CHAPTER-III

GEOMETRY OF A WORLDLINE IN BERTOTTI-ROBERTSON SPACE-TIME

1. Introduction

On the same line of previous chapter, in this chapter we exploit the NP formalism to study the geometry of the world line of the time-like particle in the Bertotti-Robertson space-time. In Sec.2 using differential calculus the tetrad components relative to Bertotti-Robertson space-time are obtained. In Sec.3, we define the components of the null tetrad vector fields. In Sec.4 we present the Christofell symbols. For the choice of the tetrad components in Sec.3, the Newman-Penrose Spin Coefficients are evaluated in the Sec.5. It is shown for Bertotti-Robertson space-time that, the spin coefficients $\kappa, \sigma, \rho, \tau, \pi, \mu, \lambda, \nu$ vanish. Consequently, on the basis of Goldberg-Sachs theorem we conclude that the Bertotti-Robertson space-time is of Petrov-type D. In Sec.6 the components of the Riemann Curvature tensor are evaluated to find the Riemann Curvature at a point of Bertotti-Robertson space-time in the Sec.7, the expression for the Riemann Curvature at a point of Bertotti-Robertson space-time for the orientation determined by two real null vector field or two complex null vector fields are obtained. In the Sec.-8, the expressions for the Curvature field $K = \frac{-1}{\sqrt{2}}\left(\frac{e}{\sqrt{2}r^2} + \frac{3}{\sqrt{2}e}\right)$, the Torsion field T = 0 and the Bitorsion field B = 0 of the world line of the time like particle are derived through NP spin coefficients. With the help of rheotetrad , it is shown that the world line of the time-like particle in the Bertotti-Robertson space-time is a torsion free plane curve.

2. TETRAD VECTORS RELATIVE TO BERTOTTI-ROBERTSON SPACE-TIME

The Bertotti-Robertson space-time metric is a solution of Einstein-Maxwell field equation with the source free non-null electromagnetic field.

The Bertotti-Robertson Space-time is given by

$$ds^2 = \left(\frac{e}{r}\right)^2 \left[du^2 + 2dudr - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$$

$$ds^2 = \left(\frac{e}{r}\right)^2 du^2 + \left(\frac{e}{r}\right)^2 dudr + \left(\frac{e}{r}\right)^2 drdu - e^2 d\theta^2 - e^2 \sin^2\theta d\phi^2 \qquad (2.1)$$

Then the covariant tensor components of the metric tensor are given by

$$g_{ab} = \begin{bmatrix} \left(\frac{e}{r}\right)^2 & \left(\frac{e}{r}\right)^2 & 0 & 0 \\ \left(\frac{e}{r}\right)^2 & 0 & 0 & 0 \\ 0 & 0 & -e^2 & 0 \\ 0 & 0 & 0 & -e^2\sin^2\theta \end{bmatrix} \qquad (2.2)$$

Here $g = |g_{ab}| = -\frac{e^8 \sin^2\theta}{r^4}$ (2.3)

The contravariant tensor components are given by

$$g^{ab} = \begin{bmatrix} 0 & \left(\frac{r}{e}\right)^2 & 0 & 0 \\ \left(\frac{r}{e}\right)^2 & -\left(\frac{r}{e}\right)^2 & 0 & 0 \\ 0 & 0 & \frac{-1}{e^2} & 0 \\ 0 & 0 & 0 & \frac{-1}{e^2 \sin^2\theta} \end{bmatrix} \tag{2.4}$$

We express the Bertotti-Robertson space-time (2.1) as

$$ds^2 = \left(\frac{e}{r}\right)^2 \left[(du + dr)^2 - dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$$

$$ds^2 = \left[\left(\left(\frac{e}{r}\right)du + \left(\frac{e}{r}\right)dr\right)^2 - \left(\frac{e}{r}\right)^2 dr^2 - e^2 d\theta^2 - e^2 \sin^2\theta d\phi^2\right]$$

which can be expressed in terms of basis 1-forms $\theta^{(\alpha)}$ as

$$dx^2 = \left(\theta^1\right)^2 - \left(\theta^2\right)^2 - \left(\theta^3\right)^2 - \left(\theta^4\right)^2 \tag{2.5}$$

where

$$\theta^1 = \left(\frac{e}{r}\right)du + \left(\frac{e}{r}\right)dr,$$

$$\theta^2 = \left(\frac{e}{r}\right)dr,$$

$$\theta^3 = ed\theta,$$

$$\theta^4 = e\sin\theta d\phi. \tag{2.6}$$

are 1-forms.

