a^b w(x)f(x)dx - \sum_{k=1}^{n} w_{nk}f(x_{nk}) \qquad \ldots(3.50)$$

where the weights and abscissas are supplied with the earlier implied index n.

We restrict our attention first to the cases where [a,b] is finite. Also, we suppose here that the weight function w(x) is non-negative in (a,b) and all the weights w_{nk} are positive so that many cases so far considered will be included. i.e. Let

$$w(x) \geq 0 \quad (a \leq x \leq b) \ , \ w_{nk} > 0 \qquad \ldots(3.51)$$

We also suppose that the degree of precision of the Gaussian formula under consideration is positive and increasing with m . Let f(x) be continuous on [a,b].

With these assumptions, for given any $\varepsilon > 0$ (however small) by the Weierstrass approximation theorem, we can find a polynomial p(x) for which

$$|f(x)-p(x)| < \epsilon , \qquad (a \le x \le b) \qquad \ldots(3.52)$$

If p(x) has degree M, we next take n sufficiently large so that the degree of precision of the Gaussian quadrature formula under consideration exceeds M, and hence

$$R_n[p(x)] = 0.$$

Now, from the linearity of the operator R_n, we have

$$R_n[f] = R_n[f-p] + R_n[p]$$

Hence for all sufficiently large n , we obtain

$$|R_n[f]| = \left| \int_a^b w(x)[f(x)-p(x)]dx - \sum_{k=1}^{n} w_{nk}[f(x_{nk}) - p(x_{nk})] \right|$$

$$\le \int_a^b w(x) | f(x)-p(x) | dx + \sum_{k=1}^{n} w_{nk} |f(x_{nk}) - p(x_{nk})|$$

$$\le \epsilon \left[\int_a^b w(x)dx + \sum_{k=1}^{n} w_{nk}\right] \qquad \text{(by 3.52)}$$

$$= 2 \epsilon \int_a^b w(x)\, dx \qquad \ldots(3.53)$$

Here we have used the assumed properties of $w(x)$ and w_{nk}. The last step appears due to the fact that $R_n[1] = 0$ implies the relation,

$$\int_a^b w(x)\, dx = \sum_{k=1}^{n} w_{nk} .$$

Thus, accordingly $|E_n|$ can be made smaller than any preassigned positive quantity ϵ by taking n sufficiently large. Hence we conclude that $E_n \to 0$ as $n \to \infty$ or

$$\lim_{n\to\infty} E_n = 0.$$

When the interval of integration is not finite, we need a less simple approach because the Weierstrass theorem is not available. However, convergence has been established in the cases of Gauss-Laguerre and Gauss-Hermite quadrature provided f(x) is continuous in every finite subinterval of $[\emptyset,\infty)$ [or $(-\infty,\infty)$] and f(x) is such that

$$w(x)|f(x)| < \frac{1}{|x|^{1+p}} ,$$

for sufficiently large x (or $|x|$) and for some $p > \emptyset$. Also, we have the following results about convergence of Gaussian Rules. [4,pp 100].

(i) If f is a bounded Riemann integrable function on [-1,1] i.e. if $f \in R[-1,1]$ then $\lim_{n\to\infty} E_n = \emptyset$.

(ii) If f(x) is piecewise constant function on [-1,1] then also $\lim_{n\to\infty} E_n = \emptyset$.

A most general theorem about the convergence of a family of rules is given by Polya. [4,pp 103] The statement of this theorem is as follows:

" Let $L_n(f) = \sum_{k=1}^{n} w_{nk} f(x_{nk})$, $\qquad a \le x_{nk} \le b$.

Then $\lim_{n\to\infty} L_n(f) = \int_a^b f(x)dx$, $\qquad \forall f \in C[a,b]$

if and only if

$$\lim_{n\to\infty} L_n(x^k) = \int_a^b x^k dx \quad , \qquad k = \emptyset,1,2,...$$

and

$$\sum_{k=1}^{n} |w_{nk}| \le M \qquad , \qquad n = 1,2,3,...$$

where, M is some constant."

