CHAPTER - I

NOTATIONS, DEFINATIONS AND PRILIMNARY RESULTS

The best way of overcoming a difficult problem is to solve in some particular easy cases. These give much light in to the general solution. By this way, Newton says, he overcame the most difficult things.

David Gregory

CHAPTER – 1

Notations, Definitions and Preliminary results

1.1 Notations, which are used in this dissertation.

Sr. No.	Notations	Meaning of Notations
1	$=$	Equal to
2	$<$	Strictly less than
3	$\leq$	Less than or equal to
4	$>$	Strictly greater than
5	$\geq$	Greater than or equal to
6	$\therefore$	Therefore
7	$\because$	Since
8	$\Rightarrow$	Implies
9	∞	Infinity
10	$\mathbb{Z}$	Set of integers.
11	$\mathbb{R}$	Set of real numbers.
12	$\delta_{i,j}$	Kronecker Delta
13	Γ	Gamma function.
14	$\hat{f}(w)$	Fourier Transform of function f(x).
15	$Y_\nu(z)$	Young's function of order ν.
16	$J_\nu(z)$	Bessel function of First kind and order ν.
17	F(a;b,z)	Confluent hypergeometric function.
18	$P_\nu^\mu(z)$	Legendre Function.
19	$\beta(a,b)$	Beta function.
20	$L^p(\mathbb{R})$	The class of all measurable functions f on $\mathbb{R}$ such that the (Lebesgue) integral $\left\{\int_{-\infty}^{\infty}\lvert f(x)\rvert^p dx\right\}^{1/p}$ is finite.

1.2 Definitions:

1. Fourier Transform:

The Fourier Transform of function $f \in L^1(\mathbb{R})$

is defined by

$$\hat{f}(w) = \int_{-\infty}^{\infty} e^{iwx} f(x)\,dx \qquad \text{where } w \in \mathbb{R} \qquad (1.2\text{-}1)$$

Let $\hat{f} \in L^1(\mathbb{R})$ be the Fourier Transform of some function

$f \in L^1(\mathbb{R})$, then the inverse Fourier Transform of $\hat{f}$ is defined by,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iwx} \hat{f}(w)\,dw \qquad (1.2\text{-}2)$$

2. Mother wavelet (Basic wavelet) :

A function $\psi \in L^2(\mathbb{R})$ is called basic wavelet if

i) $\int_{-\infty}^{\infty} \psi(t)\, dt = 0$

i.e. function integrates to zero and it also suggests that ψ is an oscillatory function .

ii) ψ satisfies admissibility condition

$$c_\psi = \int_{-\infty}^{\infty} \frac{|\hat{\psi}(w)|^2}{|w|}\, dw < \infty$$

3. Wavelet:

A wavelet is a function $\psi \in L^2(\mathbb{R})$ such that the family of the functions

$$\psi_{m,n}(t) = 2^{\frac{m}{2}} \psi(2^m t - n) \qquad (1.2\text{-}3)$$

where m and n are arbitrary integers, is an orthonormal basis in the space $L^2(\mathbb{R})$.

4. Orthonormal wavelet:

A function $\psi \in L^2(\mathbb{R})$ is called an orthonormal wavelet if $\{\psi_{j,k}\}$ satisfies the orthonormality condition

$$\langle \psi_{j,k}\ \psi_{l,m} \rangle = \delta_{j,l}\ \delta_{k,m} \qquad j,k,l,m \in \mathbb{Z} \qquad (1.2\text{-}4)$$

5. Sampling function (Interpolating function) :

A continuous function $S \in L^2(\mathbb{R})$ is a sampling function if

1. S interpolates the Kronecker sequence at integers

$$S(n) = \delta_{n,0} \qquad n = 0, \pm 1, \pm 2, \ldots\ldots\ldots \qquad (1.2\text{-}5)$$

2. Their exist a nonnegative constant A and B such that

$$0 < A \le \sum_{k=-\infty}^{\infty} \left|\hat{S}(w + 2\pi k)\right|^2 \le B < \infty \qquad (1.2\text{-}6)$$

for almost all $w \in \mathbb{R}$, where $\hat{S}$ is Fourier Transform taken in the sense of $L^2(\mathbb{R})$

