CHAPTER – II

Properties of univalent functions that are in $S_n(\alpha, \beta, \gamma)$

ABSTRACT

In this chapter, we introduce a new subfamily $S_n(\alpha, \beta, \gamma)$ of class S, of normalized univalent functions f on the unit disk $U = \{ z : |z| < 1 \}$, having Taylor's series expansion of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j \ .$$

The main theme of the present chapter is to study various properties of functions in $S_n(\alpha, \beta, \gamma)$, having functions with negative coefficients. We characterize the class and obtain distortion theorem, radius of convexity, closure properties and extreme points for the class $S_n(\alpha, \beta, \gamma)$.

In [7] Salagean obtained inclusion relations for the class $S_n(\alpha, \beta, 1)$ with negative coefficients, we propose to generalize the results for the class $S_n(\alpha, \beta, \gamma)$ with negative coefficients.

Lastly, we also try to generalize the results in Salagean [8], which gives convolution theorems for the class $S_n(\alpha, \beta, \gamma)$ with negative coefficients.

1. INTRODUCTION

We introduce a new subclass of S, that can be treated as a generalization of starlike functions.

Definition . Let $\alpha \in [0,1)$, $\beta \in (0,1]$, $\gamma \in (1/2,1]$ and let $n \in N_0$, we define the class $S_n(\alpha,\beta,\gamma)$ of n-starlike functions of order α, type β and γ by

$$S_n(\alpha,\beta,\gamma)=\{\, f \in H(U): f(0)=f'(0)-1=0 \text{ and } |J_n(f,\alpha,\gamma;z)|<\beta \ , \ z\in U \,\}$$

where

$$J_n(f,\alpha,\gamma;z) = \frac{\left(\dfrac{D^{n+1}f(z)}{D^n f(z)}-1\right)}{2\gamma\left(\dfrac{D^{n+1}f(z)}{D^n f(z)}-\alpha\right)-\left(\dfrac{D^{n+1}f(z)}{D^n f(z)}-1\right)} \ , \quad z \in U .$$

We note that $S_0(0,1,1)$ is the class of starlike functions in U , $S_0(\alpha,1,1)$ is class of starlike functions of order α and $S_1(\alpha,1,1)$ is class of convex functions of order α. The class $S_0(\alpha,\beta,\gamma)$ is studied by Kulkarni [3] and the classes $S_n(\alpha,1,1)$ and $S_n(\alpha,\beta,1)$ are studied and introduced by Salagean [6,7] .

Motivated by Silverman [9] and Salagean [8], we have introduced subclass $S_n(\alpha,\beta,\gamma)$ with negative coefficients.

Let T denote the subclass of S consisting of analytic functions whose non-zero coefficients, from the second on, are negative; that is, an univalent holomorphic function f is in T if and only if it can be expressed in the form

$$f(z) = z - \sum_{j=2}^{\infty} a_j z^j, \quad a_j \geq 0, \ j = 2, 3, \ldots \ . \tag{2.1.1}$$

We define the class $T_n(\alpha, \beta, \gamma)$ by

$$T_n(\alpha, \beta, \gamma) = S_n(\alpha, \beta, \gamma) \cap T \tag{2.1.2}$$

and obtain several interesting results for the class $T_n(\alpha, \beta, \gamma)$ in the line of Silverman [9], Salagean [7], Kulkarni [3]. We note that the class $T_0(\alpha, \beta, \gamma)$ is studied by Kulkarni [3] and the classes $T_n(\alpha, 1, 1)$ and $T_n(\alpha, \beta, 1)$ are introduced and studied by Salagean [6,7].

2. CHARACTERIZATION OF CLASS $T_n(\alpha, \beta, \gamma)$

First we state the characterization theorem which completely characterizes the member of class $T_n(\alpha, \beta, \gamma)$.

Theorem 1. *Let* $\alpha \in [0,1)$, $\beta \in (0,1]$, $\gamma \in (1/2,1]$, *and let* $n \in N_0$, *the function* f *of the form* (1.1) *is in* $T_n(\alpha, \beta, \gamma)$ *if and only if*

$$\sum_{j=2}^{\infty} j^n [\, j - 1 + \beta(1 - j + 2\gamma j - 2\gamma\alpha)\,] a_j \leq 2\beta\gamma(1-\alpha) \quad . \tag{2.2.1}$$

The result (2.2.1) is sharp.

Proof. We suppose that (2.2.1) holds. Then we have

$$\left| J_n(f,\alpha,\gamma;z) \right| = \left| \frac{\left(\frac{D^{n+1} f(z)}{D^n f(z)} - 1 \right)}{2\gamma\left(\frac{D^{n+1} f(z)}{D^n f(z)} - \alpha \right) - \left(\frac{D^{n+1} f(z)}{D^n f(z)} - 1 \right)} \right|$$

$$= \left| \frac{\sum_{j=2}^{\infty} j^n \, a_j \, (1-j) \, z^j}{2\gamma (1-\alpha) z - \sum_{j=2}^{\infty} j^n \, a_j \, (1-j+2\gamma j - 2\gamma\alpha) \, z^j} \right| .$$

Let $|z| = 1$, then

$$\left| \sum_{j=2}^{\infty} j^r \, (j-1) \, a_j \, z^j \right| - \beta \left| 2\gamma(1-\alpha) z - \sum_{j=2}^{\infty} j^n \, (1-j+2\gamma j - 2\gamma\alpha) \, a_j \, z^j \right|$$

$$\leq \sum_{j=2}^{\infty} j^n \left[(j-1) + \beta (1-j+2\gamma j - 2\gamma\alpha) \right] a_j - 2\beta\gamma(1-\alpha) \leq 0$$

where we used (2.2.1).

From the last inequality we deduce

$$\left| J_n(f,\alpha,\gamma;z) \right| \leq \beta, \qquad |z| = 1.$$

Hence $\qquad \left| J_n(f,\alpha,\gamma;z) \right| < \beta, \quad z \in U$ and $\quad f \in T_n(\alpha,\beta,\gamma).$

Conversely, we assume that $f \in T_n(\alpha,\beta,\gamma)$. Then

$$\left| J_n(f,\alpha,\gamma;z) \right| < \beta, \qquad z \in U. \tag{2.2.2}$$

For $z \in [0,1)$ the inequality (2.2.2) can be written

$$-\beta < \frac{\sum_{j=2}^{\infty} j^n (j-1) a_j z^{j-1}}{2\gamma(1-\alpha) - \sum_{j=2}^{\infty} j^n (1-j+2\gamma j-2\gamma\alpha) a_j z^{j-1}} < \beta . \quad (2.2.3)$$

We note that $E(z) = 2\gamma(1-\alpha) - \sum_{j=2}^{\infty} j^n (1-j+2\gamma j-2\gamma\alpha) a_j z^{j-1} > 0$, $z \in [0,1)$, because $E(z) \neq 0$ for $z \in [0,1)$ and $E(0) = 2\gamma(1-\alpha) > 0$. Upon clearing the denominator in (2.2.3) and letting $z \to 1$ through real values, we deduce

$$\sum_{j=2}^{\infty} j^n (j-1) a_j \leq 2\beta\gamma(1-\alpha) - \beta \sum_{j=2}^{\infty} j^n (1-j+2\gamma j-2\gamma\alpha) a_j .$$

Thus

$$\sum_{j=2}^{\infty} j^n [\, j-1+\beta(1-j+2\gamma j-2\gamma\alpha)\,] a_j \leq 2\beta\gamma(1-\alpha) .$$

The extremal functions are

$$f_j(z) = z - \frac{2\gamma\beta(1-\alpha)}{j^n [\, j-1+\beta(1-j+2\gamma j-2\gamma\alpha)]} z^j , \qquad j = 2,3,\ldots . \quad (2.2.4)$$

Corollary 1. *If* $f \in T_n(\alpha, \beta, \gamma)$, *then*

$$a_j \leq \frac{2\gamma\beta(1-\alpha)}{j^n [\, j-1+\beta(1-j+2\gamma j-2\gamma\alpha)]}, \qquad j = 2,3,\ldots .$$

The result is sharp and the extremal functions are given by (2.2.4).

We state following particular cases for Theorem 1.

Corollary 2. *A function* f *of the form (2.1.1) is starlike of order* α, *if and only if*

$$\sum_{j=2}^{\infty} (j-\alpha) a_j \leq (1-\alpha) .$$

The result is sharp. This result is due to Silverman [9].

Next is the similar characterization for the class of univalent functions with negative coefficients, studied by Kulkarni [3].

Corollary 3. *A function f of the form (1.1) is in $T_0(\alpha, \beta, \gamma)$, if and only if*

$$\sum_{j=2}^{\infty} [\, j-1+\beta(1-j+2\gamma j-2\gamma\alpha)\,]a_j \le 2\beta\gamma(1-\alpha).$$

This result is sharp.

In the same vein, we also have a corresponding result for univalent holomorphic functions studied by Salagean [7] and Gupta and Jain [1] respectively.

Corollary 4. *A function f of the form (1.1) is in $T_n(\alpha, \beta, 1)$, if and only if*

$$\sum_{j=2}^{\infty} j^n [\, j-1+\beta(1+j-2\alpha)\,]a_j \le 2\beta(1-\alpha).$$

This result is sharp.

Corollary 5. *A function f of the form (1.1) is in $T_0(\alpha, \beta, 1)$, if and only if*

$$\sum_{j=2}^{\infty} [\, j-1+\beta(1+j-2\alpha)\,]a_j \le 2\beta(1-\alpha).$$

This result is sharp.