We have the basis 1-forms $\theta^{(\alpha)}$ in the form of the 4-basis vectors $e_a^{(\alpha)}$

$$\theta^1 = e_a^1 dx^a = \left(\frac{e}{r}\right)du + \left(\frac{e}{r}\right)dr,$$

$$\theta^2 = e_a^2 dx^a = \left(\frac{e}{r}\right)dr,$$

$$\theta^3 = e_a^3 dx^a = ed\theta$$

$$\theta^4 = e_a^4 dx^a = e\sin\theta d\phi \tag{2.7}$$

From which we readily obtain the covariant components of the tetrad vector fields as

$$e_a^{(1)} = \left(\frac{e}{r}, \ \frac{e}{r}, \ 0, \ 0\right),$$

$$e_a^{(2)} = \left(0, \ \frac{e}{r}, \ 0, \ 0\right),$$

$$e_a^{(3)} = \left(0, \ 0, \ e \ 0\right),$$

$$e_a^{(4)} = \left(0, \ 0, \ 0, \ e\sin\theta\right). \tag{2.8}$$

The contravariant components of the tetrad vectors are obtained by using the equation $e^{(\alpha)a} = g^{ab}e_b^{(\alpha)}$ in the form

$$e^{(1)a} = \left(\frac{r}{e},\ 0,\ 0,\ 0\right),$$

$$e^{(2)a} = \left(\frac{r}{e},\ \frac{-r}{e},\ 0,\ 0\right),$$

$$e^{(3)a} = \left(0,\ 0,\ \frac{-1}{e},\ 0\right),$$

$$e^{(4)a} = \left(0,\ 0,\ 0,\ \frac{-1}{e\sin\theta}\right). \quad (2.9)$$

From equations (2.7) and (2.9) we notice that the tetrad vector field $e_a^{(1)}$ is a time-like vector field, while $e_a^{(2)}, e_a^{(3)}, e_a^{(4)}$ are space-like vector fields.

3. BERTOTTI-ROBERTSON SPACE-TIME IN NEWMAN-PENROSE FORMALISM.

In to take advantage of the mighty NP formalism for the description of Bertotti-Robertson space-time we choose four null vectors of the tetrad $e_{(\alpha)a} = (l_a, n_a, m_a, \bar{m}_a)$ as

$$l_a = \frac{1}{\sqrt{2}}(e_a^{(1)} - e_a^{(2)}),$$

$$n_a = \frac{1}{\sqrt{2}}(e_a^{(1)} + e_a^{(2)}),$$

$$m_a = \frac{1}{\sqrt{2}}(e_a^{(3)} + ie_a^{(4)}),$$

$$\bar{m}_a = \frac{1}{\sqrt{2}}(e_a^{(3)} - ie_a^{(4)}). \quad (3.1)$$

Using equations (2.8) in (3.1) we readily get

$$l_a = \frac{1}{\sqrt{2}}\left(\frac{e}{r}, \quad 0, \quad 0, \quad 0\right),$$

$$n_a = \frac{1}{\sqrt{2}}\left(\frac{e}{r}, \quad \frac{2e}{r}, \quad 0, \quad 0\right),$$

$$m_a = \frac{1}{\sqrt{2}}(0, \quad 0, \quad e \quad ie\sin\theta),$$

$$\bar{m}_a = \frac{1}{\sqrt{2}}(0, \quad 0, \quad e \quad -ie\sin\theta). \quad (3.2)$$

While the contravariant components of tetrad vector fields can be obtained by using the equation $e^a_{(\alpha)} = g^{ab}e_{(\alpha)b}$. Thus we have

$$l^a = \frac{1}{\sqrt{2}}\left(0, \quad \frac{r}{e}, \quad 0, \quad 0\right),$$

$$n^a = \frac{1}{\sqrt{2}}\left(\frac{2r}{e}, \quad \frac{-r}{e}, \quad 0, \quad 0\right),$$

$$m^a = \frac{1}{\sqrt{2}}\left(0, \quad 0, \quad \frac{-1}{e}, \quad \frac{-i}{e\sin\theta}\right),$$

$$\bar{m}^a = \frac{1}{\sqrt{2}}\left(0, \quad 0, \quad \frac{-1}{e}, \quad \frac{i}{e\sin\theta}\right). \quad (3.3)$$

We observe from equation (3.1) and (3.3) that the null vector fields of the tetrad satisfy the orthogonality condition.

