Chapter - 2

FINITE BOOLEAN ALGEBRAS

2. FINITE BOOLEAN ALGEBRAS

Hypothesis : Throughout this chapter, L denotes a compactly generated multiplicative lattice with 1 compact.

Customarily, complemented distributive bounded lattices are called Boolean algebras (after G. Boole). During the decades 1900-1940 Boolean algebras were intensively studied.

In this chapter, using abstract versions of Nakayama's lemma and Krull intersection theorem, some equivalent conditions for a multiplicative lattice with 1 compact to be a finite Boolean Algebra are studied.

The following lemma provides an important sufficient condition for compact primary elements to be maximal.

Lemma 2.1: Suppose L is reduced and every proper compact element of L is a zero divisor. Then every compact primary element is maximal. [3]

Proof : Let x be a compact primary element s. t. $x \leq y < 1$, for some $y \in L$.

Inasmuch as, L is compactly generated, we have

$y = \bigvee_{\alpha} a_{\alpha}$, where a_{α}'s are compact elements.

To prove x = y, suppose, if possible, $x \neq y$.

i. e., $x < y$.

Hence, $a_{\alpha} \nleq x$, for some α.

$\Rightarrow x < x \vee a_{\alpha} < 1$, for some α.

$\Rightarrow x \vee a_{\alpha}$ is a proper compact element, for some α.

$\Rightarrow \exists\, b \neq 0$ in L such that $(x \vee a_{\alpha})b = 0$, for some α. (by given data)

Again, as L is compactly generated, we have

$b = \vee_{\beta} b_{\beta}$, where b_{β}'s are non-zero compact elements.

$\Rightarrow (x \vee a_{\alpha})(\vee_{\beta} b_{\beta}) = 0.$

$\Rightarrow \vee_{\beta} [(x \vee a_{\alpha})b_{\beta}] = 0.$

$\Rightarrow (x \vee a_{\alpha})b_{\beta} = 0$, for each β.

$\Rightarrow xb_{\beta} \vee a_{\alpha}b_{\beta} = 0$, for each β.

$\Rightarrow xb_{\beta} = 0 = a_{\alpha}b_{\beta}$, for each β.

$\Rightarrow xb_{\beta} = 0$ & $a_{\alpha}b_{\beta} \leq x$, for each β.

$\Rightarrow xb_{\beta}=0$ & $(b_{\beta})^{n_{\beta}} \leq x$, for some $n_{\beta} \in Z^{+}$ & each β. ($\because$ x is primary & $a_{\alpha} \nleq x$)

$\Rightarrow (b_{\beta})^{n_{\beta}+1} \leq xb_{\beta} = 0$, for each β.

$\Rightarrow b_{\beta} = 0$, for each β. ($\because$ L is reduced)

Which contradicts $b_{\beta} \neq 0$, for each β.

Therefore our supposition must be wrong.

Hence, x = y. i. e., every compact primary element is maximal.

A salient feature of the following result is that, a complement of a maximal element must be an atom in L.

Lemma 2.2: Suppose L is reduced & $x \in L$ is maximal s. t. $xy = 0$ ($y \neq 0$), then y is an atom. [3]

Proof : Let $0 \leq z \leq y$, for some $z \in L$.

As x is maximal, we have either $z \leq x$ or $z \nleq x$.

i. e., $z \leq x$ or $x \vee z = 1$.

$\Rightarrow zy \leq xy$ or $y = y(x \vee z) = xy \vee yz$.

$\Rightarrow yz = 0$ or $y = yz \leq z$. ($\because xy = 0$)

$\Rightarrow z^2 \leq zy = 0$ or $y = z$. ($\because z \leq y$)

$\Rightarrow z^2 = 0$ or $y = z$.

$\Rightarrow z = 0$ or $y = z$. ($\because$ L is reduced)

Thus, y is an atom.

With this result, now we can easily turn to an important result. The original proof of the following lemma uses ZORN's lemma.

But it seems that the new proof given by us to this theorem is quite simple.

Lemma 2.3 : L is a finite Boolean algebra with $xy = x \wedge y$ iff every maximal element is a complemented element. [3]

Proof : *If part* : Assume every maximal element is complemented.