It should be noted that a family of approximate integration formulae,

$$L_n(f) = \sum_{k=1}^{n} w_{nk} f(x_{nk}) \quad , \quad a \le x_{nk} \le b \qquad \ldots(3.54)$$

which converges for all functions which are continuous on [a,b] will not converge automatically for all functions which are Riemann integrable on [a,b]. For example, select the weights and abscissas as follows:

$$x_{ni} = (i-1)/n \quad , \qquad i = 1,2,\ldots,n.$$

$$w_{n1} = 1,$$

$$w_{n2} = -1,$$

$$w_{ni} = 1/n, \qquad i = 3,4,\ldots,n.$$

This family of rules integrates all functions which are continuous in [Ø,1] properly in the limit but cannot integrate the function,

$$f(Ø) = 1,$$

$$f(x) = Ø, \qquad Ø < x \le 1.$$

Polya has given the necessary and sufficient conditions for convergence for all functions of class R[a,b]. Let I denote the sum of a finite number of intervals (intervals may or may not be disjoint) situated in (a,b). Let m(I) denote the sum of the lengths of the individual intervals of I. Let $\sum_I |w_{nk}|$ will be the sum taken over those w_{nk} for which $x_{nk} \in$ I. Set

$$\triangle(I) = \lim_{n\to\infty} \sup \sum_{I} |w_{nk}|$$

It can be shown that, the set function $\triangle(I)$ is non-negative, monotone: $\triangle(I_1) \le \triangle(I_1+I_2)$, and subadditive: $\triangle(I_1+I_2) \le \triangle(I_1)+\triangle(I_2)$. If for any sequence, $I_1 \supseteq I_2 \supseteq \ldots$ and $m(I_n) \to 0$, we have

$$\lim_{n\to\infty} \triangle(I_n) = 0$$

then $\triangle(I)$ is called semicontinuoûs.

Now, the necessary and sufficient condition given by Polya for convergence for all functions of class R[a,b] is as follows:

" If

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{\infty} w_{nk} f(x_{nk})$$

holds for all $f \in C[a,b]$ then it also holds for all $f \in R[a,b]$ if and only if $\triangle(I)$ is semicontinuous."

3.10 ERROR ANALYSIS

In approximate integration, the value of $\int_a^b f(x)dx$ is replaced by a finite sum $\sum_{i=1}^{n} w_i f(x_i)$. In this process, two sorts of errors involve. First, there is the truncation error E which arises due to the fact that the sum is only approximately equal to the integral:

$$\int_a^b f(x)dx = \sum_{i=1}^{n} w_i f(x_i) + E \qquad ...(3.55)$$

In the another place, there is the roundoff error R and it is due to the limitation of accuracy of the computer. Hence we compute the finite sum $\sum_{i=1}^{n} w_i f(x_i)$ approximately. Let Σ^* denote this produced value, therefore,

$$\Sigma^* = \sum_{i=1}^{n} w_i f(x_i) + R.$$

Thus, the estimate of total error is

$$\left| \int_a^b f(x)dx - \Sigma^* \right| \le |E| + |R|$$

Of course, we have assumed that f is a function defined mathematically. Hence it does not consist of experimental data. Therefore we can compute f within the accuracy of the computer word length.

Usually roundoff error is negligible, but it becomes significant when we take large values of n in the sum (3.55). First we shall discuss the effect of roundoff error in the computation of the rule

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} w_i f(x_i) \qquad ...(3.56)$$

The analysis given below is based on analysis of Wilkinson. [4,pp 208]. Suppose that the computation of right hand sum in (3.56) is carried out in floating point arithmetic on a computer which works with t binary digits. Also suppose that our computer has a single precision accumulator. Let $fl(x_1+ x_2)$, $fl(x_1x_2)$ denote the addition and multiplication of x_1 and x_2 in the floating point mode, then it can be shown that

$$fl(x_1+x_2) = x_1(1+\epsilon') + x_2(1 + \epsilon'')$$

$$\text{and} \quad fl(x_1x_2) = x_1x_2(1 +\epsilon''') \qquad \ldots(3.57)$$

where $|\epsilon'|$, $|\epsilon''| \le (3/2)2^{-t}$ and $|\epsilon'''| \le 2^{-t}$ ...(3.58)

Now we are able to analyse the error made in computing a sum of products. We shall use these basic inequalities in (3.56).

