CHAPTER – I

DEFINITIONS AND TERMINOLOGY

ABSTRACT

This chapter contains some definitions and statements of the results, which we need in the course of investigation. The relevant references are given at the end of the chapter.

DEFINITIONS AND TERMINOLOGY

Definition 1. Let $U = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \}$.

Definition 2. A complex valued function f is holomorphic or analytic in a domain D in the complex plane $\mathbb{C}$, if it has uniquely determined derivative at each point of D.

Note. We shall use words holomorphic and analytic as synonymous.

Definition 3. A function f is holomorphic in a domain D is said to be univalent in D if $f(z_1) = f(z_2)$ implies $z_1 = z_2$ for all z_1 , z_2 in D.

In other words, a single valued function f is said to be univalent (or schlicht) in a domain $D \subset \mathbb{C}$, ($\mathbb{C}$ - is a complex plane) if it never takes the same value twice, that is $f(z_1) \neq f(z_2)$ for all points z_1 , z_2 in D with $z_1 \neq z_2$.

Definition 4. The function f is said to locally univalent at a point $z_0 \in D$ if it is univalent in some neighborhood of z_0 .

Remark . A holomorphic univalent function f is a conformal mapping because of its angle preserving property.

Statement . Let D be simply connected region which is not whole plane $\mathbb{C}$ and let $a \in D$, then there is a unique analytic function f having following properties

i) $f(a) = 0$, $f'(a) > 0$.

ii) f is one one .

iii) $f(D) = \{z : |z| < 1\}$.

In other words among the simply connected regions there are only two equivalence classes; one consisting of $\mathbb{C}$ alone and other containing all proper simply connected regions.

Definition 5. Let $H(U)$ denote the class of functions of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$.

Definition 6. Let S denote the subclass of $H(U)$, which consists of functions which are univalent in U and satisfy conditions of normalization $f(0) = 0$, $f'(0) = 1$.

Remark . The leading example of the class S is the Koebe function, defined as

$$K(z) = z(1-z)^{-2} = z + 2z^2 + 3z^3 + \dots .$$

The Koebe function maps the disk U onto the entire plane minus the part of negative real axis from -1/4 to infinity . This function plays extremal role for the class S. It maximizes $|a_n| \leq n$ for every n.

Statement . (De Branges' Theorem). The coefficient of each function $f \in S$ satisfy $|a_n| \leq n$ (for $n = 2,3,\ldots$), strict inequality holds for all n unless f is the Koebe function one of its rotations.

Definition 7. A set $E \subset \mathbb{C}$ is said to be starlike with respect to $w_0 \in E$ if the line segment joining w_0 to every point $w \in E$ lies entirely in E. We note that starlike with respect to the origin will be referred as simply starlike.

Definition 8. A starlike function is a conformal mapping of the unit disk onto a domain starlike with respect to the origin.

Definition 9. Let S^* denote the subclass of S , whose members transform every disk $|z| \leq \rho$, $0 < \rho < 1$, onto a starlike domain.

The analytic description of starlike functions is given by the following statement.

Statement . Let f be holomorphic in a domain $D \subset \mathbb{C}$, with $f(0) = 0$, $f'(0) = 1$. Then $f \in S^*$ if and only if $\dfrac{z f'(z)}{f(z)} \in P$, P denoting the class of all functions ψ , which are holomorphic and having positive real part in D, with $\psi(0) = 1$.

Definition 10. Let f be holomorphic at $z = 0$ and satisfying the condition of normalizations. Then the radius of univalence is defined to be the largest value of r such that f is holomorphic and univalent for $|z| < r$.

Definition 11. Let f be normalized holomorphic function at $z = 0$, let λ be a real number such that $0 \leq \lambda < 1$ .Then f is said to be starlike of order λ if,

$$\mathrm{Re}\left\{\frac{z f'(z)}{f(z)}\right\} > \lambda .$$

Definition 12. Let f be normalized holomorphic function at $z = 0$, let λ be a real number such that $0 \leq \lambda < 1$. We define the radius of starlikeness of order λ , denoted by S_λ , to be the largest value of r such that f is holomorphic and

$$\mathrm{Re}\left\{\frac{z f'(z)}{f(z)}\right\} > \lambda \quad , \quad \text{for} \quad |z| < r \ .$$

Definition 13. A set $E \subset \mathbb{C}$ is said to be convex if it starlike with respect to each of its points, that is if the linear segment joining any two points of E lies entirely in E.

Definition 14. A convex function is a conformal mapping of the unit disk onto a convex domain.

Definition 15. Let C denote the subclass of S, whose members map every disk $|z| \leq \rho$, $0 < \rho < 1$, onto a convex domain.

The analytic description of convex functions is given by the following statement.

Statement. Let f be holomorphic in a domain $D \subset \mathbb{C}$, with $f(0) = 0,\ f'(0) = 1$. Then $f \in C$ if and only if $\left(1+\frac{z f''(z)}{f'(z)}\right) \in P$, P denoting the class of all functions ψ , which are holomorphic and having positive real part in D, with $\psi(0) = 1$.