We show that, every element of L is complemented.

Let $a \in L$.

Define, $b = \vee\{ x \in L \,/\, ax = 0 \}$. ---------------------------- ①

If $a \vee b < 1$, then $a \vee b \leq m$, for some maximal element $m \in L$.

Hence by assumption, $\exists\, m' \in L$ such that $mm'=0$ & $m \vee m'=1$.

$\Rightarrow (a \vee b)m' \leq mm' = 0$.

$\Rightarrow am' \vee bm' = 0$.

$\Rightarrow am' = 0$ & $bm' = 0$.

$\Rightarrow m' \leq b$ & $bm' = 0$. (by ①)

$\Rightarrow (m')^2 \leq bm' = 0$.

$\Rightarrow (m')^2 = 0$.

$\Rightarrow m' = 0$. ($\because$ by 1.4)

$\Rightarrow m = m \vee 0 = m \vee m' = 1$, a contradiction.

Hence, $a \vee b = 1$.

Obviously, $ab = a(\vee\{ x \,/\, xa = 0 \}) = \vee\{ ax \,/\, xa = 0 \} = 0$.

Thus, a is complemented.

i. e., every element of L is complemented.

$\Rightarrow L = C(L)$ & every element of L is compact. (by 1.6 & 1.7)

$\Rightarrow$ L is a Boolean algebra with, $xy = x \wedge y$. (by 1.9)

Now we show that, L is finite.

By R50, we get $1 = \underset{\alpha}{\vee} a_\alpha$, where a_α's are atoms.

But, 1 is compact.

Hence, $\exists\, a_{\alpha_i} \in \{a_\alpha\}$, (i = 1,---, n) such that $1 = a_{\alpha_1} \vee --- \vee a_{\alpha_n}$

This shows that, L contains a finite number of atoms. Consequently, L is finite, since by R50, every non-zero element of L is a join of atoms.

Thus, L is a finite Boolean algebra with $xy = x \wedge y$.

Only if part : Assume, L is a finite Boolean algebra with $xy = x \wedge y$.

Then evidently, every maximal element of L is complemented.

Now here follows its obvious consequence.

Corollary 2.4 : L is a Boolean algebra with $xy = x \wedge y$ iff every element of L is complemented. [3]

Proof : *If part* : Assume, every element of L is complemented.

Then, every maximal element of L is complemented.

So by lemma 2.3, L is a Boolean algebra with $xy = x \wedge y$.

Only if part : Assume, L is a Boolean algebra with $xy = x \wedge y$.

Then clearly, every element of L is complemented.

The following lemma develops an important equivalent condition for L to be a finite Boolean algebra.

Lemma 2.5 : L is a finite Boolean algebra with $xy = x \wedge y$ iff for every maximal element m, there is some complemented atom $a \in L$ s.t. $a \not\leq m$. [3]

Proof : *If part* : Assume, for every maximal element $m \in L$, there is a complemented atom $a \in L$ such that $a \not\leq m$.

Let m be a maximal element.

Then, $\exists$ an complemented atom $a \in L$ such that $a \not\leq m$.

Claim : $m = a'$, where a' is a complement of a.

Since every maximal element is prime & $aa' = 0 \leq m$ with $a \not\leq m$, we have $a' \leq m$. -- ①

But, $0 \leq am \leq a$ & a is an atom.

$\Rightarrow am = 0$ or $am = a$.

$\Rightarrow am = 0$. ($\because am \leq a \wedge m < a$, as $a \not\leq m$)

$\Rightarrow m = m.1 = m(a \vee a') = ma \vee ma' = ma' \leq a'$.

$\Rightarrow m = a'$. (by ①)

Thus, every maximal element is complemented.

Hence by lemma 2.3, L is a finite Boolean algebra with $xy = x \wedge y$.

Only if part: Assume, L is a finite Boolean algebra with $xy = x \wedge y$.

Let m be any maximal element of L.

As L is a Boolean algebra, m is complemented.

Hence, $\exists\ a \in L$ s. t. $am = 0$ and $a \vee m = 1$.

i. e., $am = 0$ & $a \neq 0$ with $a \nleq m$.

