CHAPTER – I

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1.1 Preliminary Remarks:

Day to day knowledge of science is expanding. Mathematics plays an important role in knowledge of science. In words of Philip, mathematics is a science of quantity and space. The range of mathematical tools expanded considerable through the use of theory of operators, the theory of generalized functions, the theory of functions of complex variables, topological and algebraic methods, computational mathematics and computers. All these theories were pressed into service in addition to the traditional tools of mathematics, gradually brought to further new domain, that of modern mathematical physics. Modern mathematical physics makes brief use of the latest mathematics, one of which is the theory of generalized functions. However, the concept is a convenient links connecting many aspects of analysis, functional analysis, the theory of differential equations, the representation theory and the theory of probability and statistics. The new mode of thinking gives birth to the theory of generalised functions, which put the wheel of research in several branches of mathematics in rapid motion. The impact of generalised functions on the integral transformation of has recently revolutionalised the theory of "Generalised Integral Transformations"

The history of operational methods or integral transformations and generalised functions visually testifies to fruitfulness of cross-fertilization between mathematics, the physical science, and engineering. The oldest systematic technique for the solution of the partial differential equation of mathematics and physics is the method of separation of variables, introduced by 'D' Alembert, Daniel Bernouli and Euler in the middle of the eighteenth century. It remains the method of great value today and lies at the heart of the use of integral transforms in the solution of the problems in applied mathematics.

1.2 Integral Transformations:

The theory of integral transform is a classical object in mathematics, whose literature can be traced back through at least 150 years, but the theory of generalised functions is comparatively of recent again. Every one who studied linear differential equation beyond an elementary level quickly grasps to appreciate the power of, so called integral transform technique.

Many functions in analyis can be expressed as Lebesque integrals or improper integrals of the form.

$$F(z) = \int_{-\infty}^{\infty} k(z, x) . f(x) . dx \quad \ldots (1.2.1)$$

A function defined by an equation (1.2.1) of this type is called integral transform. k(z,x) is called kernel of the integral transform, kernel k (z, x) is assumed of course that the infinite integral is convergent. In the equation (1.2.1) 'z' may be real or complex. When the range of integral transformation (1.2.1) is replaced by the finite range [a, b], F(z) is called the finite transform of f(x).

The use of integral transform will often reduce a partial differential equation in n independent variables to one in (n-1) variables. Thus reducing the difficulty of the problem under the discussion, Integral transform are imployed very extensively in both pure and applied mathematics. The particular type of kernel k(z, x) and related range of integration commonly used, there are various types of integral transformations given below:

S.No	Kernel of Integration	Range of Integration	Name of the integral Transformations
1	e^{-izx}	$(-\infty, \infty)$	Fourier transform
2	e^{-zx}	$(0, \infty)$	Laplace transform
3	sin zx	$(0, \infty)$	Fourier sine transform
4	cos zx	$(0, \infty)$	Fourier cosine transform
5	x^{z-1}	$(0, \infty)$	Mellin transform
6	zx Jv (zx)	$(0, \infty)$	Hankel transform

Integral transform are used in pure as well as applied mathematics for solving certain boundary value problem and certain type of integral equations. Integral transform gives an expression F(z) in the term of algebraic function of f. Each integral transform has an invesion theorem, which gives expression f(x) using F(z). Erdelyi A. [1], [2] and Sneddon [8] have discussed properties and applications of some integral transforms. Poularikas [6] in this newly edited book discussed newly arised as well as useful integral transformations, which are frequently used by Engineers, such as

Fourier sine transform, Fourier cosine transform. Hartely, Laplace, Hilbert, Z-transform, Radom and Abel, Hankel transform and Wavelet transformation.

A comprehensive analysis of the many important integral transforms and generalized functions together with their applications can be found in the recent book published in 1997 by R.S. Pathak [3].

1.3 Laplace Transform:

The transform theory provides a powerfull technique to solve an ordinary and partial differential equation in a direct and systematic manner. A suitable integral transform is required according to the conditions of the problems.