6. Young's Function:

Young's function of order ν (ν ≥ 0) denoted by Y_ν defined by

$$Y_\nu(z) = z^\nu \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{\Gamma(\nu + 2k + 1)} \tag{1.2-7}$$

7. Bessel function:

Bessel function of first kind and order ν denoted by J_ν and defined by

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+\nu}}{2^{\nu+2k} k! \, \Gamma(\nu + k + 1)} \tag{1.2-8}$$

8. Riesz Basis:

A function $\psi \in L^2(\mathbb{R})$ is said to generate a Riesz basis (unconditional basis) $\{\psi_{b_0;j,k}\}$ with sampling rate b_0 if

i) The linear span $\langle \psi_{b_0;j,k};\ j,k \in \mathbb{Z} \rangle$ is dense in $L^2(\mathbb{R})$.

ii) Their exist a constants A and B with $0 < A \le B < \infty$ such that

$$A \left\| \{c_{j,k}\} \right\|_{l^2}^2 \le \left\| \sum_{j,k\in\mathbb{Z}} c_{j,k} \psi_{b_0;j,k} \right\|_2^2 \le B \left\| \{c_{j,k}\} \right\|_{l^2}^2 \tag{1.2-9}$$

for all $\{c_{j,k}\} \in L^2(\mathbb{Z}^2)$

here A and B are called Riesz Bounds of $\{\psi_{b_0;j,k}\}$

9. Scaling function and Multi Resolution Analysis(MRA):

A function $\phi \in L^2(\mathbb{R})$ is called scaling function if the subspaces V_j of $L^2(\mathbb{R})$ defined by $V_j = clos_{L^2(\mathbb{R})}\langle \phi_{j,k} : k \in \mathbb{Z}\rangle \;\; j \in \mathbb{Z}$ where $\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k)$ Satisfy the properties

1. $V_j \subset V_{j+1} \;\; for\ all \;\; j \in \mathbb{Z}$

2. $clos_{L^2}\left(\bigcup_{j\in\mathbb{Z}} V_j\right) = L^2(\mathbb{R})$

3. $f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1}$

4. $f(x) \in V_j \Leftrightarrow f\left(x + \frac{1}{2^j}\right) \in V_j$

and if $\{\phi(\cdot - k) : k \in \mathbb{Z}\}$ is Riesz basis of V_0, then scaling function ϕ generates MRA $\{V_j\}$ of $L^2(\mathbb{R})$.

1.3 Preliminary results :

Usually the mother wavelet is constructed from another function called the scaling function or 'father wavelet' $\phi(t) \in L^2(\mathbb{R})$.

The farther wavelet $\phi(t) \in L^2(\mathbb{R})$ is chosen in such a way that

$$\left.\begin{array}{l} 1.\ \int \phi(t)\phi(t-n) = \delta_{0,n} \qquad n \in \mathbb{Z} \\ 2.\ \phi(t) = \sum_k c_k\,\phi(2t-k) \qquad \{c_k\}_{k\in\mathbb{Z}} \in l^2 \\ 3.\ \text{For each } f \in l^2(\mathbb{R}), \in > 0 \text{ there is function} \\ f_m(t) = \sum_k a_{mn}\phi(2^m t - n) \text{ such that } \|f_m - f\| < \in \end{array}\right\} \qquad (1.3-1)$$

Sometimes these conditions are expressed in terms of their Fourier Transform. We give a sufficient condition for (1.3-1) to hold

$$\left.\begin{array}{l} 1.\sum_k \left|\hat{\phi}(w+2\pi k)\right|^2 = 1 \\ 2.\ \textit{Self similarity property or dilation equation} \\ \hat{\phi}(w) = m(w/2)\hat{\phi}(w/2)\ \ \textit{where}\ m(w/2) = \frac{1}{2}\sum_k c_k e^{ikw/2} \\ 3.\hat{\phi}(w)\ \textit{is continuous at}\ w = 0\ \textit{and}\ \hat{\phi}(0) = 1. \end{array}\right\} \qquad (1.3-2)$$