Let $s_k = fl(a_1b_1 + a_2b_2 + \ldots + a_kb_k)$ ...(3.59)

The meaning of s_k is that we compute the products a_ib_i in floating and then add them in floating in the indicated order. In fact the computation (3.59) is the abbreviation for the computation of following steps indicated recursively.

$$s_1 = t_1 = fl(a_1b_1) ,$$

$$t_r = fl(a_rb_r),$$

$$s_r = fl(s_{r-1}+ t_r), \qquad r > 1$$

With the help of (3.57), (3.58), we obtain the above equations as

$$s_1 = t_1 = fl(a_1b_1)$$

$$t_r = a_rb_r (1 + \xi_r),$$

$$s_r = s_{r-1}(1 +\eta_r') + t_r(1 + \eta_r''), \qquad \ldots(3.60)$$

where

$$\left|\eta_r'\right|, \left|\eta_r''\right| \le \Omega_1 , \qquad \left|\xi_r\right| \le \Omega_2 \qquad \ldots(3.60A)$$

alongwith $\Omega_1 = (3/2)\ 2^{-t}$ and $\Omega_2 = 2^{-t}$. Using (3.60) recurrsively in (3.59), we find

$$s_k = a_1b_1(1+\varepsilon_1) + a_2b_2(1+\varepsilon_2) + \ldots + a_kb_k(1+\varepsilon_k) \qquad \ldots(3.61)$$

where, $(1+\varepsilon_1) = (1+\xi_1)(1+n_2') \ldots (1+n_k')$,

$$(1+\varepsilon_r) = (1+\xi_r)(1+n_r'')(1+n_{r+1}') \ldots (1+n_k'),$$

$$r = 2,\ldots,k-1$$

and $(1+\varepsilon_k) = (1+\xi_k)(1+\eta_k'')$ ...(3.62)

Using (3.60A) and (3.62), we have

$$(1-\Omega_2)(1-\Omega_1)^{k-1} \le 1+\varepsilon_1 \le (1+\Omega_2)(1+\Omega_1)^{k-1},$$

$$(1-\Omega_2)(1-\Omega_1)^{k-r+1} \le 1+\varepsilon_r \le (1+\Omega_2)(1+\Omega_1)^{k-r+1},$$

$$r = 2,3,\ldots,k.$$

This gives the uniform estimate for $r = 1,2,\ldots,k$:

$$(1-\Omega_2)(1-\Omega_1)^{k-r+1} \le 1+\varepsilon_r \le (1+\Omega_2)(1+\Omega_1)^{k-r+1} \qquad \ldots(3.63)$$

$$\text{Now, } (1+\eta)^m = 1 + m\,\eta + \frac{m(m-1)}{2!}\,\eta^2 + \ldots.$$

$$= 1 + m\,\eta\left[1 + \frac{m-1}{2!}\,\eta + \frac{(m-1)(m-2)}{3!}\,\eta^2 + \ldots\right]$$

$$\leq 1 + m\,\eta\left[1 + \frac{m}{2!}\,\eta + \frac{m^2}{3!}\,\eta^2 + \ldots\right]$$

$$= 1 + m\,\eta\left[\frac{e^{m\eta} - 1}{m\,\eta}\right] \qquad \ldots(3.64)$$

Similarly we can show that,

$$(1-\eta)^m \geq 1 - m\,\eta\left[\frac{e^{m\eta} - 1}{m\eta}\right] \qquad \ldots(3.65)$$

If we suppose that $k\eta_1 \leq 0.1$ then as $(e^{0.1}-1)/0.1 \leq 1.06$, we have from (3.63), (3.64), (3.65),

$$(1-\eta_2)[1-(k-r+1)\eta_1(1.06)] \leq 1+\epsilon_r$$
$$\leq (1+\eta_2)[1+(k-r+1)\eta_1(1.06)]$$

Therefore,

$$|\epsilon_r| \leq \eta_2 + (k-r+1)\eta_1(1+\eta_2)(1.06), \qquad r = 1,2,\ldots,k. \qquad \ldots(3.66)$$

Now, from (3.59) & (3.61), we get

$$\sum_{i=1}^{k} a_i b_i - fl\left(\sum_{i=1}^{k} a_i b_i\right) = \sum_{i=1}^{k} a_i b_i \epsilon_i \qquad \ldots(3.67)$$