Next we obtain a theorem which supplies the extreme point of the class $T_n(\alpha, \beta, \gamma)$.

Theorem 2. *Let*

$$f_1(z) = z, \qquad and$$

$$f_j(z) = z - \frac{2\gamma\beta(1-\alpha)}{j^n[\,j-1+\beta(1-j+2\gamma j-2\gamma\alpha)\,]}\, z^j. \tag{2.2.5}$$

Then $f(z) \in T_n(\alpha,\beta,\gamma)$, *if and only if it can be expressed in the form*

$$f(z) = \lambda_1 f_1(z) + \sum_{j=2}^{\infty} \lambda_j f_j(z), \tag{2.2.6}$$

where

$$\lambda_j \geq 0 \quad (j = 1,2,3,\ldots) \quad and \quad \lambda_1 + \sum_{j=2}^{\infty} \lambda_j = 1. \tag{2.2.7}$$

Proof. Suppose that

$$f(z) = \lambda_1 f_1(z) + \sum_{j=2}^{\infty} \lambda_j f_j(z)$$

$$= z - \sum_{j=2}^{\infty} \lambda_j \frac{2\gamma\beta(1-\alpha)}{j^n[\,j-1+\beta(1-j+2\gamma j-2\gamma\alpha)\,]}\, z^j.$$

Since

$$\sum_{j=2}^{\infty} j^n[\,j-1+\beta(1-j+2\gamma j-2\gamma\alpha)]\,\frac{2\gamma\beta(1-\alpha)}{j^n[\,j-1+\beta(1-j+2\gamma j-2\gamma\alpha)\,]}\,\lambda_j$$

$$= 2\beta\gamma(1-\alpha)\sum_{j=2}^{\infty} \lambda_j$$

$$\leq 2\beta\gamma(1-\alpha) \quad ,$$

by Theorem 1. $f \in T_n(\alpha, \beta, \gamma)$.

Conversely, we suppose that $f \in T_n(\alpha, \beta, \gamma)$.

Since

$$a_j \le \frac{2\gamma\beta(1-\alpha)}{j^n[\,j-1+\beta(1-j+2\gamma j-2\gamma\alpha)\,]} \qquad (j = 1,2,3,\ldots) \quad .$$

Setting

$$\lambda_j = \frac{j^n[\,j-1+\beta(1-j+2\gamma j-2\gamma\alpha)\,]}{2\gamma\beta(1-\alpha)}\, a_j \ ,$$

and

$$\lambda_1 = 1 - \sum_{j=2}^{\infty} \lambda_j \ .$$

Then we have

$$f(z) = \lambda_1 f_1(z) + \sum_{j=2}^{\infty} \lambda_j f_j(z) \quad .$$

This completes the proof of Theorem 2.

Corollary 1. The extreme points of $T_n(\alpha, \beta, \gamma)$ are the functions

$$f_1(z) = z$$

and

$$f_j(z) = z - \frac{2\gamma\beta(1-\alpha)}{j^n[\,j-1+\beta(1-j+2\gamma j-2\gamma\alpha)\,]}\, z^j \qquad (j = 1,2,3,\ldots) \ .$$

Now we give the following particular cases for above theorem.

Corollary 2. The extreme points of $T_0(\alpha, 1, 1)$ are the functions

$f_1(z) = z$ *and*

$$f_j(z) = z - \frac{(1-\alpha)}{(j-\alpha)} z^j \qquad (j = 1, 2, 3, \ldots) .$$

Which same as studied by Silverman [9].

Corollary 3. The extreme points of $T_0(\alpha, \beta, \gamma)$ are the functions

$$f_1(z) = z$$

and

$$f_j(z) = z - \frac{2\gamma\beta(1-\alpha)}{[j-1+\beta(1-j+2\gamma j-2\gamma\alpha)]} z^j \qquad (j = 1, 2, 3, \ldots) .$$

This is due to the class studied by Kulkarni [3].

Lastly, we also state the corollaries for the class of the functions introduced by Salagean [7] , Jain and Gupta [1], respectively.

Corollary 4. The extreme points of $T_n(\alpha, \beta, 1)$ are the functions

$$f_1(z) = z$$

and

$$f_j(z) = z - \frac{2\beta(1-\alpha)}{j^n[j-1+\beta(1+j-2\alpha)]} z^j \quad (j = 1, 2, 3, \ldots) .$$

Corollary 5. The extreme points of $T_0(\alpha, \beta, 1)$ are the functions

$$f_1(z) = z$$

and

$$f_j(z) = z - \frac{2\beta(1-\alpha)}{[j-1+\beta(1+j-2\alpha)]} z^j \qquad (j = 1, 2, 3, \ldots) .$$

3. SOME PROPERTIES OF CLASS $T_n(\alpha, \beta, \gamma)$

Now we prove some properties of class $T_n(\alpha, \beta, \gamma)$, like distortion theorem , radius of convexity and closure theorems.

Theorem 3. *Let* $\alpha \in [0,1)$, $\beta \in (0,1]$, $\gamma \in (1/2,1]$ *and let* $n \in N_0$, *if* $f \in T_n(\alpha, \beta, \gamma)$, *then for* $0 < |z| = r < 1$, *we have*

$$r - \frac{\gamma\beta(1-\alpha)}{2^{n-1}[1+\beta(4\gamma-2\gamma\alpha-1)]} r^2 \le |f(z)| \le r + \frac{\gamma\beta(1-\alpha)}{2^{n-1}[1+\beta(4\gamma-2\gamma\alpha-1)]} r^2 \tag{2.3.1}$$

with equality for

$$f(z) = z - \frac{\gamma\beta(1-\alpha)}{2^{n-1}[1+\beta(4\gamma-2\gamma\alpha-1)]} z^2 \qquad (z = \pm r).$$

Proof. From (2.2.1), we have

$$2^{n-k}[1+\beta(4\gamma-2\gamma\alpha-1)] \sum_{j=}^{\infty} j^k a_j$$

$$\le \sum_{j=2}^{\infty} j^n [j-1+\beta(1-j+2\gamma j-2\gamma\alpha)] \le 2\gamma\beta(1-\alpha)$$

$$\sum_{j=2}^{\infty} j^k a_j \le \frac{\gamma\beta(1-\alpha)}{2^{n-k-1}[1+\beta(4\gamma-2\gamma\alpha-1)]} . \tag{2.3.2}$$

Using (2.3.2) with k = 0, for $0 < |z| = r < 1$, we obtain

$$|f(z)| \le r + \sum_{j=2}^{\infty} a_j r^j \le r + \frac{\gamma\beta(1-\alpha)}{2^{n-1}[1+\beta(4\gamma-2\gamma\alpha-1)]} r^2,$$

and

$$|f(z)| \geq r - \sum_{j=2}^{\infty} a_j r^j \geq r - \frac{\gamma\beta(1-\alpha)}{2^{n-1}[1+\beta(4\gamma-2\gamma\alpha-1)]} r^2.$$

Thus (2.3.1) follows.

Keeping our intension in view, we go to state some special cases of Theorem 3.1.

Corollary 1. *A function* $f \in T_0(\alpha,1,1)$, *that is starlike of order* α then

$$r - \left[\frac{1-\alpha}{2-\alpha}\right] r^2 \leq |f(z)| \leq r + \left[\frac{1-\alpha}{2-\alpha}\right] r^2 .$$

This result is sharp. This result is due to Silverman [9].

Corollary 2. *If* $f \in T_n(\alpha,\beta,1)$, *then*

$$r - \frac{\beta(1-\alpha)}{2^{n-1}[1+\beta(3-2\alpha)]} r^2 \leq |f(z)| \leq r + \frac{\beta(1-\alpha)}{2^{n-1}[1+\beta(3-2\alpha)]} r^2$$

This result is sharp. This result is obtained by Salagean [7].

Next we obtain the corollary which gives the result for the class introduced and studied by Kulkarni [3].

Corollary 3. *If* $f \in T_0(\alpha,\beta,\gamma)$, *then*

$$r - \frac{2\gamma\beta(1-\alpha)}{[1+\beta(4\gamma-2\gamma\alpha-1)]} r^2 \leq |f(z)| \leq r + \frac{2\gamma\beta(1-\alpha)}{[1+\beta(4\gamma-2\gamma\alpha-1)]} r^2 .$$

The result is sharp.

We now state the theorem which gives the disk contained in the range set of functions in class $T_n(\alpha, \beta, \gamma)$.

Theorem 4. *The disk* $|z| < 1$ *is mapped onto a domain that contains the disk*

$$|w| < 1 - \frac{\gamma\beta(1-\alpha)}{2^{n-1}[1+\beta(4\gamma-2\gamma\alpha-1)]}$$

by any $f \in T_n(\alpha, \beta, \gamma)$.

Proof. The result follows upon by letting $r \to 1$ in Theorem 3.

In the next theorem we obtain the distortion property of derivative of the normalized univalent functions in the class $T_n(\alpha, \beta, \gamma)$.