4. CHRISTOFFEL SYMBOLS FOR THE BERTOTTI-ROBERTSON SPACE-TIME

Using the formula for the Christoffel symbols of first kind and second kinds given in equations (4.1) and (4.2) in chapter II, we find the Christoffel symbols for the Bertotti-Robertson space-time. These components will be used in the sequel to find the components of the curvature tensor and the expression for the curvature of the space-time at a point.

$$\Gamma^1_{11} = g^{1b}\Gamma_{11,b}$$

$$= g^{12}\Gamma_{11,2}$$

$$= \frac{g^{12}}{2}(g_{12,1} + g_{12,1} - g_{11,2})$$

On using (2.2) and (2.4) we obtain

$$= \frac{-1}{2}\left(\frac{r}{e}\right)^2\left(\frac{e^2}{r^2}\right)_{,r}$$

$$\Gamma^1_{11} = \frac{-1}{r}$$

All other Christoffel symbols are similarly derived. We record the non-vanishing Christoffel symbols here for further reference.

$$\Gamma^2_{11} = \frac{1}{r},$$

$$\Gamma^2_{12} = \frac{-1}{r},$$

$$\Gamma^2_{22} = \frac{-2}{r},$$

$$\Gamma^3_{44} = -\sin\theta\cos\theta,$$

$$\Gamma^4_{34} = \cot\theta \quad . \tag{4.1}$$

All other Christoffel symbols are zero.

i.e. $\Gamma^1_{12} = \Gamma^1_{13} = \Gamma^1_{14} = \Gamma^1_{22} = \Gamma^1_{23} = \Gamma^1_{24} = \Gamma^1_{33} = \Gamma^1_{34} = \Gamma^1_{44} = 0,$

$\Gamma^2_{13} = \Gamma^2_{14} = \Gamma^2_{23} = \Gamma^2_{24} = \Gamma^2_{33} = \Gamma^2_{34} = \Gamma^2_{44} = 0,$

$\Gamma^3_{11} = \Gamma^3_{12} = \Gamma^3_{13} = \Gamma^3_{14} = \Gamma^3_{22} = \Gamma^3_{23} = \Gamma^3_{24} = \Gamma^3_{33} = \Gamma^3_{34} = 0,$

$$\Gamma^4_{11} = \Gamma^4_{12} = \Gamma^4_{13} = \Gamma^4_{14} = \Gamma^4_{22} = \Gamma^4_{23} = \Gamma^4_{24} = \Gamma^4_{33} = \Gamma^4_{44} = 0. \tag{4.2}$$

5. NEWMAN-PENROSE SPIN COEFFICIENTS

The Newman-Penrose spin coefficients with respect to the chosen basis are obtained as follows. We have by definition,

$$\rho = l_{a;b} m^a \overline{m}^b$$

$$= m^a [l_{a,b} \overline{m}^b - \Gamma^c_{ab} l_c \overline{m}^b]$$

$$= -m^a [\Gamma^1_{a3} l_1 \overline{m}^3 + \Gamma^2_{a3} l_2 \overline{m}^3 + \Gamma^1_{a4} l_1 \overline{m}^4 + \Gamma^2_{a4} l_2 \overline{m}^4]$$

Using (3.2), (3.3), (4.1) and (4.2) we get

$$\rho = 0$$

Similarly $$\alpha = \frac{1}{2}(l_{a;b} n^a \overline{m}^b - m_{a;b} \overline{m}^a \overline{m}^b)$$

$$\alpha = \frac{1}{2}\overline{m}^b \left[\left(l_{a,b} n^a - \Gamma^c_{ab} l_c n^a \right) - \left(m_{a,b} \overline{m}^a - \Gamma^c_{ab} m_c \overline{m}^a \right) \right]$$

Noting the non-vanishing components of the null vector fields from (3.2), (3.3) and using (4.1) and (4.2) we get,

$$= \frac{-1}{2}\overline{m}^3 [m_{4,3}\overline{m}^4 - \Gamma^4_{34} m_4 \overline{m}^4] - \frac{1}{2}\overline{m}^4 [-\Gamma^4_{43} m_4 \overline{m}^3 - \Gamma^3_{44} m_3 \overline{m}^4]$$

Using (3.2), (3.3) , (4.1) and (4.2) we get,

$$\alpha = -\frac{1}{2}\left(\frac{-1}{\sqrt{2}e}\right)\left[\left(\frac{ie\sin\theta}{\sqrt{2}}\right)_{,\theta}\left(\frac{i}{\sqrt{2}e\sin\theta}\right) - \cot\theta\left(\frac{ie\sin\theta}{\sqrt{2}}\right)\left(\frac{i}{\sqrt{2}e\sin\theta}\right)\right] +$$

$$- \theta\frac{1}{2}\left(\frac{+i}{\sqrt{2}e\sin\theta}\right)\left[\cot\theta\left(\frac{ie\sin\theta}{\sqrt{2}}\right)\left(\frac{-1}{\sqrt{2}e}\right) - (-\sin\theta\cos\theta)\left(\frac{e}{\sqrt{2}}\right)\left(\frac{i}{\sqrt{2}e\sin\theta}\right)\right]$$

i.e. $$\alpha = \frac{i}{2\sqrt{2}e\sin\theta}\left[-i\sin\theta\cot\theta\right]$$

$$\alpha = \frac{\cot\theta}{2\sqrt{2}e} \qquad (5.1)$$

We list here all other non-vanishing NP spin coefficients derived for further reference in the next sections.