The Mother wavelet is obtained from (1.3-1,2) or (1.3-2,2) via $\phi(t)$

$$\left.\begin{array}{l} \psi(t) = \sum_k (-1)^{k+1}\overline{c}_{1-k}\phi(2t-k) \\ \hat{\psi}(w) = e^{iw/2}\overline{m}(\frac{w}{2}+\pi)\,\hat{\phi}(\frac{w}{2}) \end{array}\right\} \qquad (1.3-3)$$

Example (1.3-1):

The Sinc function $\phi(t) = \frac{\sin \pi t}{\pi t}$ satisfies (1.3-1) and we have

$\hat{\phi}(w) = \chi_{[-\pi\ \pi]}(w)$ the characteristic function of $[-\pi\ \ \pi]$.

In (1.3-2,2) $m(w/2)$ is 4π periodic extension of $\hat{\phi}(w)$.

Therefore $\overline{m}\left(\frac{w}{2}+\pi\right)\hat{\phi}\left(\frac{w}{2}\right) = \begin{cases} 1 & \pi \le w \le 2\pi \\ 0 & otherwise \end{cases}$

which leads to the Shannon wavelet,

$$\psi(t) = \frac{\sin 2\pi(t-1/2) - \sin \pi(t-1/2)}{\pi(t-1/2)} \qquad (1.3-4)$$

Lemma (1.3-1):

Let S(t) be a function such that both S and $\hat{S}$ belong to $L^1(\mathbb{R})$ and $S(n)=\delta_{n,0}$

then S(t) is sampling function iff $\sum_{k=-\infty}^{\infty} \hat{S}(w+2\pi k)=1$ a.e.

Moreover if $\hat{S}(w)\geq 0$ then S generates an orthonormal family of functions $\{\phi_n(t)=\phi(t-n)\}_{n\in\mathbf{Z}}$ in $L^2(\mathbb{R})$ where $\hat{\phi}(w)=\sqrt{\hat{S}(w)}$.

Proof:

If S(t) is sampling function then we have nonnegative constants A and B such that $0<A\leq\sum_{k=-\infty}^{\infty}\left|\hat{S}(w+2\pi k)\right|^2\leq B<\infty$ for almost all $w\in\mathbb{R}$

Therefore for A = B = 1

$\sum_{k=-\infty}^{\infty} \hat{S}(w+2\pi k)=1$ a.e.

Conversely if $\sum_{k=-\infty}^{\infty} \hat{S}(w+2\pi k)=1$ a.e.,then obviously their exist nonnegative constants A and B such that

$0<A\leq\sum_{k=-\infty}^{\infty}\left|\hat{S}(w+2\pi k)\right|^2\leq B<\infty$ for almost all $w\in\mathbb{R}$

Therefore S(t) is sampling function .

Now consider

$$\delta_{k,0} = \int_R \phi(x)\bar{\phi}(x-k)\,dx$$

By Plancherel theorem $\langle f\ \ g\rangle = \frac{1}{2\pi}\langle \hat{f}\ \ \hat{g}\rangle$

$$\therefore\quad \delta_{k,0} = \frac{1}{2\pi}\int_R \left|\hat{\phi}(w)\right|^2 e^{ikw}\,dw$$

$$= \frac{1}{2\pi}\sum_{l=-\infty}^{\infty}\int_{2l\pi}^{2(l+1)\pi} \left|\hat{\phi}(w)\right|^2 e^{ikw}\,dw$$

$$= \frac{1}{2\pi}\sum_{l=-\infty}^{\infty}\int_0^{2\pi} \left|\hat{\phi}(w+2l\pi)\right|^2 e^{ikw}\,dw$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{l\in\mathbb{Z}}\left|\hat{\phi}(w+2l\pi)\right|^2\right) e^{ikw}\,dw$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{l\in\mathbb{Z}}\hat{S}(w+2l\pi)\right) e^{ikw}\,dw \quad (\because given\,\hat{\phi}(w) = \sqrt{\hat{S}(w)}\ )$$

$$= 1 \quad a.e.$$

Thus $\delta_{k,0} = 1$ *a.e.* $k \in \mathbb{Z}$

$\Rightarrow \{\phi_n(t) = \phi(t-n)\}_{n\in\mathbb{Z}}$ is an orthonormal family of functions.