Theorem 5. *Let* $\alpha \in [0,1)$, $\beta \in (0,1]$, $\gamma \in (1/2,1]$ *and let* $n \in N_0$, *if* $f \in T_n(\alpha, \beta, \gamma)$, *then for* $0 < |z| = r < 1$, *we have*

$$1 - \frac{2\gamma\beta(1-\alpha)}{2^{n-1}[1+\beta(4\gamma-2\gamma\alpha-1)]} r \le |f'(z)| \le 1 + \frac{2\gamma\beta(1-\alpha)}{2^{n-1}[1+\beta(4\gamma-2\gamma\alpha-1)]} r. \quad (2.3.3)$$

Equality holds for

$$f(z) = z - \frac{\gamma\beta(1-\alpha)}{2^{n-1}[1+\beta(4\gamma-2\gamma\alpha-1)]} z^2 \qquad (z = \pm r).$$

Proof. Using (2.3.2) with k = 1, for $0 < |z| = r < 1$ we obtain

$$\sum_{j=2}^{\infty} j\, a_j \le \frac{2\beta\gamma(1-\alpha)}{2^{n-1}[1+\beta(4\gamma-2\gamma\alpha-1)]}.$$

Thus

$$|f'(z)| \le 1+\sum_{j=2}^{\infty} j\, a_j\, r^{j-1} \le 1+\frac{2\gamma\beta(1-\alpha)}{2^{n-1}[1+\beta(4\gamma-2\gamma\alpha-1)]}\, r,$$

and

$$|f'(z)| \ge 1-\sum_{j=2}^{\infty} j\, a_j\, r^{j-1} \ge 1-\frac{2\gamma\beta(1-\alpha)}{2^{n-1}[1+\beta(4\gamma-2\gamma\alpha-1)]}\, r.$$

Thus Theorem 5 follows.

Now we state several special cases of above theorem.

Corollary 1. *A function* $f \in T_0(\alpha,1,1)$ *that is starlike of order* α , then

$$1-\left[\frac{2(1-\alpha)}{2-\alpha}\right] r \le |f'(z)| \le 1+\left[\frac{2(1-\alpha)}{2-\alpha}\right] r .$$

This result is sharp. This result is due to Silverman [9].

Corollary 2. *If* $f \in T_n(\alpha,\beta,1)$, *then*

$$1-\frac{2\beta(1-\alpha)}{2^{n-1}[1+\beta(3-2\alpha)]}\, r \le |f'(z)| \le 1+\frac{2\beta(1-\alpha)}{2^{n-1}[1+\beta(3-2\alpha)]}\, r .$$

This result is sharp. This result is obtained by Salagean [7].

Next we obtain the corollary which gives the result for the class introduced and studied by Kulkarni [3].

Corollary 3. *If* $f \in T_0(\alpha,\beta,\gamma)$, *then*

$$1-\frac{4\gamma\beta(1-\alpha)}{[1+\beta(4\gamma-2\gamma\alpha-1)]}\, r \le |f'(z)| \le 1+\frac{4\gamma\beta(1-\alpha)}{[1+\beta(4\gamma-2\gamma\alpha-1)]}\, r .$$

The result is sharp.

In the next theorem we determine the radius of convexity for the functions in $T_n(\alpha,\beta,\gamma)$.

Theorem 6. *If the function* $f(z) = z-\sum_{j=2}^{\infty} a_j z^j \quad a_j \geq 0, j = 2,3,...$ *is in* $T_n(\alpha,\beta,\gamma)$, *then* f *is convex in the disk*

$$|z| < r = r(\alpha,\beta,\gamma,n) = \inf_j \left(\frac{j^{n-2}[\,j-1+\beta(1-j+2\gamma j-2\gamma\alpha)\,]}{2\gamma\beta(1-\alpha)} \right)^{\frac{1}{j-1}}, \qquad (2.3.4)$$

$$(j = 2,3,...).$$

This result is sharp, with the extremal functions as given in (2.2.4).

Proof. It suffices to show that $\left|\frac{z f''(z)}{f'(z)}\right| \leq 1$ for $|z| \leq r(\alpha,\beta,\gamma,n)$. We have

$$\left|\frac{z f''(z)}{f'(z)}\right| = \left|\frac{\sum_{j=2}^{\infty} j(j-1)a_j z^{j-1}}{1-\sum_{j=2}^{\infty} j a_j z^{j-1}}\right| \leq \frac{\sum_{j=2}^{\infty} j(j-1)a_j |z|^{j-1}}{1-\sum_{j=2}^{\infty} j a_j |z|^{j-1}}.$$

Thus $\left|\frac{z f''(z)}{f'(z)}\right| \leq 1$ if

$$\sum_{j=2}^{\infty} j(j-1)a_j |z|^{j-1} \leq 1-\sum_{j=2}^{\infty} j a_j |z|^{j-1}$$

or

$$\sum_{j=2}^{\infty} j^2 a_j |z|^{j-1} \leq 1. \qquad (2.3.5)$$

According to Theorem 1,

$$\sum_{j=2}^{\infty} \frac{j^n [\, j-1+\beta(1-j+2\gamma j-2\gamma\alpha)\,]}{2\gamma\beta(1-\alpha)} a_j \leq 1 \quad .$$

Hence (2.3.5) will be true if

$$j^2 |\, z\,|^{j-1} \leq \frac{j^n [\, j-1+\beta(1-j+2\gamma j-2\gamma\alpha)\,]}{2\gamma\beta(1-\alpha)} \quad .$$

Solving for $|\, z\,|$, we obtain

$$|\, z\,| \leq \left(\frac{j^{n-2} \, [j-1+\beta(1-j+2\gamma j-2\gamma\alpha)\,]}{2\gamma\beta(1-\alpha)} \right)^{\frac{1}{j-1}} \qquad (j = 2, 3, \ldots). \quad (2.3.6)$$

Setting $|\, z\,| = r(\alpha, \beta, \gamma, n)$ in (2.3.6), the result follows.

We state some particular cases of above theorem.

Corollary 1. *If* $f \in T_0(\alpha, 1, 1)$, *that is starlike of order* α, *then* f *is convex in the disk*

$$|\, z\,| < r = r(\alpha) = \inf_j \left(\frac{j-2}{j^2\,(1-\alpha)} \right)^{\frac{1}{j-1}} .$$

This result is sharp. This result is same as obtained by Silverman [9].

Next we obtain the corollaries which gives the result for the class introduced and studied by Salagean [7] and Kulkarni [3].

Corollary 2. *If* $f \in T_n(\alpha, \beta, 1)$, *then* f *is convex in the disk*

$$|z| < r = r(\alpha, \beta, 1, n) = \inf_j \left(\frac{j^{n-2}[\, j-1+\beta(1+j-2\alpha)\,]}{2\beta(1-\alpha)} \right)^{\frac{1}{j-1}} .$$

This result is sharp.

Corollary 3. *If* $f \in T_0(\alpha, \beta, \gamma)$, *then* f *is convex in the disk*

$$|z| < r = r(\alpha, \beta, \gamma, 0) = \inf_j \left(\frac{[\, j-1+\beta(1-j+2\gamma j-2\gamma\alpha)\,]}{2\gamma\beta(1-\alpha)j^2} \right)^{\frac{1}{j-1}} .$$

The result is sharp. This result is due to Kulkarni [3].

Now we shall prove the following result for the closure of function in the class $T_n(\alpha,\beta,\gamma)$.

Let the function $f_k(z)$ be defined for $k \in \{1, 2, 3, ..., m\}$, by

$$f_k(z) = z - \sum_{j=2}^{\infty} a_{j,k} z^j , \quad (a_{j,k} \geq 0) \tag{2.3.7}$$

for $z \in U$.

Theorem 7. *Let the function* $f_k(z)$ *defined by (2.3.7) be in class* $T_n(\alpha, \beta, \gamma)$ *for every* $k \in \{1, 2, 3, ..., m\}$. *Then the function* $F(z)$ *defined by,*

$$F(z) = z - \sum_{j=2}^{\infty} b_j z^j , \quad (b_j \geq 0) \tag{2.3.8}$$

is a member of class $T_n(\alpha, \beta, \gamma)$, *where*

$$b_j = \frac{1}{m}\sum_{k=1}^{m} a_{j,k} \qquad (j = 2, 3, \ldots). \tag{2.3.9}$$

Proof. Since $f_k(z) \in T_n(\alpha, \beta, \gamma)$, it follows from Theorem 1, that

$$\sum_{j=2}^{\infty} j^n \left[\, j-1+\beta(1-j+2\gamma j-2\gamma\alpha)\,\right] a_{j,k} \le 2\beta\gamma(1-\alpha)$$

for every $k \in \{1, 2, 3, \ldots, m\}$. Hence,

$$\sum_{j=2}^{\infty} j^n \left[\, j-1+\beta(1-j+2\gamma j-2\gamma\alpha)\,\right] b_j$$

$$\le \sum_{j=2}^{\infty} j^n \left[\, j-1+\beta(1-j+2\gamma j-2\gamma\alpha)\,\right]\left(\frac{1}{m}\sum_{k=1}^{m} a_{j,k}\right)$$

$$= \frac{1}{m}\sum_{k=1}^{m}\left(\sum_{j=2}^{\infty} j^n \left[\, j-1+\beta(1-j+2\gamma j-2\gamma\alpha)\,\right] a_{j,k}\right)$$

$$\le \frac{1}{m}\sum_{k=1}^{m} 2\gamma\beta(1-\alpha) = 2\gamma\beta(1-\alpha),$$

which (in view of Theorem 1) implies that $F(z) \in T_n(\alpha, \beta, \gamma)$.