$$\beta = \frac{-\cot\theta}{2\sqrt{2}e},$$

$$\gamma = \frac{3}{4\sqrt{2}e},$$

$$\varepsilon = \frac{e}{2\sqrt{2}r^2}. \tag{5.2}$$

All other NP spin coefficients vanish

i.e. $\kappa = \sigma = \rho = \tau = \pi = \mu = \lambda = \nu = 0$. (5.3)

Here as $\kappa, \sigma, \lambda, \nu$ vanish, it confirms the type-D characteristic of the space-time. The vanishing of the spin coefficients κ, σ shows that the null geodesic l_a is shear-free, while the null geodesic n_a is also shear-free as λ, ν vanish. Then by Goldberg-Sachs theorem, the shear-free character of the null geodesic congruences l_a and n_a shows that the Bertotti-Robertson space-time is of Petrov-type D.

6. COMPONENTS OF RIEMANN CURVATURE TENSOR

In V_4, the 20 independent components of the Riemann-Curvature tensor are obtain by using the formula

$$R_{hijk} = \frac{1}{2}(g_{hk,ij} + g_{ij,hk} - g_{hj,ik} - g_{ik,hj}) + g_{ab}\Gamma^a_{hk}\Gamma^b_{ij} - g_{ab}\Gamma^a_{hj}\Gamma^b_{ik} \tag{6.1}$$

For h,l,j,k = 1,2,3,4 we obtain all non-vanishing components of the then the non-vanishing components are Riemann-Curvature tensor.

For example

$$R_{1212} = \frac{1}{2}(g_{12,21} + g_{21,12} - g_{11,22} - g_{22,11}) - g_{12}\Gamma^1_{11}\Gamma^2_{22}$$

Using (2.2),(4.1) and (4.2) we get,

$$R_{1212} = \frac{1}{2}\left[\left(\frac{e}{r}\right)^2_{,ur} + \left(\frac{e}{r}\right)^2_{,ru} - \left(\frac{e}{r}\right)^2_{,rr}\right] - \left(\frac{e}{r}\right)^2\left(\frac{-1}{r}\right)\left(\frac{-2}{r}\right)$$

$$R_{1212} = \frac{-5e^2}{r^4} \tag{6.2}$$

The other non-vanishing components of Riemann-Curvature tensor are

$$R_{3434} = -e^2 \sin^2\theta \tag{6.3}$$

We record below the vanishing components of Riemann-Curvature tensor

$$R_{1213} = R_{1214} = R_{1223} = R_{1224} = R_{1234} = 0,$$

$$R_{1313} = R_{1314} = R_{1323} = R_{1324} = R_{1334} = 0,$$

$$R_{1414} = R_{1423} = R_{1424} = R_{1434} = 0,$$

$$R_{2323} = R_{2424} = R_{2434} = R_{3234} = 0. \tag{6.4}$$

7. CURVATURE OF BERTOTTI_ROBERTSON SPACE-TIME

Using the equation (7.3) from Chapter-II and using the properties of R_{abcd} we find the curvature determined by the real null vector fields l^a and n^a as

$$\mathrm{K} = \frac{5}{e^2} \tag{7.1}$$

Similarly the Riemann curvature at a point of the Bertotti-Robertson space-time for the orientation determined by the complex null vector fields m^a and $\overline{m}^a$ is given by

$$K = \frac{-1}{e^2} \tag{7.2}$$

8.CURVATURE, TORSION, BITORSION FIELDS OF THE WORLDLINE OF THE TME-LIKE VECTOR FIELD IN BERTOTTI-ROBERTSON SPACE-TIME

Here we exploit the Rheotetrad introduced in Chapter-I, to study the geometry of the world line of the time-like particle in the Bertotti-Robertson space-time. In this section we explicitly find the expression for the Curvature, Torsion and Bitiorsion of the world line of the particle by using the expressions (3.7),(3.10) and (3.16) of chapter-I. Using the equations (5.2) and (5.3) we get,

$$K = \frac{-1}{\sqrt{2}}\left(\frac{e}{2\sqrt{2}r^2} + \frac{e}{2\sqrt{2}r^2} + \frac{3}{4\sqrt{2}e} + \frac{3}{4\sqrt{2}e}\right)$$

$$K = \frac{-1}{\sqrt{2}}\left(\frac{e}{\sqrt{2}r^2} + \frac{3}{\sqrt{2}e}\right), \tag{8.1}$$

$$T = 0, \tag{8.2}$$

$$B = \frac{-1}{\sqrt{2}}\left(0 - 0 + \frac{3}{4\sqrt{2}e} - \frac{3}{4\sqrt{2}e}\right)$$

$$B = 0 \; . \tag{8.3}$$

As, T=0, by the conclusions from chapter-I, the world line of the particle in Bertotti-Robertson space-time is a torsion free plane curve.