1.4 Young's function:

W.H. Young introduced Young's functions in 1912 [3] in his investigation of nonconverging Fourier series. Young's function can be calculated numerically to a high degree of precision. Young's function of integral order is an entire function of exponential type that reduced to the cosine and sine function, when its order ν is zero and one respectively.

Therefore putting ν = 0 and ν =1 in the formula (1.2-7); we get,

$Y_0(z)$ = Cos Z and $Y_1(z)$ = Sin Z.

Also by using formula (1.2-7) we get the relations,

$$Y_v(x) = \frac{d}{dx} Y_{v+1}(x)$$
$$\text{and} \quad Y_v(x) = \frac{x^{v-2}}{\lceil v-1} - Y_{v-2}(x)$$

and combining together we get the differential equation for Y_v,

$$y'' + y = \frac{x^{v-2}}{\lceil v-1} \qquad v > 1.$$

For $v = 0$ and 1 we have, $y'' + y = 0$.

Another important special function that will be used in subsequent chapters is the following integral of Young's function,

$I_{v,\alpha}$ defined by

$$I_{v,\alpha}(x) = \int x^\alpha Y_v(x)\,dx = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+v-\alpha+1}}{(2k+v+\alpha+1)\lceil v+2k+1} \tag{1.4-1}$$

In particular $I_{v,0}(x) = \int Y_v(x)\,dx = Y_{v+1}(x)$ and

$$\left.\begin{aligned} \int \sqrt{x}\cos x\,dx &= I_{0,1/2}(x) \\ \int \sqrt{x}\sin x\,dx &= I_{1,1/2}(x) \end{aligned}\right\} \tag{1.4-2}$$

CHAPTER - II

WAVELET CONSTRUCTION

All depends, then, on finding these easire problems, and solving them by means of devices as perfect as possible and concepts capable of generalization.

David Hilbert

CHAPTER - 2

WAVELET CONSTRUCTION

2.1 Introduction:

In this chapter we have studied a general procedure for constructing band-limited wavelets known as h-construction of wavelets. This is based on earlier work by Walter [4].

Lemma (2.1-1):

Let h be a function satisfying the following conditions

1) $h \in L^1(\mathbb{R})$

2) $h \geq 0$

3) $\int_{-\infty}^{\infty} h(x)\,dx = 1$

4) $h(x)$ *is even*;

5) *Support* $h \subset \left[\frac{-\pi}{3}, \frac{\pi}{3}\right]$

$$Let\ \hat{\phi}(w) = \int_{w-\pi}^{w+\pi} h(x)\,dx \qquad (2.1\text{-}1)$$

then $\hat{\phi}(w)$ is nonnegative, even , continuous function with support in

$\left[-\frac{4\pi}{3}, \frac{4\pi}{3}\right]$ and $\hat{\phi}(w)=1\ on\left[-\frac{2\pi}{3}, \frac{2\pi}{3}\right]$

Moreover, $\sum_{k=-\infty}^{\infty} \hat{\phi}(w+2\pi k)=1$ (2.1-2)

Proof:

1) As $0 \leq h$ We have $\hat{\phi}(w) = \int_{w-\pi}^{w+\pi} h(x)\,dx \quad \geq 0$

i.e.$\hat{\phi}$ is nonnegative function.