We will apply this identity to $\sum_{i=1}^{n} w_i f(x_i)$. Let $\bar{f}_i$ denote the result of computing $f(x_i)$ in the floating point mode, and suppose that $\bar{f}_i = f(x_i)(1 + \theta_i)$, $\quad i = 1,2,\ldots,n$

$$\ldots(3.68)$$

with $\quad |\theta_i| \le \theta$, $\quad i = 1,2,\ldots,n \quad \ldots(3.69)$

where θ is a small quantity like η_1. We have

$$R = \sum_{i=1}^{n} w_i f(x_i) - fl\left(\sum_{i=1}^{n} w_i \bar{f}_i \right).$$

$$\Rightarrow R = \sum_{i=1}^{n} w_i f(x_i) - \sum_{i=1}^{n} w_i \bar{f}_i + \sum_{i=1}^{n} w_i \bar{f}_i \epsilon_i, \quad \text{(using 3.67)}.$$

$$\Rightarrow |R| \le \theta \sum_{i=1}^{n} |w_i||f(x_i)| + (1+\theta) \sum_{i=1}^{n} |w_i||f(x_i)||\epsilon_i|,$$

$$\text{(using3.68,3.69)} \qquad \ldots(3.70)$$

Let $M = \max \left\{ |f(x)| \;/\; a \le x \le b \right\}$. Then (3.70) becomes

$$|R| \le \theta M \sum_{i=1}^{n} |w_i| + (1 + \theta) M \sum_{i=1}^{n} |w_i||\epsilon_i|$$

Further using (3.66) we get

$$|R| \le \theta M \sum_{i=1}^{n} |w_i| + (1 + \theta) M \eta_2 \sum_{i=1}^{n} |w_i| +$$

$$(1.06)\ (1+\theta)\ M\ \eta_1 (1 + \eta_2) \sum_{i=1}^{n} |w_i| (n-i+1) \qquad \ldots(3.71)$$

Let us have some assumptions:

$$w_i \ge 0\ , \quad \sum_{i=1}^{n} w_i = b - a, \quad w_i \le \frac{A}{n}, \qquad i = 1,2,\ldots,n.$$

where A is a constant independent of n. Many rules satisfy these assumptions. With these, we estimate (3.71) as

$$|R| \le \theta(b-a)M + (1+\theta)\ M\Omega_2(b-a) +$$

$$1.06(1+\theta)M\Omega_1(1+\Omega_2)A^{\frac{(n+1)}{2}}$$

$$= M(b-a)[\theta+(1+\theta)\Omega_2] + 1.06(1+\theta)M\Omega_1(1+\Omega_2)\ A^{\frac{(n+1)}{2}} \quad \ldots(3.72)$$

Here exact weights are assumed. If θ' is bound on relative error in each weight, then the term $M(1+\theta)n\theta'$ is to added to the right hand side of (3.72) as well as (3.71).

From this analysis, we see that the roundoff error in approximate integration can be expected to increase as the first power of n. Hence these errors will not harm more because, in general, n varies within the range 10^1 to 10^2. These errors become still less if double precision computing is used. However we have to take care when vast numbers of abscissas are used.

In the approximation (3.56), the total error committed consists of both types of the errors: roundoff error as well as truncation error. The truncation error is $\sim n^{-1}$ or better in case of usual employed rules. As n increases, roundoff error also increases but truncation error decreases. Hence n should be properly selected at some intermediate value.

<u>Trunction Error Through Peano's Theorem</u>

Let the truncation error be designated as

$$E = E(f) = \int_a^b f(x)dx - \sum_{i=1}^{n} w_i f(x_i).$$

We shall treat E(f) as a linear functional defined over a certain class of functions for the sake of convenience. Linear functional E have the following property,

$$E(af(x) + bg(x)) = a\, E(f(x)) + b\, E(g(x))$$

for some constants a and b. Let f be a function whose n-th derivative exists and is absolutely continuous on [a,b]. Now the Peano's Theorem is as follows:

"Let $E(P(x)) = \emptyset$ whenever $P(x) \in p_n$.
Then for all $f(x) \in C^{n+1}[a,b]$,

$$E(f) = \int_a^b f^{(n+1)}(t)k(t)dt$$

where
$$k(t) = \frac{1}{n!} E_x[(x-t)_+^n]$$

and
$$(x-t)_+^n = \begin{cases} (x-t)^n & x \geq t \\ \emptyset & x < t \end{cases}$$"

Here the symbol E_x means that the linear functional E is applied to the variable x in $(x-t)_+^n$. The function k(t) is known as the Peano Kernel for E or influence function for E. If k(t) does not change its sign on [a,b] then we get

$$E(f) = \frac{f^{(n+1)}(\xi)}{(n+1)!} E(x^{n+1}), \qquad \xi \in (a,b).$$

For an arbitrary rule of approximate integration (3.56) there is no reason for k(t) to have one sign.