The Laplace transform is a mathematical tool that greatly facilities the solution of the constant coefficients linear differential equation. Although the method was developed over 150 years ago by P.S. de Laplace. Only in recent years it has achieved widespread use. The Laplace transformation method enables differential equation to be transformed into relatively simple algebraic equations that can be multipulated until the desired form is obtained. They may be transformed back into complete solution of the original differential equation. Initial conditions or boundary values may be readily introduced.

In the integral transform, if the kernel $k(z, t)$ is e^{-zx} and range of integration is 0 to ∞, the relation defined by the integral,

$$L[f(x)] = \int_0^\infty e^{-zx}.f(x).dx....(1.3.1)$$

$$z = (a + ib) \qquad a, b \in R.$$

is called Laplace transform. It is denoted by $\phi(z)$ or $L[f(x)]$. We can find the inverse Laplace transform of $\phi(z)$, i.e. the original function f(x).

The inversion formula for Laplace transform is given by the relation.

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \phi(z).e^{zx}.dz.....(1.3.2)$$

where c is positive constant and $\phi(z)$ is Laplace transform of f(x). Sneddon [8] discussed some important classical properties of the

Laplace transform and the definitive work of distributional properties of Laplace transform is in the book of Zemanian [14] Here we discussed some important properties of Laplace transform

1.4 Classical Properties of Laplace transform.

1) Linear Property.

If $f_1(x)$ and $f_2(x)$ are any two functions and a_1, a_2 are any pair of constants then ,

$$L[a_1 f_1(x) + a_2 f_2(x)] = a_1 L[f_1(x)] + a_2 L[f_2(x)] \quad \ldots\ldots(1.4.1)$$

2) Change of Scale Property.

If $L[(x)] = \phi(z)$ then,

$$L[f(ax)] = 1/a \;\; \phi(z/a) \quad \ldots\ldots(1.4.2)$$

where a is non-zero constant .

3) First Translation or Shifting Theorem.

If $L[f(x)] = \phi(z)$ when $\mathrm{Re}\, z > \alpha$, α is positive number

then, $[e^{-ax}.f(x)] = \phi(z - a) \quad \ldots\ldots(1.4.3)$

$\mathrm{Re}\, z > \alpha + a$, a is non zero constant.

4) Second Translation or Shifting Theorem.

If $L[f(x)] = \phi(z)$ and

$$G(x) = \begin{cases} f(x-a), & x > a \\ 0 & x < a \end{cases}$$

then $L[G(x)] = e^{-az} . \phi(z)$....(1.4.4)

5) Multiplication by x.

If f(x) is piecewise continuous and exponential order function, and

if $L[f(x)] = \phi(z)$

then $L[x f(x)] = \phi'(z)$.......(1.4.5)

$\phi'(z)$ denotes differentiation of $\phi(z)$

6) Multiplication by x^n.

It f(x) is piecewise continuous and exponential order function and if $L[f(x)] = \phi(z)$

then,

$$L[x^n f(x)] = (-1)^n . \frac{d^n}{dz^n} \phi(z) \quad ....(1.4.6)$$

where n = 1,2,3,.....

7) Division by x.

If $L[f(x) = \phi(z)$ then,

$$L\left[\frac{1}{x} f(x)\right] = \int_z^{\infty} \phi(u).du \quad .....(1.4.7)$$

provided $\lim_{x \to 0} \int \frac{f(x)}{x}$ ... exists

8) Division by x^n.

If $L[f(x)] = \phi(z)$ then,

$$L\left[\frac{f(x)}{x^n}\right] = \int_z^{\infty} ... \int_z^{\infty} \phi(u).(du)^n \quad .....(1.4.8)$$

n^{th} repeated integrals ....

n=1,2,3......

9) Laplace Transform of Derivative of f(x).

Let f(x) be continuous and be of exponential order in interval $0 < x < \infty$ and if f '(x) is also continuous and exponential order and if $L[f(x)] = \phi(z)$

then Laplace transform of $f'(x)$ exists when

$\operatorname{Re} z > a$

$$L[f'(x)] = z\,L[f(x)] - f(0) \quad \text{.........(1.4.9)}$$

10) Laplace Transform of the n^{th} Order Derivative of f(x).