Thus, a is a complemented atom such that $a \nleq m$. (by 2.2)

Here now we have another useful theorem which gives an equivalent condition for the lattice L to be a finite Boolean algebra.

The following theorem is little bit proved in different manner.

Theorem 2.6 : L is a finite Boolean algebra with $xy = x \wedge y$ iff L satisfies the conditions : [3]

(i) L is reduced,

(ii) Every proper compact element of L is a zero divisor,

(iii) 0 is the product of a finite number of compact primary elements.

Proof : *If part* : Assume, L satisfies the given conditions.

Then we have, $0 = a_1 a_2 \ldots a_n$,where a_i's are compact primary elements.

But by 2.1, each a_i is a compact maximal element.

So by the condition (ii), for each $a_i \ \exists\ b_i \neq 0$ in L such that $a_i b_i = 0$

Hence, each b_i is an atom. (by 2.2)

Clearly, $b_i \nleq a_i$ ($\because b_i \leq a_i \Rightarrow b_i^2 \leq a_i b_i = 0 \Rightarrow b_i = 0$, by (i))

$\Rightarrow a_i \vee b_i = 1$, for each i, as a_i is maximal.

$\Rightarrow$ each b_i is a complemented atom.

Of course, these are the only maximal elements of L. Since, if there is any another maximal element $m_0 \in L$ s. t. $m_0 \neq a_i$, for each i = 1,2,---, n, then $a_1 a_2 \ldots a_n = 0 \leq m_0$ & hence by R1 & R28, we have $a_i = m_0$, a contradiction.

Thus for each maximal element $a_i \in L$, there is some complemented atom $b_i \in L$ s. t. $b_i \nleq a_i$.

Consequently, by 2.5, L is finite Boolean algebra with $xy = x \wedge y$.

Only if part : Assume, L is a finite Boolean algebra with $xy = x \wedge y$.---①

Hence, $x^2 = x \wedge x = x$, for each $x \in L$.

Thus, every element of L is idempotent.

$\Rightarrow$ Only nilpotent element is zero.

$\Rightarrow$ L is reduced. -- ②

Further, let $x \in L$ be a proper compact element.

Then, $\exists\ y \in L$ s. t. $xy = 0$ & $x \vee y = 1$. (by ①)

Clearly, $y \neq 0$, since if $y = 0$, then $1 = x \vee y = x \vee 0 = x$, a contradiction.

Thus, for any compact element $x \in L$, $\exists\ y \neq 0$ in L s. t. $xy = 0$.

i. e., every proper compact element of L is a zero divisor.

As L is finite, L satisfies DCC & hence r^* is nilpotent. (by R35)

But L is reduced. Hence $r^* = 0$.

i. e., $0 = \wedge_i \{m_i \in L \mid m_i$ is a maximal element$\}$

$= \bigwedge_{i=1}^{n} \{m_i \in L \mid m_i$ is a maximal element$\}$ ($\because$ L is finite)

$= m_1 m_2 \ldots. m_n$ ($\because xy = x \wedge y$)

Of course, as L is finite, each m_i is compact & hence by R28 & R24, each m_i is compact primary element.

Thus, 0 is the finite product of compact primary elements.

Interestingly, the following theorem focuses on the fact that, if L is a Boolean algebra, then L must be a finite lattice. (i. e., L has a finite number of elements).

Theorem 2.7 : The following statements on L are equivalent :

(i) L is a finite Boolean algebra with $xy = x \wedge y$.

(ii) L is a Boolean algebra with $xy = x \wedge y$.

(iii) L is reduced and every proper element of L is a zero divisor. [3]

Proof : *(i) $\Rightarrow$ (ii)* is obvious.

(ii) $\Rightarrow$ (iii) : Assume, L is a Boolean algebra with $xy = x \wedge y$.

Then, $x^2 = x \wedge x = x$, for each $x \in L$.

Thus, every element of L is idempotent.

i. e., only nilpotent element is zero.

Thus, L is reduced.

Now, let $x \in L$ be a proper element.

Then, $\exists\ y \in L$ s. t. $xy = 0$ & $x \vee y = 1$.

Clearly $y \neq 0$, since if $y = 0$, then $1 = x \vee y = x \vee 0 = x$, a contradiction.

