CHAPTER – III

Inversion Theorem

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INVERSION THEOREM

3.1 Introduction:

In the preceding chapter we have derived some important properties and results concerning the mixed Stieltjes transform m(t, z) at a function f(x). Our objective in this chapter is to derive the inversion formula for the mixed Stieltjes transform.

Transform analysis deals with different kind of integral transforms and corresponding to each integral transforms there must be inversion formula for the transform. To derive the inversion formula for the mixed Stieltjes transform, we shall consider converse problem – that of deriving information about the original function f(x), when we have known its mixed Stieltjes transform. When m(t, z) is prescribed any formula enabling us to derive the form of the original function f(x) is called inversion formula for the mixed Stieltjes transform. The mixed Stieltjes transform defined by

M. Giona and O. Patierno [1] by the equation

$$m\,[f(x)] = \int_0^\infty \frac{e^{-zx}}{1+tx} f(x)\cdot dx \quad \ldots (3.1.1)$$

where f(x) is piecewise continuous and locally integeable function over interval $0 < x < \infty$, z is complex variable and t is real variable on interval $0 < t < \infty$. To derive the inversion formula for the mixed Stieltjes transform (3.1.1), we need some transforms and their inversion formulae. The Laplace Transform is defined by the equation,

$$L\,[f(x)] = \int_0^\infty e^{-zx} f(x)\,dx$$

where z is a complex number.

The inversion formula is given by

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{zx}\,\phi(z)\cdot dz$$

where f(x) is continuos and exponential order in the interval $0 < x < \infty$ and $\phi(z)$ is the Laplace transform of f(x).

The Stieltjes transform is defined by the equation.

$$S[f(x)] = \int_0^{\infty} \frac{f(x)}{t+x} f(x)\,dx$$

where t is a real or complex valuable and f(x) is locally integrable function in interval $0 < x < \infty$.

The inversion formula for the Stieltjes transform [2], is given by the equation,

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{S(t)\cdot x^{-t}}{\Gamma t \Gamma(1-t)} \cdot dt$$

where S(t) is the Stieltjes transform of the function f(x).

Also the Mellin transform is defined by the equation,

$$M[f(x)] = \int_0^{\infty} x^{p-1} f(x)\,dx$$

and its inversion is given by,

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} M(p)\cdot x^{-p}\,dp \ldots (3.1.2)$$

where c is a positive constant and M(p) denotes the Mellin transform of f(x).

To give derivation of an inversion formula we have been use the methods of inversion formulae for the Laplace transform and the Stieltjes transform. The method of an inversion of the Laplace transform is given by Sneddon [3]. We use this method to give the proof inversion formula for the mixed Stieltjes transform.

3.2 Inversion Theorem.

Statement: Let f(x) be continuous and $x^{-1} f(x)$ be a locally integrable function in the interval $0 < x < \infty$, If $m[f(x)] = m(t,z)$

then,

$$f(x) = \frac{x^{(1-\frac{1}{x})}}{(2\pi i)^2} \int_{c-i\infty}^{c+i\infty} \int_{c-i\infty}^{c+i\infty} m(t,z) \cdot e^{zx} x^{-t} \, dt \cdot dz$$

where 'c' is positive constant

Proof: To prove the theorem, we have to show that the integral

$$\frac{x^{(1-\frac{1}{x})}}{(2\pi i)^2} \int_{c-i\tau}^{c+i\tau} \int_{c-i\tau}^{c+i\tau} m(t,z) \cdot e^{zx} x^{-t} \, dt \cdot dz$$

converges to $f(x)$ as $\tau \longrightarrow \infty$.