Let $f(x)$ and its derivatives $f'(x)$, $f''(x)$, ..., $f^{(n-1)}(x)$, $f^{(n)}(x)$, be continuous function and be exponential order in interval $0 < x < \infty$ then

Laplace transform of $f^{(n)}(x)$ exists when $\operatorname{Re} z > a$ and given by

$$L[f^{(n)}(x)] = z^{n}L[f(x)] - z^{(n-1)}f(0) - z^{(n-2)}f'(0) - \ldots f^{(n-1)}(0) \quad \text{....(1.4.10)}$$

1.5 Stieltjes Transform.

The Stieltjes transform is introduced for the first time in 1894 by J.J. Stieltjes, in connection with this work on continued fractions. Latter on many eminent mathematicians certain properties of Stieltjes transformation. The important properties of Stieltjes transformation is discussed by Erdelyi [1] and Sneddon [8].

The integral transform defined by the equation,

$$S[f(x)] = \int_0^\infty \frac{f(x)}{t+x} dx \quad \text{.....(1.5.1)}$$

where t is point of complex plane cut along the negative real axis, is called the Stieltjes transform. It is denoted by S(t) or S [f(x)]. The Stieltjes transform is obtained by first iteration of the Laplace transformation. The theoretical foundation of the Stielties transform have been set down by Widder [11] and Titmarsh [9].

Widder [10], [11] Studies the successive iterates of Stieitjes kernel expressed in the form of the elementary function. Harry Pollard [4] has obtained an inversion formula for the Stieltjes transform. N. Pollard [5] also studied the Stieltjes transform. Widder [10] give some inversion formulae for the Stieltjes transform and generalized Stieltjes transform.

The complex inversion formula [7] for the Stieltjes transform of function f(x) is given by the integral

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{S(t).x^{-t}}{\Gamma t \Gamma (1-t)} . dt \quad \text{....(1.5.2)}$$

where c is positive constant and Γ denotes the gamma function given by (0.2.13). and S(t) is given by (1.5.1). The different kinds of inversion theorem for Stieltjes transformation are given by Kapoor [2A] , Woolcock [13] and Zimering [15].

1.6 Classical Properties of the Stieltjes Transform.

The properties of the Stieltjes transform are discussed by Sneddon [8] and given in together by Erdelyi A. [2]. Out of these properties some of important properties are as follows.

1) Linear Property.

If $f_1(x)$ and $f_2(x)$ are any two functions and a_1, a_2 are any pair of constants then,

$$S[\, a_1 f_1(x) + a_2 f_2(x)\,] = a_1 S[\, f_1(x)] + a_2 S[\, f_2(x)]$$

2) Change of Scale Property.

If f(x) is piecewise continuous function and if

$$S[f(x)] = S(t) \quad \text{then,}$$

$$S[f(ax)] = S(at)$$

a is non-zero constant.

3) Division by x .

If f(x) is piecewise continuous and locally integrable function and

if $S[f(x)] = S(t)$ then,

$$S\left[\frac{f(a/x)}{x}\right] = t^{-1} S(a/t)$$

where a is non-zero constant.

4) Multiplication by x.

If function f(x) is continuous and locally integrable on interval $0 < x < \infty$, and if $S[f(x)] = S(t)$ then,

$$S[x.f(x)] = \int_0^\infty f(x).dx - t.S(t)$$

5) Stieltjes Transform of Derivative.

If function $f'(x)$ is continuous and locally integrable on the interval $0 < x < \infty$ then,

$$S[f'(x)] = -t^{-1} f(0) - S'(t)$$

$S'(t)$ denotes derivative of $S(t)$.

6) Stieltjes Transform of $[f(x) - f(a)] / (x-a)$.

If f(x) is locally integrable function on interval $0 < t < \infty$ and if

$S[f(x)] = S(t)$ then,

$$S[\frac{f(x)-f(a)}{x-a}] = (t+a)^{-1}[(1/2).S(ae^{i\pi}) + (1/2).S(ae^{-i\pi})$$

$$-f(a)\log(t/a) - S(t)]$$

where a is positive constants.

7) Stieltjes transform of $(x+a)^{-1} f(x)$.

If f(x) is locally integrable function on $0 < x < \infty$

then,

$$S[(x+a)^{-1} f(x)] = (t-a)^{-1}[S(a) - S(t)]$$

$$a > 0.$$

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