Thus, every proper element of L is a zero divisor.

(iii) $\Rightarrow$ *(i)*: Assume, L is reduced & every proper element is a zero divisor.

Let $a \in L$.

Define, $b = \vee\{x \in L \ / \ ax = 0\}$. ---------------- ①

Then, $ab = a(\vee\{x \in L \ / \ ax = 0\}) = \vee\{\ ax \in L \ / \ ax = 0\ \} = 0$.

We now claim that, $a \vee b = 1$.

Suppose, if possible, $a \vee b < 1$.

Then by assumption, $\exists\ c \neq 0$ in L s. t. $(a \vee b)c = 0$.

$\Rightarrow \exists\ c \neq 0$ s. t. $ac \vee bc = 0$.

$\Rightarrow \exists\ c \neq 0$ s. t. $ac = 0 = bc$.

$\Rightarrow \exists\ c \neq 0$ s. t. $c \leq b$ & $bc = 0$. (by ①)

$\Rightarrow \exists\ c \neq 0$ s. t. $c^2 \leq bc = 0$.

$\Rightarrow \exists\ c \neq 0$ s. t. $c = 0$. ($\because$ L is reduced)

Which is a contradiction .

Thus, $ab = 0$ and $a \vee b = 1$.

i. e., a is complemented.

Thus, every element of L is complemented.

$\Rightarrow$ every maximal element of L is complemented.

$\Rightarrow$ L is a finite Boolean algebra with $xy = x \wedge y$. (by 2.3)

Theorem 2.8 : If S is a semisimple lattice with 1 compact and satisfies the descending chain condition (DCC), S is a finite Boolean algebra. [2]

Proof : By 1.19, S contains only a finite number of maximal elements, say, m_1, ---, m_n such that $\bigwedge_{i=1}^{n} m_i = 0$.

Let $a \in S$.

Define, $a^* = \wedge\{ m_i \in S \,/\, a \nleq m_i \}$.

Then, $a \wedge a^* = 0$

To prove, $a \vee a^* = 1$, suppose, if possible, $a \vee a^* < 1$.

Then $a \vee a^* \leq m_o$, for some maximal element $m_o \in S$.

$\Rightarrow a \leq m_o \;\&\; a^* \leq m_o$

$\Rightarrow a \leq m_o \;\&\; \wedge\{ m_i \in S \,/\, a \nleq m_i \} \leq m_o$

$\Rightarrow a \leq m_o \;\&\; m_i \leq m_o$, for some i, such that $a \nleq m_i$ (by R28, R1)

$\Rightarrow a \leq m_o \;\&\; m_i = m_o$, for some i, such that $a \nleq m_i$ ($\because$ m_i is maximal)

$\Rightarrow a \leq m_o \;\&\; a \nleq m_o$

Which is a contradiction.

Hence, $a \vee a^* = 1$.

i. e., a^* is a complement of a, in S.

Thus, every element of S is complemented.

$\Rightarrow$ every element of S is compact s. t. $S = C(S)$. (by 1.6)

$\Rightarrow$ S is a compactly generated Boolean algebra with $xy = x \wedge y$. (by 1.9)

$\Rightarrow$ S is a finite Boolean algebra. (by 2.7)

This result leads us to understand the following.

It is well-known that, if R is a commutative ring with identity, then L(R) is a Boolean algebra iff R is reduced and satisfies the descending chain condition. The coming theorem offers just an abstract version of this result.

Theorem 2.9 : Suppose S is a join principally generated lattice with 1 compact. S is finite Boolean algebra iff S is reduced & satisfies DCC. [2]

Proof : *If Part* : Assume, S is reduced and satisfies DCC.

By R35, r^* is nilpotent.

$\Rightarrow r^* = 0$. ($\because$ S is reduced)

$\Rightarrow$ S is semisimple.

Thus, S is semisimple and satisfying DCC.

Consequently, S is a finite Boolean algebra. (by 2.8)

Only if Part : Assume, S is a finite Boolean algebra.

Then, every element of S is complemented.

⇒ every element of S is idempotent. (by 1.4)

⇒ 0 is the only nilpotent.

⇒ S is reduced.