Theorem 8. *The class* $T_n(\alpha, \beta, \gamma)$ *is closed under convex linear combination.*

Proof. Let the function $f_k(z)$ defined by (2.3.7) be in class $T_n(\alpha, \beta, \gamma)$ for every $k \in \{1, 2, 3, \ldots, m\}$, it is suffices to prove that the function

$$H(z) = \lambda f_1(z) + (1-\lambda) f_2(z) \quad , \quad (0 \le \lambda \le 1)$$

is also in class $T_n(\alpha, \beta, \gamma)$.Since , for $0 \le \lambda \le 1$,

$$H(z) = z - \sum_{j=2}^{\infty} \{ \lambda a_{j,1} + (1-\lambda) a_{j,2} \} z^j , \qquad (0 \le \lambda \le 1).$$

We observe that

$$\sum_{j=2}^{\infty} j^n [j-1+\beta(1-j+2\gamma j-2\gamma\alpha)] \{ \lambda a_{j,1} + (1-\lambda) a_{j,2} \}$$

$$= \lambda \sum_{j=2}^{\infty} j^n [j-1+\beta(1-j+2\gamma j-2\gamma\alpha)] a_{j,1}$$

$$+ (1-\lambda) \sum_{j=2}^{\infty} j^n [j-1+\beta(1-j+2\gamma j-2\gamma\alpha)] a_{j,2}$$

$$\le \lambda\, 2\gamma\beta(1-\alpha) + (1-\lambda) 2\gamma\beta(1-\alpha) = 2\gamma\beta(1-\alpha),$$

with the aid of Theorem 1. Hence $H(z) \in T_n(\alpha,\beta,\gamma)$. This completes the proof of Theorem 8.

4. INCLUSION THEOREMS

We prove some inclusion relations for the class $T_n(\alpha,\beta,\gamma)$, which generalizes the result of Salagean [7].

Theorem 9. *Let* $\alpha \in [0,1)$, $\beta \in (0,1]$, $\gamma \in (1/2,1]$ *and let* $n \in N_0$, *then*

$$T_{n+1}(\alpha,\beta,\gamma) \subset T_n(\alpha',\beta,\gamma),$$

where

$$\alpha' = \alpha'(\alpha,\beta,\gamma) = 1 - \frac{(1-\beta+2\beta\gamma)(1-\alpha)}{2(1-\beta+3\beta\gamma-\alpha\beta\gamma)} \tag{2.4.1}$$

$$= \frac{2(1-\beta+2\beta\gamma) - (1-\alpha)(1-\beta)}{2(1-\beta+3\beta\gamma-\alpha\beta\gamma)}$$

The result is sharp and $1 > \alpha' > \alpha$.

Proof. Let $f \in T_{n+1}(\alpha,\beta,\gamma)$, then from Theorem 1, we have

$$\sum_{j=2}^{\infty} \frac{j^{n+1}[\,j-1+\beta(1-j+2\gamma j-2\gamma\alpha)\,]}{2\beta\gamma(1-\alpha)} a_j \leq 1 \quad . \qquad (2.4.2)$$

We determine the largest α' such that,

$$\sum_{j=2}^{\infty} \frac{j^{n}[\,j-1+\beta(1-j+2\gamma j-2\gamma\alpha')\,]}{2\beta\gamma(1-\alpha')} a_j \leq 1 \quad . \qquad (2.4.3)$$

If

$$\frac{[\,j-1+\beta(1-j+2\gamma j-2\gamma\alpha')\,]}{(1-\alpha')} \leq \frac{j\,[\,j-1+\beta(1-j+2\gamma j-2\gamma\alpha)\,]}{(1-\alpha)} \leq 1 \quad , \qquad (2.4.4)$$

$j = 2,3,\ldots$, then (2.4.2) implies (2.4.3). But the inequalities (2.4.4) are equivalent to

$$\alpha' \leq \frac{(1-\beta+2\beta\gamma)\,j - (1-\alpha)(1-\beta)}{(1-\beta+2\beta\gamma)\,j + 2\beta\gamma(1-\alpha)} = 1 - \frac{(1-\beta+2\beta\gamma)(1-\alpha)}{(1-\beta+2\beta\gamma)\,j + 2\beta\gamma(1-\alpha)},$$

$$j = 2,3,\ldots . \qquad (2.4.5)$$

We have $\quad d(2) \le d(j) = \dfrac{(1-\beta+2\beta\gamma)\,j - (1-\alpha)(1-\beta)}{(1-\beta+2\beta\gamma)\,j + 2\beta\gamma(1-\alpha)}$

$j = 2, 3, \ldots .$

We choose

$$\alpha'(\alpha,\beta,\gamma) = d(2) = 1 - \frac{(1-\beta+2\beta\gamma)(1-\alpha)}{2(1-\beta+3\beta\gamma-\alpha\beta\gamma)} \quad . \tag{2.4.6}$$

We have $\alpha < \alpha' < 1$, because

$$\alpha'(\alpha,\beta,\gamma) - \alpha = \frac{(1-\alpha)[(1-\beta)+2\beta\gamma(2-\alpha)]}{2\,[(1-\beta)+\beta\gamma(3-\alpha)]} > 0$$

and

$$1-\alpha'(\alpha,\beta,\gamma) = \frac{(1-\beta+2\beta\gamma)(1-\alpha)}{2\,[(1-\beta)+\beta\gamma(3-\alpha)]} > 0.$$

The extremal function is $f_2 \in T_{n+1}(\alpha,\beta,\gamma)$,

$$f_2(z) = z - \frac{\gamma\beta(1-\alpha)}{2^n\,[1+\beta(-1+4\gamma-2\gamma\alpha)]}\, z^2 \quad . \tag{2.4.7}$$

Corollary 1. $T_{n+1}(\alpha,\beta,\gamma) \subset T_n(\alpha,\beta,\gamma)$.

Corollary 2. *If* $f \in T_{n+1}(\alpha,\beta,\gamma)$, *then* $f \in T_n(\rho,1,1)$ *where*

$$\rho = \rho(\alpha,\beta,\gamma) = \frac{1-\beta+2\beta\gamma}{1-\beta+3\beta\gamma-\alpha\beta\gamma} \; .$$

Proof. A simple computation yields

$$|J_n(f,\alpha,\gamma;z)| < \beta \quad \text{implies} \quad \left|\frac{D^{n+1}f(z)}{D^n f(z)}-1\right| \le \frac{2\beta\gamma(1-\alpha)}{1+2\beta\gamma-\beta} \quad .$$

That is

$$\mathrm{Re}\left\{\frac{D^{n+1}f(z)}{D^n f(z)}\right\} > \sigma(\alpha,\beta,\gamma) \quad ,$$

where

$$\sigma(\alpha,\beta,\gamma) = \frac{1+2\alpha\beta\gamma-\beta}{1+2\beta\gamma-\beta} \quad .$$

Let $f \in T_{n+1}(\alpha,\beta,\gamma)$, then by Theorem 9 $f \in T_n(\alpha',\beta,\gamma)$, where

$$\alpha' = 1 - \frac{(1-\beta+2\beta\gamma)(1-\alpha)}{2(1-\beta+3\beta\gamma-\alpha\beta\gamma)} \quad .$$

Therefore

$$\sigma(\alpha',\beta,\gamma) = \rho(\alpha,\beta,\gamma) = \frac{1-\beta+2\beta\gamma}{1-\beta+3\beta\gamma-\alpha\beta\gamma} \quad .$$

Hence

$$f \in T_n(\sigma(\alpha',\beta,\gamma),1,1) = T_n(\rho(\alpha,\beta,\gamma),1,1).$$

We now give some particular cases for Theorem 9.

Corollary 3. *Let* $\alpha \in [0,1)$, $\beta \in (0,1]$, , *and let* $n \in N_0$, *then*

$$T_{n+1}(\alpha,\beta,1) \subset T_n(\alpha',\beta,1),$$

where

$$\alpha' = \alpha'(\alpha, \beta, 1) = 1 - \frac{(1+\beta)(1-\alpha)}{2(1+2\beta-\alpha\beta)}$$

$$= \frac{2(1+\beta) - (1-\alpha)(1-\beta)}{2(1+2\beta-\alpha\beta)}$$

The result is sharp .This result is same as obtained by Salagean [7].

Next corollary gives the result for the class introduced and studied by Salagean [6] .

Corollary 4. *Let* $\alpha \in [0,1)$ *and let* $n \in N_0$, *then*

$$T_{n+1}(\alpha, 1, 1) \subset T_n(\alpha', 1, 1),$$

where

$$\alpha' = \alpha'(\alpha, 1, 1) = 1 - \frac{1-\alpha}{3-\alpha} = \frac{2}{(3-\alpha)}$$

The result is sharp .

Theorem 10. *Let* $\alpha \in [0,1)$, $\beta \in (0,1]$, $\gamma \in (1/2,1]$ *and let* $n \in N_0$, *then*

$$T_{n+1}(\alpha, \beta, \gamma) \subset T_n(\alpha, \beta', \gamma),$$

where

$$\beta' = \beta'(\alpha, \beta, \gamma) = \frac{\beta}{2+\beta(-1+4\gamma-2\alpha\gamma)} \qquad (2.4.9)$$

The result is sharp and $0 < \beta' < \beta$.