Definition 16. Let f be normalized holomorphic function at $z = 0$, let λ be a real number such that $0 \le \lambda < 1$ .Then f is said to be convex of order λ if

$$\operatorname{Re}\left\{1 + \frac{z f''(z)}{f'(z)}\right\} > \lambda .$$

Definition 17. Let f be normalized holomorphic function at $z = 0$, let λ be a real number such that $0 \le \lambda < 1$. We define the radius of convexity of order λ , denoted by C_λ , to be the largest value of r such that f is holomorphic and satisfying

$$\operatorname{Re}\left\{1 + \frac{z f''(z)}{f'(z)}\right\} > \lambda \ , \quad \text{for} \quad |z| < r \ .$$

A close analytic connection between convex and starlike mapping was first observed by Alexander [1] in 1915.

Statement. Let f be holomorphic in D, with $f(0) = 0,\ f'(0) = 1$.Then $f \in C$ if and only if $z f'(z) \in S^*$.

We now turn to an interesting subclass of S which contains S^* and has a simple geometric description. This is the class of close-to- convex functions, introduced by Kaplan [4] in 1952.

Definition 18. A function f analytic in the unit disk U is said to be close-to-convex if there is a convex function g such that

$$\operatorname{Re}\left\{\frac{f'(z)}{g'(z)}\right\} > 0 \quad \text{for all} \quad z \in U.$$

We shall denote by K the class of close-to- convex functions f normalized by the usual conditions $f(0) = 0,\ f'(0) = 1$. We note that f is not required a priori to be univalent and also the associated function g need not be normalized .

These remarks are summarized by the chain of proper inclusion relations as

$$C \subset S^* \subset K \subset S.$$

Definition 19. Let f be normalized holomorphic function at $z = 0$. We define the radius of close-to-convexity to be the largest value of r such that f is holomorphic and close-to-convex, for $|z| < r$.

Remark . Every convex function is obviously close-to- convex . More generally ,every starlike function is close-to- convex.

Close-to-convex function can be stamped by a geometric condition is similar to defining properties of convex and starlike functions. Let f be holomorphic in D and let C_r denote the image of the unit circle $|z| = r$, under the mapping f, lying between 0 and 1. Then roughly speaking , f is close-to-convex if and only if none of the curves C_r makes a "reverse hairpin turn ". In this connection , Kaplan [4] has stated the following definition of close-to-convex function , which is known as Kaplan's theorem.

Kaplan's Theorem . Let f be analytic and locally univalent in D , then f is close-to-convex if and only if

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left[1 + \frac{z f''(z)}{f'(z)} \right] d\theta > -\pi \quad , \quad z = r e^{i\theta}$$

for each r, $0 < r < 1$ and for each pair of real numbers θ_1 and θ_2 , with $\theta_1 < \theta_2$.

Remark . A function satisfying either the close-to-convex condition, the starlike condition or the convex condition is univalent.

There is another natural generalization of starlikeness which again leads to useful criteria for univalence. We refer to the class of spiral like functions introduced by Spacek [9] in 1933.

A logarithmic spiral is curve in the complex plane of the form

$$w = w_0 \, e^{-\lambda t} \quad , \quad (-\infty < t < \infty).$$

Where w_0 and λ are complex constants with $w_0 \neq 0$, $\mathrm{Re}\{\lambda\} \neq 0$. There is no loss of generality in assuming that $\lambda = e^{i\alpha}$ with $(-\pi/2 < \alpha < \pi/2)$. The curve is then called as α - spiral.

Definition 20. A domain D containing the origin is said to be α -spiral like if for each point $w_0 \neq 0$ in D , the arc of the α –spiral from w_0 to the origin lies entirely in D .

Definition 21. A function f holomorphic and univalent in the unit disk with $f(0) = 0$ is said to be α -spiral like function , if its range is α -spiral like.

Remark . The 0-spiral like functions are simply the starlike functions.

Definition 22. Let f be holomorphic in D , with $f(0) = 0$, $f'(0) \neq 0$ and $f(z) \neq 0$ for $0 < |z| < 1$. Let α lie in the interval $-\pi/2 < \alpha < \pi/2$ .Then f is α -spiral like if and only if

$$\operatorname{Re}\left\{ e^{i\alpha} \frac{z f'(z)}{f(z)} \right\} > 0 \quad , \quad |z| < 1.$$

Now we define an operator defined by Salagean [8] which we have used to define new subclasses. Also we define the concept of δ -neighborhoods introduced by St. Ruscheweyh [7]. We denote N_0, the set of all non-negative integers ($N_0 = (0,1,2,\ldots)$).

Definition 23 [8]. We define the operator $D^n : H(U) \to H(U)$, $n \in N_0$, by

$$D^0 f(z) = f(z).$$

$$D^1 f(z) = zf'(z).$$

In general for $n \geq 2$

$$D^n f(z) = D(D^{n-1} f(z)).$$

Definition 24 [7]. We define the δ-neighborhood of a function $f \in S$ by

$$N_\delta(f) = \{ g \in S : g(z) = z + \sum_{j=2}^{\infty} b_j z^j \quad and \quad \sum_{j=2}^{\infty} j \left| a_j - b_j \right| \leq \delta \}.$$

In particular, for the identity function

$$e(z) = z,$$

we immediately have

$$N_\delta(e) = \{ g \in S : g(z) = z - \sum_{j=2}^{\infty} b_j z^j \quad and \quad \sum_{j=2}^{\infty} j \left| b_j \right| \leq \delta \}.$$

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