2) Consider $\hat{\phi}(-w) = \int_{-w-\pi}^{-w+\pi} h(x)\,dx$

Let $-t = x + 2w \Rightarrow dx = -dt$

Also when $x = -w - \pi$, $t = -w + \pi$ and when $x = w + \pi$, $t = -w - \pi$

$$\therefore \hat{\phi}(-w) = \int_{-w+\pi}^{-w-\pi} h(-t-2w)\,(-dt)$$

$$= \int_{-w+\pi}^{-w-\pi} h(t+2w)\,(-dt) \quad (\because h \text{ is even function.})$$

$$= \int_{-w-\pi}^{-w+\pi} h(t+2w)\,dt$$

Let $t + 2w = y \Rightarrow dt = dy$

When $t = -w - \pi$, $y = w - \pi$ and when $t = -w + \pi$, $t = w + \pi$

$$\therefore \hat{\phi}(-w) = \int_{w-\pi}^{w+\pi} h(y)\,dy = \hat{\phi}(w) \left(\because \text{By defination } \hat{\phi}.\right)$$

Thus $\hat{\phi}(-w) = \hat{\phi}(w)$

$\Rightarrow \hat{\phi}$ is even function .

3) Let $w < -\frac{4\pi}{3}$ then $w - \pi < \frac{-7\pi}{3}$ and $w + \pi < -\frac{\pi}{3}$

$\Rightarrow [w-\pi \quad w+\pi] \subset \left(-\infty \quad -\frac{\pi}{3}\right]$

But $\hat{\phi}(w)=\int_{w-\pi}^{w+\pi} h(x)dx$ *and* $supp\, h \subset \left[-\frac{\pi}{3}\ \frac{\pi}{3}\right]$

$\Rightarrow \hat{\phi}(w)=0$ *for* $w<-\frac{4\pi}{3}$

Similary $\hat{\phi}(w)=0$ *for* $w>\frac{4\pi}{3}$

$\therefore supp\,\hat{\phi} \subset \left[-\frac{4\pi}{3}\ \frac{4\pi}{3}\right]$

4) Let

$\frac{-2\pi}{3} \le w \le \frac{2\pi}{3}$

$\therefore \frac{-5\pi}{3} \le w-\pi \le \frac{-\pi}{3}$ and $\frac{\pi}{3} \le w+\pi \le \frac{5\pi}{3}$

Also $\int_{-\infty}^{\infty} h(x)\,dx = \int_{-\pi/3}^{\pi/3} h(x)\,dx = 1$ $\quad \therefore \hat{\phi}(w)=\int_{w-\pi}^{w+\pi} h(x)dx = 1$ *on* $\left[-\frac{2\pi}{3}\ \frac{2\pi}{3}\right]$

5) Let $|w_1-w_2|<\delta$

Consider $\left|\hat{\phi}(w_1)-\hat{\phi}(w_2)\right|$

$$= \left| \int_{w_1-\pi}^{w_1+\pi} h(x)dx - \int_{w_2-\pi}^{w_2+\pi} h(x)dx \right|$$

$$= \left| \int_{w_1-\pi}^{0} h(x)dx + \int_{0}^{w_1+\pi} h(x)dx - \int_{w_2-\pi}^{0} h(x)dx - \int_{0}^{w_2+\pi} h(x)dx \right|$$

$$= \left| \int_{w_1-\pi}^{w_2-\pi} h(x)dx - \int_{w_1+\pi}^{w_2+\pi} h(x)dx \right|$$

Let $x+\pi=t_1 \Rightarrow dx=dt_1$
and when $x=w_1-\pi,\ t_1 = w_1$
when $x=w_2-\pi,\ t_1 = w_2$

Let $x-\pi=t_2 \Rightarrow dx=dt_2$
and when $x=w_1+\pi,\ t_2 = w_1$
when $x=w_2+\pi,\ t_2 = w_2$

$$\therefore \left|\hat{\phi}(w_1)-\hat{\phi}(w_2)\right| = \left|\int_{w_1}^{w_2} h(t_1-\pi)\,dt_1 - \int_{w_1}^{w_2} h(t_2+\pi)\,dt_2\right|$$

$$= \left|\int_{w_1}^{w_2} \left[h(t-\pi) - h(t+\pi)\right] dt\right|$$

$$\leq \int_{w_1}^{w_2} \left|h(t-\pi) - h(t+\pi)\right| \left|dt\right|$$

As $h \in L^1(\mathbb{R})$

$$\therefore \left|\hat{\phi}(w_1)-\hat{\phi}(w_2)\right| \leq m\left|w_2 - w_1\right| = m\delta = \in$$