Proof. Let $f \in T_{n+1}(\alpha, \beta, \gamma)$, then from Theorem 1 we have

$$\sum_{j=2}^{\infty} \frac{j^{n+1}[\, j-1+\beta(1-j+2\gamma j-2\gamma\alpha)\,]}{2\,\beta\gamma\,(1-\alpha)}\, a_j \le 1 \quad , \qquad (2.4.10)$$

we determine the smallest β' such that

$$\sum_{j=2}^{\infty} \frac{j^{n}[\, j-1+\beta'(1-j+2\gamma j-2\gamma\alpha)\,]}{2\,\beta'\gamma\,(1-\alpha)}\, a_j \le 1 \quad . \qquad (2.4.11)$$

If

$$\frac{[\, j-1+\beta'(1-j+2\gamma j-2\gamma\alpha)\,]}{\beta'} \le \frac{j\,[\, j-1+\beta(1-j+2\gamma j-2\gamma\alpha)\,]}{\beta} \quad ,$$

$$j = 2, 3, \ldots , \qquad (2.4.12)$$

then (2.4.10) implies (2.4.11). But the inequalities (2.4.12) are equivalent to

$$\beta' \ge \frac{\beta}{j + \beta(1-j+2\gamma j-2\alpha\gamma)} , \qquad j = 2, 3, \ldots . \qquad (2.4.13)$$

We have $c(2) \ge c(j) = \dfrac{\beta}{j + \beta(1-j+2\gamma j-2\alpha\gamma)}$ $\quad j = 2, 3, \ldots .$

Therefore we choose

$$\beta'(\alpha, \beta, \gamma) = c(2) = \frac{\beta}{2 + \beta(-1+4\gamma-2\alpha\gamma)} . \qquad (2.4.14)$$

We have $0 < \beta' < \beta$, because

$$\beta - \beta'(\alpha,\beta,\gamma) = \frac{\beta[(1-\beta)+2\beta\gamma(2-\alpha)]}{[(2-\beta)+2\beta\gamma(2-\alpha)]} > 0.$$

The extremal function is $f_2 \in T_{n+1}(\alpha,\beta,\gamma)$ given by (2.4.7).

Corollary 1. $T_{n+1}(\alpha,\beta,\gamma) \subset T_n(\alpha,\beta,\gamma)$.

Corollary 2. *If* $f \in T_{n+1}(\alpha,\beta,\gamma)$, *then* $f \in T_n(\rho,1,1)$ *where*

$$\rho = \rho(\alpha,\beta,\gamma) = \frac{1-\beta+2\beta\gamma}{1-\beta+3\beta\gamma-\alpha\beta\gamma} .$$

Proof. A simple computation yields

$$|J_n(f,\alpha,\gamma;z)| < \beta \quad \text{implies} \quad \left|\frac{D^{n+1}f(z)}{D^n f(z)}-1\right| \le \frac{2\beta\gamma(1-\alpha)}{1+2\beta\gamma-\beta} .$$

That is

$$\mathrm{Re}\left\{\frac{D^{n+1}f(z)}{D^n f(z)}\right\} > \sigma(\alpha,\beta,\gamma) ,$$

where

$$\sigma(\alpha,\beta,\gamma) = \frac{1+2\alpha\beta\gamma-\beta}{1+2\beta\gamma-\beta} .$$

Let $f \in T_{n+1}(\alpha,\beta,\gamma)$ then by Theorem 10, $f \in T_n(\alpha,\beta',\gamma)$, where

$$\beta' = \beta'(\alpha, \beta, \gamma) = \frac{\beta}{2+\beta(-1+4\gamma-2\alpha\gamma)} .$$

Therefore

$$\sigma(\alpha, \beta', \gamma) = \rho(\alpha, \beta, \gamma) = \frac{1-\beta+2\beta\gamma}{1-\beta+3\beta\gamma-\alpha\beta\gamma} .$$

Hence

$$f \in T_n(\sigma(\alpha, \beta', \gamma), 1, 1) = T_n(\rho(\alpha, \beta, \gamma), 1, 1).$$

Corollary 3. *Let* $\alpha \in [0,1)$, $\beta \in (0,1]$, *let* $n \in N_0$, *then*

$$T_{n+1}(\alpha, \beta, 1) \subset T_n(\alpha, \beta', 1),$$

where

$$\beta' = \beta'(\alpha, \beta, \gamma) = \frac{\beta}{2+3\beta-2\alpha}$$

The result is sharp .This result is same as obtained by Salagean [7].

Theorem 11. *Let* $\alpha \in [0,1)$, $\beta \in (0,1]$, $\gamma \in (1/2,1]$ *and let* $n \in N_0$, *then*

$$T_{n+1}(\alpha, \beta, \gamma) \subset T_n(\alpha, \beta, \gamma'),$$

where

$$\gamma' = \gamma'(\alpha, \beta, \gamma) = \frac{\gamma(1-\beta)}{2(1-\beta)+2\gamma\beta(2-\alpha)} \qquad (2.4.15)$$

The result is sharp and $0 \leq \gamma' < \gamma$.

Proof. Let $f \in T_{n+1}(\alpha,\beta,\gamma)$, then from Theorem 1, we have

$$\sum_{j=2}^{\infty} \frac{j^{n+1}[\, j-1+\beta(1-j+2\gamma j-2\gamma\alpha)\,]}{2\,\beta\gamma\,(1-\alpha)}\, a_j \le 1 \quad . \tag{2.4.16}$$

We determine the smallest γ' such that

$$\sum_{j=2}^{\infty} \frac{j^{n}[\, j-1+\beta(1-j+2\gamma' j-2\gamma'\alpha)\,]}{2\,\beta\gamma'(1-\alpha)}\, a_j \le 1 \quad . \tag{2.4.17}$$

If

$$\frac{[\, j-1+\beta(1-j+2\gamma' j-2\gamma'\alpha)\,]}{\gamma'} \le \frac{j\,[\, j-1+\beta(1-j+2\gamma j-2\gamma\alpha)\,]}{\gamma} \quad ,$$

$$j = 2,3,\dots , \tag{2.4.18}$$

then (2.4.16) implies (2.4.17). But the inequalities (2.4.18) are equivalent to

$$\gamma' \ge \frac{\gamma\,(1-\beta)}{j\,(1-\beta)+2\gamma\beta(\,j-\alpha\,)}\, , \qquad j = 2,3,\dots . \tag{2.4.19}$$

We have

$$b(2) \ge b(j) = \frac{\gamma\,(1-\beta)}{2\,(1-\beta)+2\gamma\beta(\,2-\alpha\,)} \qquad j = 2,3,\dots .$$

Therefore we choose

$$\gamma'(\alpha,\beta,\gamma) = b(2) = \frac{\gamma(1-\beta)}{2(1-\beta) + 2\gamma\beta(2-\alpha)} . \tag{2.4.20}$$

We have $0 \leq \gamma' < \gamma$, because

$$\gamma - \gamma'(\alpha,\beta,\gamma) = \frac{\gamma[(1-\beta)+2\beta\gamma(2-\alpha)]}{[2(1-\beta)+2\beta\gamma(2-\alpha)]} > 0.$$

The extremal function is $f_2 \in T_{n+1}(\alpha,\beta,\gamma)$ given by (2.4.7).

Corollary 1. $T_{n+1}(\alpha,\beta,\gamma) \subset T_n(\alpha,\beta,\gamma)$.

Corollary 2. *If* $f \in T_{n+1}(\alpha,\beta,\gamma)$, *then* $f \in T_n(\rho,1,1)$ *where*

$$\rho = \rho(\alpha,\beta,\gamma) = \frac{1-\beta+2\beta\gamma}{1-\beta+3\beta\gamma-\alpha\beta\gamma} .$$

Proof. A simple computation yields

$$|J_n(f,\alpha,\gamma;z)| < \beta \quad \text{implies} \quad \left|\frac{D^{n+1}f(z)}{D^n f(z)} - 1\right| \leq \frac{2\beta\gamma(1-\alpha)}{1+2\beta\gamma-\beta} .$$

That is

$$\text{Re}\left\{\frac{D^{n+1}f(z)}{D^n f(z)}\right\} > \sigma(\alpha,\beta,\gamma) ,$$

where

$$\sigma(\alpha,\beta,\gamma) = \frac{1+2\alpha\beta\gamma-\beta}{1+2\beta\gamma-\beta} .$$

Let $f \in T_{n+1}(\alpha, \beta, \gamma)$ then by Theorem 11, $f \in T_n(\alpha, \beta, \gamma')$, where

$$\gamma' = \gamma'(\alpha, \beta, \gamma) = \frac{\gamma(1-\beta)}{2(1-\beta) + 2\gamma\beta(2-\alpha)} .$$

Therefore

$$\sigma(\alpha, \beta, \gamma') = \rho(\alpha, \beta, \gamma) = \frac{1-\beta+2\beta\gamma}{1-\beta+3\beta\gamma-\alpha\beta\gamma} .$$

Hence

$$f \in T_n(\sigma(\alpha, \beta, \gamma'), 1, 1) = T_n(\rho(\alpha, \beta, \gamma), 1, 1).$$

5. CONVOLUTION THEOREMS

Motivated by Salagean [8] we now prove convolution theorems for the class $T_n(\alpha, \beta, \gamma)$.

Let

$$f(z) = z - \sum_{j=2}^{\infty} a_j z^j \quad \text{and} \quad g(z) = z - \sum_{j=2}^{\infty} b_j z^j .$$

We define the Hadmard product or convolution of f and g by

$$(f * g)(z) = z - \sum_{j=2}^{\infty} a_j b_j z^j .$$

Theorem 12. *Let $f, g \in T_n(\alpha, \beta, \gamma)$, $\alpha \in [0,1)$, $\beta \in (0,1]$, $\gamma \in (1/2, 1]$ and $n \in N_0$, then $f * g \in T_n(\alpha', \beta, \gamma)$, where*

$$\alpha' = \alpha'(\alpha,\beta,\gamma,n) = 1 - \frac{2\beta\gamma(1-\alpha)^2(1-\beta+2\beta\gamma)}{[1+\beta(-1+4\gamma-2\alpha\gamma)]^2\, 2^n - 4\beta^2\gamma^2(1-\alpha)^2}$$

(2.5.1)

and $\alpha < \alpha'(\alpha,\beta,\gamma,n) < 1$ *.The result is sharp, the extremal functions are* $f = g = f_2$, *where* f_2 *is given in Theorem 1.*

Proof. Let $f, g \in T_n(\alpha,\beta,\gamma)$, then from Theorem 1 we have

$$\sum_{j=2}^{\infty} \frac{j^n[\,j-1+\beta(1-j+2\gamma j-2\gamma\alpha)\,]}{2\,\beta\gamma(1-\alpha)}\, a_j \le 1 \quad , \tag{2.5.2}$$

and

$$\sum_{j=2}^{\infty} \frac{j^n[\,j-1+\beta(1-j+2\gamma j-2\gamma\alpha)\,]}{2\,\beta\gamma(1-\alpha)}\, b_j \le 1 \quad . \tag{2.5.3}$$

From Theorem 1, we also have $f*g \in T_n(\alpha',\beta,\gamma)$ if and only if

$$\sum_{j=2}^{\infty} \frac{j^n[\,j-1+\beta(1-j+2\gamma j-2\gamma\alpha')\,]}{2\,\beta\gamma(1-\alpha')}\, a_j\, b_j \le 1 \quad . \tag{2.5.4}$$

We wish to determine the largest $\alpha' = \alpha'(\alpha,\beta,\gamma,n)$, such that (2.5.4) holds.