To prove, S satisfies DCC, suppose, if possible, S does not.

Then, ∃ a descending chain of distinct elements $\{a_i\} \subseteq S$ s. t.

$a_1 > a_2 > a_3 >$ ----- which does not terminate anywhere.

Then clearly, $\{a_i\}$ is an infinite set in S.

Which contradicts the finiteness of S.

Hence, S satisfies DCC.

R. P. Dilworth [9] *proved that, a Noether lattice in which multiplication is the meet operation, is a finite Boolean algebra. K P. Bogart* [6] *has also proved, if a Noether lattice in which every maximal element is idempotent, then it is Boolean algebra. A generalization of these two results is given by Anderson and others* [2]. *Here it is.*

Theorem 2.10 : Suppose S is a join principally generated multiplicative lattice with 1 compact. If every maximal element of S is a compact idempotent element, then S is a finite Boolean algebra. [2]

Proof : Let m be a maximal element of S.

Then, m is a compact idempotent element.

⇒ m is a finite join of join principal elements and $m^2 = m$.

⇒ $m \vee (0:m) = 1$. (by 1.12)

Thus, for every maximal element $m \in S$, $m \vee 0:m = 1$. ------------ ①

We prove that, every element of S is complemented.

Suppose, if possible, some elements are not complemented. -----②

Define, $\mathcal{F} = \{ x \in S \ / \ x$ is not complemented $\}$.

Then by ②, $\mathcal{F} \neq \varnothing$.

Let $\{a_i\}$ be an ascending chain in $\mathcal{F}$. ---------------------------- ③

If $\bigvee_i a_i$ is complemented, then by 1.4, $\bigvee_i a_i$ is compact.

Hence, $\bigvee_i a_i \le \bigvee_i a_i \Rightarrow \bigvee_i a_i \le \bigvee_{j=1}^{n} a_j$, for some $a_j \in \{a_i\}$.

$\Rightarrow \bigvee_i a_i = \bigvee_{j=1}^{n} a_j$ ------------------------------ ④

But by ③, $\exists\ a_m \in \{a_j\}_{j=1}^{n}$ such that $a_j \le a_m$, for each $j = 1, \text{---}, n$.

$\Rightarrow \bigvee_i a_i = \bigvee_{j=1}^{n} a_j = a_m$ (by ④)

$\Rightarrow \bigvee_i a_i$ is not complemented, which is a contradiction. ($\because a_m \in \mathcal{F}$)

Thus, every chain in $\mathcal{F}$ has an upper bound in $\mathcal{F}$.

Hence, by Zorn's lemma, $\mathcal{F}$ contains a maximal element, say $a \in \mathcal{F}$.

If $0{:}a \not\le a$, then $a < a \vee 0{:}a$ & hence $a \vee 0{:}a \notin \mathcal{F}$.

$\Rightarrow a \vee 0{:}a$ is complemented.

$\Rightarrow \exists\ b \in S$ such that $(a \vee 0{:}a)b = 0$ & $(a \vee 0{:}a) \vee b = 1$.

$\Rightarrow ab \vee (0{:}a)b = 0$ & $a \vee 0{:}a \vee b = 1$.

$\Rightarrow ab = 0 = (0{:}a)b$ & $a \vee 0{:}a \vee b = 1$.

$\Rightarrow a(0{:}a \vee b) = a(0{:}a) \vee ab = 0 \vee 0 = 0$ & $a \vee (0{:}a \vee b) = 1$.

$\Rightarrow$ a is complemented, which is impossible.

Hence, $0{:}a \le a$.

As a is not complemented, $a < 1$.

Hence $a \le m_0$, for some maximal element $m_0 \in S$.

$\Rightarrow 0{:}m_0 \le 0{:}a \le a \le m_0$.

$\Rightarrow m_0 \vee 0{:}m_0 = m_0 \ne 1$.

Which is a contradiction to ①.

Hence every element of S is a complemented element.

$\Rightarrow S = C(S)$.

$\Rightarrow$ S is a Boolean algebra with $xy = x \wedge y$. (by 1.9)

$\Rightarrow$ S is a finite Boolean algebra with $xy = x \wedge y$. (by 2.7)