Thus for given $\delta > 0$ with $\left|w_2 - w_1\right| < \delta$ their exit $\in > 0$ such that

$$\left|\hat{\phi}(w_1)-\hat{\phi}(w_2)\right| < \in$$

$\Rightarrow \hat{\phi}$ *is continous function .*

6) Consider

$$\sum_{k=-\infty}^{\infty} \hat{\phi}(w+2\pi k)$$

$$= \sum_{k=-\infty}^{\infty} \int_{w+2\pi k-\pi}^{w+2\pi k+\pi} h(x)\,dx$$

$$= \sum_{k=-\infty}^{\infty} \int_{w+(2k-1)\pi}^{w+(2k+1)\pi} h(x)\,dx$$

$$= \int_{-\infty}^{\infty} h(x)\,dx = 1$$

Theorem (2.1-1):

The function $\phi(t)$ defined by $\hat{\phi}(w)=\int_{w-\pi}^{w+\pi} h(x)\ dx$

Where $h \in L^1(\mathbb{R})$, $h \geq 0$, $\int_{-\infty}^{\infty} h(x)dx = 1$, h (x) is even and

supp h $\subset\left[-\frac{\pi}{3}\ \frac{\pi}{3}\right]$ is both a sampling function and a scaling function of

MRA and corresponding mother wavelet may be given by

$$\psi\left(t+\frac{1}{2}\right)=2\phi(2t)-\phi(t) \tag{2.1-3}$$

Proof:

1) The function $\phi(t)$ is sampling function follows from equation (2.1-2) and Lemma (1.3-1).

2) Since, by Lemma (2.1-1)

$$\hat{\phi}(w)=\begin{cases} 1 & on\left[\frac{-2\pi}{3}\ \frac{2\pi}{3}\right] \\ 0 & outside \quad \left[\frac{-4\pi}{3}\ \frac{4\pi}{3}\right] \end{cases}$$

$$\therefore \hat{\phi}\left(\frac{w}{2}\right)=\begin{cases} 1 & on\left[\frac{-4\pi}{3}\ \frac{4\pi}{3}\right] \\ 0 & outside \quad \left[\frac{-8\pi}{3}\ \frac{8\pi}{3}\right] \end{cases}$$

Let $m\left(\frac{w}{2}\right)$ be 4π periodic extension of $\hat{\phi}(w)$.

$$\therefore m\left(\frac{w}{2}\right)=\begin{cases}\hat{\phi}(w) & on\left[\frac{-4\pi}{3}\ \frac{4\pi}{3}\right]\\ 0 & on\frac{4\pi}{3}\le|w|\le\frac{8\pi}{3}\end{cases}$$

Thus we have $\hat{\phi}(w)=m(w/2)\hat{\phi}(w/2)$, which is the dilation equation.

3) Now to show that $\{\phi(t-n)\}$ is a Riesz basis we have to show that their exist 0 < A, B < ∞ such that $A\le\sum_k\left|\hat{\phi}(w+2\pi k)\right|^2\le B$.

Let $g(w)=\sum_k\left|\hat{\phi}(w+2\pi k)\right|^2$

Therefore $g(w)$ is even function and periodic with period 2π.

Thus it is sufficient to consider its behavior on the interval [0 2π].

Since for each w the series contains at most two nonzero terms it is bounded above, also $\hat{\phi}(w)$ is positive on [- π π], therefore $g(w)$ is also bounded below by a constant.