From (2.5.2) and (2.5.3), we get by means of Cauchy-Schwarz inequality

$$\sum_{j=2}^{\infty} \frac{j^n[\,j-1+\beta(1-j+2\gamma j-2\gamma\alpha)\,]}{2\,\beta\gamma(1-\alpha)}\, \sqrt{a_j\, b_j} \le 1 \tag{2.5.5}$$

which implies

$$\sqrt{a_j b_j} \le \frac{2\beta\gamma(1-\alpha)}{j^n[\,j-1+\beta(1-j+2\gamma j-2\gamma\alpha)\,]}\,,$$

$$j = 2, 3, \ldots . \tag{2.5.6}$$

We note that the next inequalities

$$\frac{j^n[\,j-1+\beta(1-j+2\gamma j-2\gamma\alpha')\,]}{2\beta\gamma(1-\alpha')}\,a_j b_j$$

$$\le \frac{j^n[\,j-1+\beta(1-j+2\gamma j-2\gamma\alpha)\,]}{2\beta\gamma(1-\alpha)}\sqrt{a_j b_j}\;,\; j = 2, 3, \ldots \tag{2.5.7}$$

imply (2.5.4). But the inequalities (2.5.7) are equivalent to

$$\frac{j-1+\beta(1-j+2\gamma j-2\gamma\alpha')}{(1-\alpha')}\sqrt{a_j b_j}$$

$$\le \frac{j-1+\beta(1-j+2\gamma j-2\gamma\alpha)}{(1-\alpha)}\;,\; j = 2, 3, \ldots\;, \tag{2.5.8}$$

by using (2.5.6) we have

$$\frac{j-1+\beta(1-j+2\gamma j-2\gamma\alpha')}{(1-\alpha')}\sqrt{a_j b_j}$$

$$\le \frac{2\beta\gamma(1-\alpha)[\,j-1+\beta(1-j+2\gamma j-2\gamma\alpha')\,]}{(1-\alpha')\,j^n[\,j-1+\beta(1-j+2\gamma j-2\gamma\alpha)\,]}\;,\; j = 2, 3, \ldots\;,$$

In order to obtain (2.5.8) it will sufficient to show that

$$\frac{2\beta\gamma(1-\alpha)[j-1+\beta(1-j+2\gamma j-2\gamma\alpha')]}{(1-\alpha')j''[j-1+\beta(1-j+2\gamma j-2\gamma\alpha)]} \le \frac{j-1+\beta(1-j+2\gamma j-2\gamma\alpha)}{(1-\alpha)},$$

$$j = 2,3,\ldots$$

that is

$$2\beta\gamma(1-\alpha)^2[j-1+\beta(1-j+2\gamma j-2\gamma\alpha')]$$

$$\le (1-\alpha')[j-1+\beta(1-j+2\gamma j-2\gamma\alpha)]^2 j'', \qquad j = 2,3,\ldots \tag{2.5.9}$$

The inequalities (2.5.9) are equivalent to $A\alpha' \le B$, where

$$A = -4\beta^2\gamma^2(1-\alpha)^2 + [j-1+\beta(1-j+2\gamma j-2\gamma\alpha)]^2 j'' > 0$$

and

$$B = [j-1+\beta(1-j+2\gamma j-2\gamma\alpha)]^2 j'' - 2\beta\gamma(1-\alpha)^2(j-1)$$

$$-2\beta^2\gamma(1-\alpha)^2(1-j+2\gamma j),$$

we obtain

$$\alpha' \le \frac{B}{A} = 1 - \frac{2\beta\gamma(1-\alpha)^2(j-1)(1-\beta+2\beta\gamma)}{[j-1+\beta(1-j+2\gamma j-2\gamma\alpha)]^2 j'' - 4\beta^2\gamma^2(1-\alpha)^2} = c(j).$$

We have $c(2) \le c(j)$, $j = 2, 3, \ldots$ and we choose

$$\alpha' = \alpha'\left(\alpha, \beta, \gamma, n\right) = c(2).$$

We have $\alpha < \alpha'$, because

$$\alpha'(\alpha, \beta, \gamma, n) - \alpha \ge \alpha'(\alpha, \beta, \gamma, 0) - \alpha$$

$$= \left[\frac{(1-\alpha)(1-\beta)\left(1-\beta+2\beta\gamma(3-\alpha)\right)+4\beta^2\gamma^2(1-\alpha)(2-\alpha)}{(1-\beta)\left(1-\beta+2\beta\gamma(3-\alpha)\right)+4\beta^2\gamma^2(2-\alpha)+2\beta\gamma(1-\alpha)(1-\beta+2\beta\gamma)} \right] > 0$$

,

and

$$1-\alpha'(\alpha, \beta, \gamma, n) = \frac{2\gamma\beta(1-\alpha)^2(1-\beta+2\beta\gamma)}{\left[1+\beta(-1+4\gamma-2\alpha\gamma)\right]^2 2^n - \left(2\beta\gamma(1-\alpha)\right)^2} > 0.$$

The extremal functions are $f = g = f_2$. Indeed

$(f_2 * f_2)(z) = z - c_2 z^2 \in T_n(\alpha', \beta, \gamma)$, where

$$c_2 = \frac{2^{2-2n}\beta^2\gamma^2(1-\alpha)^2}{\left[1+\beta(-1+4\gamma-2\alpha\gamma)\right]^2} \quad \text{and} \quad \alpha' = \alpha'\left(\alpha, \beta, \gamma, n\right) \quad \text{because}$$

$$\sum_{j=2}^{\infty} \frac{j^n\left[j-1+\beta(1-j+2\gamma j-2\gamma\alpha')\right]}{2\beta\gamma(1-\alpha')} c_j =$$

$$= \frac{2^n\left[1+\beta(-1+4\gamma-2\gamma\alpha')\right]}{2\beta\gamma(1-\alpha')} \frac{2^{2-2n}\beta^2\gamma^2(1-\alpha)^2}{\left[1+\beta(-1+4\gamma-2\alpha\gamma)\right]^2} = 1 .$$

Corollary 1. *If* $f, g \in T_n(\alpha, \beta, \gamma)$, *then* $f * g \in T_n(\alpha, \beta, \gamma)$.

Corollary 2. *If* $f, g \in T_n(\alpha, \beta, \gamma)$, *then* $f * g \in T_n(\rho, 1, 1)$, *where*

$$\rho = \rho(\alpha, \beta, \gamma) = 1 - \frac{(1-\alpha)^2 \beta^2 \gamma^2}{2^{n-2}\left[1+\beta(-1+4\gamma-2\alpha\gamma)\right]^2 - \beta^2(1-\alpha)^2} .$$

Proof. A simple computation yields

$$|J_n(f,\alpha,\gamma;z)| < \beta \quad \text{implies} \quad \left| \frac{D^{n+1}f(z)}{D^n f(z)} - 1 \right| \leq \frac{2\beta\gamma(1-\alpha)}{1+2\beta\gamma-\beta} ,$$

therefore

$$\operatorname{Re}\left\{ \frac{D^{n+1}f(z)}{D^n f(z)} \right\} > \sigma(\alpha, \beta, \gamma) ,$$

where

$$\sigma(\alpha, \beta, \gamma) = \frac{1-\beta+2\alpha\beta\gamma}{1-\beta+2\beta\gamma} .$$

Let $f, g \in T_n(\alpha, \beta, \gamma)$ then $f * g \in T_n(\alpha', \beta, \gamma)$, *where*

$$\alpha' = \alpha'(\alpha, \beta, \gamma, n) = 1 - \frac{2\beta\gamma(1-\alpha)^2(1-\beta+2\beta\gamma)}{\left[1+\beta(-1+4\gamma-2\alpha\gamma)\right]^2 2^n - 4\beta^2\gamma^2(1-\alpha)^2} ,$$

therefore

$$\sigma(\alpha', \beta, \gamma) = \sigma(\alpha'(\alpha, \beta, \gamma, n), \beta, \gamma) = \rho(\alpha, \beta, \gamma, n) .$$

Hence

$$f \in T_n(\sigma(\alpha', \beta, \gamma), 1, 1) = T_n(\rho(\alpha, \beta, \gamma), 1, 1).$$

We now give some particular cases for Theorem 12.

