CHAPTER I

CHAPTER – I

INTRODUCTION AND BASIC CONCEPTS

1.1 INTRODUCTION :-

Differential equations are commonly used for mathematical modeling in science and engineering. If there is no known analytical solution, then numerical approximations are used. Not all initial value problems can be solved explicitly and offen it is very difficult to find a formula for the solution. e.g. there is no 'closed form expression ' for the solution to

$$\frac{dy}{dx} = x^3 + y^2 \text{ with } y(0) = 0.$$

Hence for engineering and scientific purposes it is necessary to have methods which approximate the solution. If a solution with many significant digits is required, then more computing efforts and sophisticated alogorithms are used.

1.2 NUMERICAL METHODS.

Consider, the first order initial value differential equation

$$\frac{dy}{dt} = f(t,y) , \qquad y(t_o) = y_o. \qquad \ldots(1.1)$$

The function $f(t,y)$ may be linear or non-linear.

A numerical method for the solution of differential Eq. (1.1) is an algorithm which gives a set of approximate values of $y(t)$ at certain equally spaced points called grid, nodal, net or mesh points along the t-cordinate. The mesh points are related by

$$t_k = t_o + kh \ , \ k = 1,2,...N,$$

where h is called the step-size.

The approximate solution y(t) may contain the following errors:

1.2.1 ROUND-OFF ERROR :

For numerical calculation we are using computing apparatus such as mathematical tables, calculators or the digital compu ters. Thus most of the numerical computations are not exact. Due to these limitations, the numbers have to be rounded-off, which gives rounding errors. In calculations, such inherent errors can be minimized by obtaing better data, by correcting obvious errors in the data, by using well known programs and high speed computers.

Thus the round-off error is the quantity E which added to the computed number in order to make it the exact number.

$$y(exact) = y(computed) + E.$$

1.2.1 Truncation errors :

In computing we may use approximate formulae, e.g. to evaluate f(x) one can use infinite series in x, after trunca ting it after finite number of terms. Thus there exists an error known as Truncation error.

Hence the truncation error is the quantity T which added to the true value represents the computed quantity so that the result may be exact. i.e.

y(exact) = y(true representation) + T

The numerical methods for finding the solution of differential equation (1.1) are probably classified into two types :

i) Single-step methods :

These methods are recurrence relations, which express the value y(t) & y'(t) at t_{n+1}, when y_n, y'_n, and the stepsize h are known.

ii) Multi-step methods:

These methods are recurrence relations, which express the value y(t) at t_{n+1} in terms of function value y(t) & y'(t) at t_{n+1} and at previous mesh points.

Now we will discuss the numerical methods & related basic concepts for the initial value first order differential equation

$$\frac{dy}{dt} = \lambda y, \qquad y(t_o) = y_o , \qquad t \in [t_o, b] \qquad \ldots(1.2)$$

The exact solution of this equation is

$$y = A\, e^{\lambda t} ,$$

where A is arbitrary constant. Using given initial condition, we have

$$y(t) = y(t_o)\, e^{\lambda(t-t_o)}$$

Using the value of y(t) at nodal points $t_j = t_o + jh$

We have

$$y(t_{j+1}) = e^{\lambda h}\, y(t_j) \;, \quad j = 0,1,2,....N-1. \qquad ...(1.3)$$

Which contain exponential function,hence difficult to obtain exact solution. Thus consider a suitable approximation to $e^{\lambda h}$. Let us denote the approximation to $e^{\lambda h}$ by $E(\lambda h)$. The numerical method to find y_j for $y(t_j)$ can be written as

$$y_{j+1} = E(\lambda h)\, y_j \;, \quad j = 0,1,.....,N-1 \qquad ...(1.4)$$

1.3 Stability Analysis:

In nmerical method ,if the effect of any single fixed round-off error is bounded, independent of number of nodal points, then the method is said to be stable.

Defination: 1.1 A numerical method (1.4) is called absolutely stable if $| E(\lambda h)| \le 1$ . ...(1.5)

Defination 1.2 A Numerical method (1.4) is called relatively stable if $| E(\lambda h) | \le e^{\lambda h}$. ...(1.6)

1.4 Useful Results :

In the text that follows, we require the following formulae and theorems :

1.4.1 Definitions of some operators :

The forward difference operator

$$\Delta f(t) = f(t+h) - f(t)$$

The Backward difference operator

$$\nabla f(t) = f(t) - f(t-h)$$

The central difference operator

$$\delta f(t) = f(t+h/2) - f(t-h/2)$$

1.4.2 <u>THEOREM</u> 1.1 :

We assume that $f(t,y)$ satisfies the following conditions:

i) $f(t,y)$ is real function,

ii) $f(t,y)$ is defined and continuous in the strip

$$t \in [t_o,b] \; , \; y \in (-\infty, \infty) \; ,$$

iii) There exists a constant L such that for any $t \in [t_o,b]$ and for any two numbers y_1 and y_2

$$\left| f(t, y_1) - f(t, y_2) \right| \le L \left| y_1 - y_2 \right| ,$$

where L is called the Lipschitz constant. Then, for any y_o, the initial value problem (1.1) has a unique solution $y(t)$ for $t \in [t_o, b]$.

1.4.3 Newton Backward Interpolation Formula :

$$y_n(x) = y_n + p\nabla y_n + \frac{P(P+1)}{2!}\nabla^2 y_n + \ldots.$$

$$+\frac{P(P+1)\ \ldots.(P+n-1)}{n!}\nabla^n y_n + \frac{P(P+1)\ \ldots.(P+n)}{(n+1)!}h^{n+1}y^{(n+1)}(\xi)$$

$$\text{Where } P = \frac{x-x_n}{h} \quad \& \ x_o < \xi < x_n \ \& \ x_n = x_o + nh \qquad (1.7)$$

1.4.4 Spectral Norm :

Let λ_j be eigen values of a matrix A, then $\|A\| = \max|\lambda_j|$. This norm is called spectral norm.

1.4.5 Von Neumann necessary condition for Stability:

A necessary condition for stability is

$$\rho^n \leq c_1, \text{ for } 0 < k < t_1, 0 \leq nk \leq T.$$

where c_1 is a positive constant and ρ denote the spectral radius. (1.8)

1.4.6 Taylor's Theorem

a) In one variable :

If f(x) is continuous and $(n+1)^{th}$ continuous derivatives exists in an interval including x = a, then in that interval wehave

$$f(x) = f(a) + (x-a)\, f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) +$$

$$\ldots. + \frac{(x-a)^n}{n!} f^{(n)}(a) + E(x)$$

where $E(x) = \frac{1}{n!} \int_a^x (x-s)^n f^{(n+1)}(s)\, ds.$

and if $(x-s)^n$ does not change sign as s varies from a to x , then

$$E(x) = \frac{f^{(n+1)}(\xi)}{n!} \int_a^x (x-s)^n ds.$$

$$= \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-a)^{n+1}, \quad (a < \xi < x).$$

Here $f^{(n+1)}(x)$ is continuous. ...(1.9)

b) For two variables :

$$f(x+h, y+k) = f(x,y) + h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + \frac{1}{2!}\left[h^2 \frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + 2k\frac{\partial^2 f}{\partial y^2} + \ldots.\right]$$

$$= f(x,y) + \left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right] f + \frac{1}{2!} \left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right]^2 f$$

$$+ \quad + \frac{1}{n!} \left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right]^n f + \ldots. \qquad (1.10)$$