Consider the integral

$$\lim_{\tau\to\infty}\frac{x^{(1-\frac{1}{x})}}{(2\pi i)^2}\int_{c-i\tau}^{c+i\tau}\int_{c-i\tau}^{c+i\tau} m(t,z)\cdot e^{zx}x^{-t}dt\cdot dz$$

Substituting value of the mixed Stieltjes transform from the equation (3.1.1)

$$L.H.S.=\lim_{\tau\to\infty}\frac{x^{(1-\frac{1}{x})}}{(2\pi i)^2}\int_{c-i\tau}^{c+i\tau}\int_{c-i\tau}^{c+i\tau}\left\{\int_0^\infty\frac{e^{-zy}}{1+ty}\cdot f(y)\cdot dy\right\}$$

$$\cdot e^{zx}\cdot x^{-t}\cdot dt\cdot dz$$

Since function f(x) is continuos

$$L.H.S.=\lim_{\tau\to\infty}\frac{x^{(1-\frac{1}{x})}}{(2\pi i)^2}\int_{c-i\tau}^{c+i\tau}\int_{c-i\tau}^{c+i\tau}\int_0^\infty\frac{e^{(x-y)z}}{1+ty}$$

$$\cdot x^{-t}f(y)\cdot dy\cdot dt\cdot dz$$

Since the limits of integration are not the function of variables t, y , z and the integrand is continuous.

by using change of order of integrations (0.2.11)

We have,

$$\text{L.H.S.} = \lim_{\tau\to\infty} \frac{x^{(1-\frac{1}{x})}}{(2\pi i)^2}\int_0^{\infty}\int_{c-i\tau}^{c+i\tau}\left\{\int_{c-i\tau}^{c+i\tau} e^{(x-y)z}dz\right\}$$

$$\cdot\frac{x^{-t}f(y)}{1+ty}dt\cdot dy$$

Carring out inner integration.

$$\frac{1}{2i}\int_{c-i\tau}^{c+i\tau} e^{(x-y)z}dz = \frac{\sin\tau(x-y)}{(x-y)}\cdot e^{(x-y)c}$$

$$\text{L.H.S.} = \lim_{\tau\to\infty} \frac{x^{(1-\frac{1}{x})}}{(2\pi i)^2}\int_0^{\infty}\int_{c-i\tau}^{c+i\tau}\left\{\frac{2i\cdot\sin\tau(x-y)}{(x-y)}\cdot e^{(x-y)c}\right\}.$$

$$\cdot\frac{x^{-t}f(y)}{1+ty}dt\cdot dy$$

$$\text{L.H.S.} = \lim_{\tau\to\infty} \frac{x^{(1-\frac{1}{x})}}{2i\pi^2} \int_0^\infty \int_{c-i\tau}^{c+i\tau} \left\{ \frac{x^{-t} e^{(x-y)c} \sin\tau(x-y)}{y(\frac{1}{y}+t)(x-y)} \right\} \cdot f(y)\cdot dt \cdot dy$$

$$\text{L.H.S.} = \lim_{\tau\to\infty} \frac{x^{(1-\frac{1}{x})}}{2i\pi^2} \int_0^\infty \left\{ \lim_{\tau\to\infty} \int_{c-i\tau}^{c+i\tau t} x^{-t} \left[\frac{1}{\frac{1}{y}+t} \right] \cdot dt \right\}$$

$$\frac{e^{(x-y)c} \sin\tau(x-y)}{y(x-y)} \cdot f(y)\cdot dy$$

Carrying out inner integration,

$$\therefore \lim_{\tau\to\infty} \frac{1}{2\pi i} \int_{c-i\tau}^{c+i\tau} x^{-t} \left[\frac{1}{\frac{1}{y}+t} \right] \cdot dt = x^{\frac{1}{y}}$$