Corollary 3. *Let* $f, g \in T_n(\alpha, \beta, 1)$, $\alpha \in [\,0,1\,)$, $\beta \in (\,0,1\,]$ *and* $n \in N_0$, *then* $f * g \in T_n(\alpha', \beta, 1)$, *where*

$$\alpha' = \alpha'(\,\alpha, \beta, 1, n\,) = 1 - \frac{2\beta(\,1-\alpha\,)^2\ (\,1+\beta\,)}{[\,1+\beta(\,3-2\alpha\,)\,]^2\ 2^n - 4\,\beta^2\ (\,1-\alpha\,)^2}$$

and $\alpha < \alpha'(\,\alpha, \beta, 1, n\,) < 1$ *.The result is sharp .This result is due to Salagean* [8].

Next corollaries gives the results for the class introduced and studied by Salagean [6] and Kulkarni [3], respectively.

Corollary 4. *Let* $f, g \in T_n(\alpha, 1, 1)$, $\alpha \in [\,0,1\,)$ *and* $n \in N_0$, *then* $f * g \in T_n(\alpha', 1, 1)$, *where*

$$\alpha' = \alpha'(\,\alpha, 1, 1, n\,) = 1 - \frac{(\,1-\alpha\,)^2}{(\,2-\alpha\,)^2\ 2^n - \ (\,1-\alpha\,)^2}$$

and $\alpha < \alpha'(\,\alpha, 1, 1, n\,) < 1$ *.The result is sharp .*

Corollary 5. *Let* $f, g \in T_0(\alpha, \beta, \gamma)$, $\alpha \in [\,0,1\,)$, $\beta \in (\,0,1\,]$, $\gamma \in (\,1/2,1\,]$, *then* $f * g \in T_0(\alpha', \beta, \gamma)$, *where*

$$\alpha' = \alpha'(\,\alpha, \beta, \gamma, 0\,) = 1 - \frac{2\beta\gamma(\,1-\alpha\,)^2\ (\,1-\beta+2\beta\gamma\,)}{[\,1+\beta(\,-1+4\gamma-2\alpha\gamma\,)\,]^2\ - 4\,\beta^2\,\gamma^2\,(\,1-\alpha\,)^2}$$

and $\alpha < \alpha'(\,\alpha, \beta, \gamma, 0\,) < 1$ *.The result is sharp*

Theorem 13. *Let* $f, g \in T_n(\alpha, \beta, \gamma)$, $\alpha \in [0,1)$, $\beta \in (0,1]$, $\gamma \in (1/2, 1]$ *and* $n \in N_0$, *then* $f * g \in T_n(\alpha, \beta', \gamma)$, *where*

$$\beta'(\alpha, \beta, \gamma, n) = \frac{2\beta^2\gamma(1-\alpha)}{[1+\beta(-1+4\gamma-2\alpha\gamma)]^2 2^n - 2\beta^2\gamma(1-\alpha)(-1+4\gamma-2\alpha\gamma)} \tag{2.5.10}$$

and $0 < \beta'(\alpha, \beta, \gamma, n) < \beta$ *.The result is sharp, the extremal functions are* $f = g = f_2$, *where* f_2 *is given in Theorem 1.*

Proof. If $f, g \in T_n(\alpha, \beta, \gamma)$, then (2.3.2) and (2.3.3) hold. By Theorem 1 we have $f * g \in T_n(\alpha, \beta', \gamma)$ if and only if

$$\sum_{j=2}^{\infty} \frac{j^n[\, j-1+\beta'(1-j+2\gamma j-2\gamma\alpha)\,]}{2\beta'\gamma(1-\alpha)} a_j b_j \le 1 \ , \tag{2.5.11}$$

We wish to determine the smallest $\beta' = \beta'(\alpha, \beta, \gamma, n)$, such that (2.5.11) holds.

We note that the next inequalities

$$\frac{j-1+\beta'(1-j+2\gamma j-2\gamma\alpha)}{\beta'} \sqrt{a_j b_j}$$

$$\le \frac{j-1+\beta(1-j+2\gamma j-2\gamma\alpha)}{\beta} \ , \ j = 2, 3, \ldots \ . \tag{2.5.12}$$

imply (2.5.11).

By (2.5.6), we obtain

$$\frac{j-1+\beta'(1-j+2\gamma j-2\gamma\alpha)}{\beta'}\sqrt{a_j b_j}$$

$$\leq \frac{2\beta\gamma(1-\alpha)[j-1+\beta'(1-j+2\gamma j-2\gamma\alpha)]}{\beta' j^n[j-1+\beta(1-j+2\gamma j-2\gamma\alpha)]}, \quad j = 2,3,\ldots .$$

In order to obtain (2.5.12) it will sufficient to show that

$$\frac{2\beta\gamma(1-\alpha)[j-1+\beta'(1-j+2\gamma j-2\gamma\alpha)]}{\beta' j^n[j-1+\beta(1-j+2\gamma j-2\gamma\alpha)]}$$

$$\leq \frac{j-1+\beta(1-j+2\gamma j-2\gamma\alpha)}{\beta}, \quad j = 2,3,\ldots$$

These last inequalities are equivalent to

$$\beta' \geq \frac{2\beta^2\gamma(1-\alpha)(j-1)}{[1-j+\beta(-1+4\gamma-2\alpha\gamma)]^2 j^n - 2\beta^2\gamma(1-\alpha)(1-j+2\gamma j-2\alpha\gamma)} = d(j)$$

We choose $\beta' = \beta'(\alpha,\beta,\gamma,n) = d(2)$, because $d(2) \geq d(j)$, $j = 2,3,\ldots$.

We have $\beta' < \beta$, because

$$\beta - \beta'(\alpha,\beta,\gamma,n) \geq \beta - \beta'(\alpha,\beta,\gamma,0) =$$

$$= \beta\left[\frac{(1-\beta)(1-\beta+2\beta\gamma(3-\alpha))+4\beta^2\gamma^2(2-\alpha)}{(1-\beta)(1-\beta+2\beta\gamma(3-\alpha))+4\beta^2\gamma^2(2-\alpha)+2\beta\gamma(1-\alpha)}\right] > 0$$

and $\beta'(\alpha,\beta,\gamma,n) > 0$ because

$$\left[1+\beta(-1+4\gamma-2\alpha\gamma)\right]^2 2^n - 2\beta^2\gamma(1-\alpha)(-1+4\gamma-2\alpha\gamma)$$

$$\geq \left[1+\beta(-1+4\gamma-2\alpha\gamma)\right]^2 - 2\beta^2\gamma(1-\alpha)(-1+4\gamma-2\alpha\gamma)$$

$$= (1-\beta)\,(1-\beta+2\beta\gamma(3-\alpha))+4\beta^2\gamma^2(2-\alpha)+2\beta\gamma(1-\alpha) > 0$$

The extremal functions are $f = g = f_2$.

Indeed

$$(f_2 * f_2)(z) = z - c_2 z^2 \in T_n(\alpha,\beta',\gamma),$$

where c_2 is given in the proof of Theorem 12 and $\beta' = \beta'(\alpha,\beta,\gamma,n)$ because

$$\sum_{j=2}^{\infty} \frac{j^n\left[j-1+\beta'(1-j+2\gamma j-2\gamma\alpha)\right]}{2\beta'\gamma(1-\alpha)}\, c_j = \frac{2^n\left[1+\beta'(-1+2\gamma j-2\gamma\alpha)\right]}{2\beta'\gamma(1-\alpha)}\, c_2 = 1.$$

Corollary 1. *If $f,g \in T_n(\alpha,\beta,\gamma)$, then $f*g \in T_n(\alpha,\beta,\gamma)$.*

Corollary 2. *If $f,g \in T_n(\alpha,\beta,\gamma)$, then $f*g \in T_n(\rho,1,1)$, where*

$$\rho = \rho(\alpha,\beta,\gamma) = 1 - \frac{(1-\alpha)^2\beta^2\gamma^2}{2^{n-2}\left[1+\beta(-1+4\gamma-2\alpha\gamma)\right]^2 - \beta^2(1-\alpha)^2}.$$

Proof. A simple computation yields

$$|J_n(f,\alpha,\gamma;z)| < \beta \quad \text{implies} \quad \left| \frac{D^{n+1}f(z)}{D^n f(z)} - 1 \right| \leq \frac{2\beta\gamma(1-\alpha)}{1+2\beta\gamma-\beta}$$

Therefore

$$\mathrm{Re}\left\{ \frac{D^{n+1}f(z)}{D^n f(z)} \right\} > \sigma(\alpha,\beta,\gamma) \ ,$$

where

$$\sigma(\alpha,\beta,\gamma) = \frac{1-\beta+2\alpha\beta\gamma}{1-\beta+2\beta\gamma} \ .$$

Let $f, g \in T_n(\alpha,\beta,\gamma)$, then $f * g \in T_n(\alpha,\beta',\gamma)$, *where*

$$\beta'(\alpha,\beta,\gamma,n) = \frac{2\beta^2\gamma(1-\alpha)}{\left[1+\beta(-1+4\gamma-2\alpha\gamma)\right]^2 2^n - 2\beta^2\gamma(1-\alpha)(-1+4\gamma-2\alpha\gamma)}$$

therefore

$$\sigma(\alpha,\beta',\gamma) = \sigma(\alpha,\beta'(\alpha,\beta,\gamma,n),\gamma) = \rho(\alpha,\beta,\gamma,n) \ .$$

Hence

$$f \in T_n(\sigma(\alpha,\beta',\gamma),1,1) = T_n(\rho(\alpha,\beta,\gamma),1,1).$$

We now give some particular cases for Theorem 13.