Peano's Kernels for Gaussian Rules.

Let $E(f) = \int_{-1}^{1} w(x)f(x)\,dx - \sum_{k=1}^{n} w_k f(x_k)$, $x_k \in [-1,1]$

and $E(f) = \emptyset$ for $f \in p_r$ then $k(t)$, Peano Kernel, for E is given explicitly by

$$K_r(t) = \frac{1}{r!} E_x\,[(x-t)_+^r]$$

or equivalently,

$$r!k_r(t) = \int_{-1}^{1} w(x)(x-t)_+^r\,dx - \sum_{x_k>t} w_k(x_k-t)^r.$$

As a special case, when $w(x) = 1$, we have

$$r!k_r(t) = \frac{(1-t)^{r+1}}{r+1} - \sum w_k(x_k-t)^r$$

Here x_k's and w_k's are abscissas and weights for the Gaussian rules. Further if this rule is exact for $f \in P_{2n-1}$ then for each r, [$r = \emptyset,1,\ldots,(2n-1)$] we can find a kernel of order r and corresponding error estimate. Also the constants

$$e_r = \int_{-1}^{1} |k_{r-1}(t)|\,dt, \qquad r = 1,2,\ldots,2n$$

are computed by Stroud and Secrest. These constants are usually called as Peano error constants [4,pp 223]. Stroud has given the following result which is useful in computing error for Gaussian n-point formula :

"If $f^{(2n)}(x)$ is continuous on [a,b] then there is a point $\xi \in (a,b)$ so that

$$E[f] = \frac{1}{(2n)!} f^{(2n)}(\xi) \int_a^b w(x)[P_n(x)]^2 dx,$$

where $P_n(x)$ is the orthogonal polynomial (with leading coefficient unity)."

3.11 CRITICAL EVALUATION OF GAUSSIAN QUADRATURE

There are many points of merit for Gaussian quadrature. First we have freedom in the selection of base points. In other words base points are not pre-assigned. It is found that [10,pp 187; 4,pp 75] Gauss rules are best in the sense that n-point Gauss rule gives the more accurate result than the corresponding n-point Newton-Cotes formula for quadrature. Also for large n, say for n>20, the Newton-Cotes formulae fail to give the good results but Gaussian quadrature formulae can compute the integral more accurately when n is large. In fact, they integrate polynomials of much higher degrees exactly. The Gauss rule contains positive weights which are useful in keeping down the round-off errors. Another important thing is that the Gauss rules are Riemann sums. Hence continuous functions are exactly integrated by Gauss Rules. In most cases, a sequence of Gaussian quadratures converges to the true value of the integral. However, it is not true that a Gauss formula is always the best. For example, for the evaluation of

$$\int_0^1 \frac{2}{2+\sin(10\pi x)} dx$$

the n-point trapezoidal rule is much better than the n-point Gauss Legendre formula. Also, in general, the weights and absissas of the Gauss rules are irrational numbers, therefore, it is difficult as well as an error prone nuisance to deal with many digits, in case of hand computation. On the other hand, the digitial computers do not make difference between 'simple' numbers like 0.6000000 and more "complicated" numbers as 0.598250269.

Now a days, because of the use of computers, Gauss rules are very popular. But, the old difficulty of rational versus irrational is still present. Hence program should be prepared with the requirements of the typing up and checking of many irrational numbers. Further, the weights and abscissas of any Gaussian rule of one order are different from those of any other order (except that zero is an abscissa in each rule of odd order). Therefore, in the computation of m-point formula from n-point formula (m > n), almost all the information obtained in the case of n-point formula will get discarded. Kronrod has a device [4,pp 82] which is developed further by Patterson which solves this difficulty up to some extent. Due to this device we are able to add new abscissas to a given set of abscissas to create a new rule of higher accuracy, but still we can't get optimally higher accuracy.