[Since by definition of inverse of Mellin transformation [3.1.2]

$$\text{L.H.S.} = \lim_{\tau\to\infty} \frac{x^{(1-\frac{1}{x})}}{2i\pi^2} \int_0^\infty \left\{ 2\pi i x^{\frac{1}{y}} \right\} \frac{e^{(x-y)c}}{y(x-y)}$$

$$\cdot\sin\tau(x-y)\cdot f(y)\cdot dy$$

$$\text{L.H.S.} = \lim_{\tau \to \infty} \frac{x^{(1-\frac{1}{x})}}{\pi} \int_0^\infty \frac{x^{\frac{1}{y}} e^{(x-y)c}}{y(x-y)}$$

$$\cdot \sin \tau(x-y) \cdot f(y) \cdot dy$$

Put $y - x = u \Rightarrow dy = du$

$y=0 \Rightarrow u = -x$

$y = \infty \Rightarrow u = \infty$

$$\text{L.H.S.} = \lim_{\tau \to \infty} \frac{x^{(1-\frac{1}{x})}}{\pi} \int_{-x}^{\infty} \frac{x^{\left(\frac{1}{x+u}\right)}}{(x+u)} \frac{e^{-cu}}{u} \cdot \sin \tau u \cdot f(x+u) \cdot du$$

$$= \lim_{\tau \to \infty} \frac{x^{(1-\frac{1}{x})}}{\pi} \int_{-x}^{\infty} \left\{ \frac{x^{(\frac{1}{x+u})} e^{-cu} f(x+u)}{x+u} \right\} \cdot \frac{\sin \tau u}{u} \cdot du$$

$$= x^{(x-\frac{1}{x})} \lim_{\tau \to \infty} \frac{1}{\pi} \int_{-x}^{\infty} \left\{ \frac{x^{(\frac{1}{x+u})} e^{-cu} f(x+u)}{x+u} \right\} \frac{\sin \tau u}{u} . du$$

Let ,

$$g(x+u) = \frac{x^{\left(\frac{1}{x+u}\right)} f(x+u)}{x+u}$$

Since we can write the integral.

$$\frac{1}{\pi}\int_{-x}^{\infty} e^{-cu}\, g(x+u)\cdot\frac{\sin\tau u}{u}\cdot du$$

$$=\frac{1}{\pi}\int_{-\infty}^{\infty} e^{-cu}\, g(x+u)\cdot\frac{\sin\tau u}{u}\cdot du$$

We have,

$$\text{L.H.S.}= x^{\left(1-\frac{1}{x}\right)}\lim_{\tau\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty} e^{-cu} g(x+u)\cdot\frac{\sin\tau u}{u}\cdot du$$

The function $g(x+u)$ is continuous on $-\infty < x < \infty$,

$\therefore$ we have the result [0.2.8] and [3]

$$\lim_{\tau\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty} e^{-cu}\, g(x+u)\cdot\frac{\sin\tau u}{u}\cdot du = g(x)$$

where ‘c’ is positive constant.

therefore, we have,

$$L.H.S. = x^{\left(1-\frac{1}{x}\right)} \cdot g(x)$$

$$= x^{\left(1-\frac{1}{x}\right)} \cdot \left[\frac{x^{1/x}}{x} \cdot f(x) \right]$$

$$= x^{\left(1-\frac{1}{x}\right)} \left[x^{\left(\frac{1}{x}-1\right)} \right] \cdot f(x)$$

$$= f(x)$$

Therefore, the integral,

$$\frac{x^{\left(1-\frac{1}{x}\right)}}{(2\pi i)^2} \int_{c-i\tau}^{c+i\tau} \int_{c-i\tau}^{c+i\tau} m(t,z) \cdot e^{zx} \cdot x^{-1} \cdot dt \cdot dz$$

Converges to $f(x)$ as $\tau \longrightarrow \infty$

$$\therefore f(x) = \frac{x^{\left(1-\frac{1}{x}\right)}}{(2\pi i)^2} \int_{c-i\infty}^{c+i\infty} \int_{c-i\infty}^{c+i\infty} m(t,z) \cdot e^{zx} \cdot x^{-1} \cdot dt \cdot dz$$

where c is positive constant,

Which is inversion formula for the mixed Stieltjes transformation.

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