Corollary 3. *Let* $f, g \in T_n(\alpha,\beta,1)$, $\alpha \in [0,1)$, $\beta \in (0,1]$ *and* $n \in N_0$, *then* $f * g \in T_n(\alpha,\beta',1)$, *where*

$$\beta'(\alpha,\beta,1,n) = \frac{2\beta^2(1-\alpha)}{[1+\beta(3-2\alpha)]^2 2^n - 2\beta^2(1-\alpha)(3-2\alpha)}$$

and $0 < \beta'(\alpha,\beta,1,n) < \beta$ *.The result is sharp . This is due to Salagean* [8].

Next corollary gives the result for the class introduced and studied by Kulkarni [3].

Corollary 4. *Let* $f, g \in T_0(\alpha,\beta,\gamma)$, $\alpha \in [0,1)$, $\beta \in (0,1]$, $\gamma \in (1/2,1]$ *then* $f*g \in T_0(\alpha,\beta',\gamma)$, *where*

$$\beta'(\alpha,\beta,\gamma,0) = \frac{2\beta^2\gamma(1-\alpha)}{[1+\beta(-1+4\gamma-2\alpha\gamma)]^2 - 2\beta^2\gamma(1-\alpha)(-1+4\gamma-2\alpha\gamma)}$$

and $0 < \beta'(\alpha,\beta,\gamma,0) < \beta$ *.The result is sharp.*

Theorem 14. *Let* $f, g \in T_n(\alpha,\beta,\gamma)$, $\alpha \in [0,1)$, $\beta \in (0,1]$, $\gamma \in (1/2,1]$ *and* $n \in N_0$, *then* $f*g \in T_n(\alpha,\beta,\gamma')$, *where*

$$\gamma' = \gamma'(\alpha,\beta,\gamma,n) = \frac{2\beta\gamma^2(1-\alpha)(1-\beta)}{[1+\beta(-1+4\gamma-2\alpha\gamma)]^2 2^n - 4\beta^2\gamma^2(1-\alpha)(2-\alpha)} \tag{2.5.13}$$

and $0 \le \gamma'(\alpha,\beta,\gamma,n) < \gamma$ *.The result is sharp, the extremal functions are* $f = g = f_2$, *where* f_2 *is given in Theorem 1.*

Proof. If $f, g \in T_n(\alpha, \beta, \gamma)$, then (2.5.2) and (2.5.3) hold. By Theorem 1 we have $f * g \in T_n(\alpha, \beta, \gamma')$ if and only if

$$\sum_{j=2}^{\infty} \frac{j^n [\, j-1+\beta(1-j+2\gamma' j-2\gamma'\alpha)\,]}{2\,\beta\gamma'(1-\alpha)} a_j b_j \le 1 , \tag{2.5.14}$$

we wish to determine the smallest $\gamma' = \gamma'(\alpha, \beta, \gamma, n)$ such that (2.5.14) holds.

We note that the next inequalities

$$\frac{j-1+\beta(1-j+2\gamma' j-2\gamma'\alpha)}{\gamma'} \sqrt{a_j b_j}$$

$$\le \frac{j-1+\beta(1-j+2\gamma j-2\gamma\alpha)}{\gamma} , \quad j = 2, 3, \ldots \, . \tag{2.5.15}$$

imply (2.5.14).

By (2.5.6) we obtain

$$\frac{j-1+\beta(1-j+2\gamma' j-2\gamma'\alpha)}{\gamma'} \sqrt{a_j b_j}$$

$$\le \frac{2\,\beta\gamma(1-\alpha)[\, j-1+\beta(1-j+2\gamma' j-2\gamma'\alpha)\,]}{\gamma'\, j^n [\, j-1+\beta(1-j+2\gamma j-2\gamma\alpha)\,]} , \quad j = 2, 3, \ldots \, .$$

In order to obtain (2.5.15) it will sufficient to show that

$$\frac{2\,\beta\gamma\,(1-\alpha)[\,j-1+\beta(1-j+2\gamma' j-2\gamma'\alpha)\,]}{\gamma'\ j^{n}\,[\,j-1+\beta(1-j+2\gamma j-2\gamma\alpha)\,]}$$

$$\leq \frac{j-1+\beta(1-j+2\gamma j-2\gamma\alpha)}{\gamma},\quad j = 2,3,\ldots$$

These last inequalities are equivalent to

$$\gamma' \geq \frac{2\beta\,\gamma^2\,(1-\alpha)\,(1-\beta)(\,j-1)}{[\,1-j+\beta(-1+4\gamma-2\alpha\gamma)\,]^2\, j^{n} \,-\, 4\,\beta^2\,\gamma^2\,(1-\alpha)(\,j-\alpha)} = e(\,j)$$

We choose $\gamma' = \gamma'(\,\alpha,\beta,\gamma,n\,) = e(2)$, because $e(2) \geq e(\,j)\ ,\ j = 2,3,\ldots$

.

We have $\gamma' < \gamma$, because

$$\gamma - \gamma'(\,\alpha,\beta,\gamma,n\,) \geq \gamma - \gamma'(\,\alpha,\beta,\gamma,0\,) =$$

$$= \gamma\left[\frac{(1-\beta)(1-\beta+2\beta\gamma(3-\alpha))+4\beta^2\gamma^2(2-\alpha)}{(1-\beta)(1-\beta+2\beta\gamma(3-\alpha))+4\beta^2\gamma^2(2-\alpha)+2\beta\gamma(1-\alpha)(1-\beta)}\right] > 0$$

and $\gamma'(\,\alpha,\beta,\gamma,n\,) \geq 0$ because

$$[\,1+\beta(-1+4\gamma-2\alpha\gamma)\,]^2\, 2^{n} \,-\, 4\,\beta^2\,\gamma^2\,(1-\alpha)(2-\alpha)$$

$$\geq [\,1+\beta(-1+4\gamma-2\alpha\gamma)\,]^2 \,-\, 4\,\beta^2\,\gamma^2\,(1-\alpha)(2-\alpha)$$

$$= (1-\beta)\,(1-\beta+2\beta\gamma(3-\alpha))+4\beta^2\gamma^2(2-\alpha)+2\beta\gamma(1-\alpha)(1-\beta) > 0$$

The extremal functions are $f = g = f_2$.

Indeed

$$(f_2 * f_2)(z) = z - c_2 z^2 \in T_n(\alpha,\beta,\gamma'),$$

where c_2 is given in the proof of Theorem 12 and $\gamma' = \gamma'(\alpha,\beta,\gamma,n)$ because

$$\sum_{j=2}^{\infty} \frac{j^n [\, j-1+\beta(1-j+2\gamma' j-2\gamma'\alpha)\,]}{2\,\beta\gamma'(1-\alpha)}\, c_j = \frac{2^n [\, 1+\beta(-1+2\gamma' j-2\gamma'\alpha)\,]}{2\,\beta\gamma'(1-\alpha)}\, c_2 = 1.$$

Corollary 1. *If $f, g \in T_n(\alpha,\beta,\gamma)$, then $f * g \in T_n(\alpha,\beta,\gamma)$.*

Corollary 2. *If $f, g \in T_n(\alpha,\beta,\gamma)$, then $f * g \in T_n(\rho,1,1)$, where*

$$\rho = \rho(\alpha,\beta,\gamma) = 1 - \frac{(1-\alpha)^2\beta^2\gamma^2}{2^{n-2}[\,1+\beta(-1+4\gamma-2\alpha\gamma)\,]^2-\beta^2(1-\alpha)^2}\,.$$

Proof. A simple computation yields

$$|\, J_n(f,\alpha,\gamma;z)\,| < \beta \quad \text{implies} \quad \left| \frac{D^{n+1}f(z)}{D^n f(z)} - 1 \right| \le \frac{2\beta\gamma(1-\alpha)}{1+2\beta\gamma-\beta}\,,$$

therefore

$$\operatorname{Re}\left\{ \frac{D^{n+1}f(z)}{D^n f(z)} \right\} > \sigma(\alpha,\beta,\gamma)\;,$$

where

$$\sigma(\alpha,\beta,\gamma) = \frac{1-\beta+2\alpha\beta\gamma}{1-\beta+2\beta\gamma} \ .$$

Let $f, g \in T_n(\alpha,\beta,\gamma)$ then $f * g \in T_n(\alpha,\beta,\gamma')$, *where*

$$\gamma' = \gamma'(\alpha,\beta,\gamma,n) = \frac{2\beta\gamma^2(1-\alpha)(1-\beta)}{\left[1+\beta(-1+4\gamma-2\alpha\gamma)\right]^2 2^n - 4\beta^2\gamma^2(1-\alpha)(2-\alpha)},$$

therefore

$$\sigma(\alpha,\beta,\gamma') = \sigma(\alpha,\beta,\gamma'(\alpha,\beta,\gamma,n)) = \rho(\alpha,\beta,\gamma,n) \ .$$

Hence

$$f \in T_n(\sigma(\alpha,\beta,\gamma'), 1, 1) = T_n(\rho(\alpha,\beta,\gamma), 1, 1).$$

Next corollary gives the result for class introduced and studied by Kulkarni [3] .

Corollary 3. *Let* $f, g \in T_0(\alpha,\beta,\gamma)$, $\alpha \in [0,1)$, $\beta \in (0,1]$, $\gamma \in (1/2,1]$, *then* $f * g \in T_0(\alpha,\beta,\gamma')$, *where*

$$\gamma' = \gamma'(\alpha,\beta,\gamma,0) = \frac{2\beta\gamma^2(1-\alpha)(1-\beta)}{\left[1+\beta(-1+4\gamma-2\alpha\gamma)\right]^2 - 4\beta^2\gamma^2(1-\alpha)(2-\alpha)}$$

and $0 \le \gamma'(\alpha,\beta,\gamma,0) < \gamma$ *.The result is sharp*

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