As $\hat{\psi}(w)=e^{i(w/2)}\overline{m}\left(\frac{w}{2}+\pi\right)\hat{\phi}\left(\frac{w}{2}\right)$

We have $e^{-i(w/2)}\hat{\psi}(w)=\hat{\phi}\left(\frac{w}{2}\right)\left[\hat{\phi}(w-2\pi)+\hat{\phi}(w+2\pi)\right]$

$$=\hat{\phi}\left(\frac{w}{2}\right)\left[\hat{\phi}(w-2\pi)+\hat{\phi}(w+2\pi)+\hat{\phi}(w)-\hat{\phi}(w)\right]$$

$$=\hat{\phi}\left(\frac{w}{2}\right)\left[1-\hat{\phi}(w)\right]$$

$$=\hat{\phi}\left(\frac{w}{2}\right)-\hat{\phi}(w)$$

$$e^{-i(w/2)}\hat{\psi}(w) = \hat{\phi}\left(\frac{w}{2}\right) - \hat{\phi}(w) \tag{2.1-4}$$

Since $\hat{\phi}\left(\frac{w}{2}\right) = 1$ on support of $\hat{\phi}(w)$

By taking inverse Fourier Transform

$$\psi\left(t+\frac{1}{2}\right) = 2\phi(2t) - \phi(t).$$

Corollary (2.1-1):

The mother wavelet is given by $\psi\left(t+\frac{1}{2}\right) = 2\phi(2t) - \phi(t)$

is the sampling function at the half integers $\psi\left(n+\frac{1}{2}\right) = \delta_{0,n}$

Proof:

As $\psi\left(n+\frac{1}{2}\right) = 2\phi(2n) - \phi(n)$

$$\begin{aligned}\psi\left(n+\frac{1}{2}\right) &= 2\delta_{0,n} - \delta_{0,n} \\ &= \delta_{0,n}\end{aligned}$$

Therefore the mother wavelet ψ is sampling function at the half integers.

We can use the same construction to obtain an orthonormal scaling function by taking non-negative square root of $\hat{\phi}(w)$.

i.e. $\hat{\phi}_o(w) = \sqrt{\hat{\phi}(w)}$ (say)

Such a $\hat{\phi}_o(w)$ satisfies the equation (1.3-2, 1) and dilation equation (1.3-2, 2).

The mother wavelet $\psi_o(t)$ obtain from (1.3-3) will then generate an orthonormal basis.

Unfortunately except for the example

$$\phi_1(t) = \frac{\sin\pi(1-\beta)t - 4\beta t\cos\pi(1+\beta)t}{\pi t\left[1-(4\beta t)^2\right]} \tag{2.1-5}$$

Where $0 < \beta < \frac{1}{3}$ and it's associated mother wavelet

$$\begin{aligned}\psi_1\left(t+\frac{1}{2}\right) = & \frac{\sin\pi(1+\beta)t - 4\beta t\cos\pi(1-\beta)t}{\pi t\left[(4\beta t)^2-1\right]} \\ & -\frac{\sin 2\pi(1-\beta)t + 8\beta t\cos\pi(1+\beta)t}{\pi t\left[(4\beta t)^2-1\right]}\end{aligned} \tag{2.1-6}$$

This does not lead to a closed form expression.

A modification of this approach was introduced by Liu (as reported in [5]) where he constructs scaling function, which is simultaneously sampling functions and orthonormal. That is $\hat{\phi}(w)$ satisfied both (1.3-2, 1) and (2.1-2). Most scaling functions were not in closed form except for one case as reported in [2] .

$$\phi_2(t) = \frac{\sin\pi(1-\beta)t + \sin\pi(1+\beta)t}{2\pi t[1+2\beta t]} \tag{2.1-7}$$

Thus we have two raised cosine wavelets $\phi_1(t)$ and $\phi_2(t)$ in closed form expression. In next section we study another approach, which gives many additional examples.

2.2 Orthonormal wavelets in closed form:

In the previous section we did not get orthonormal wavelets in closed form except in special cases. Therefore we introduce a new approach, which is based on modification of an orthonormal scaling function to obtain new orthonormal wavelets basis.

Proposition (2.2.-1) : Let $\theta(w)$ be real valued, odd measurable, function on $\mathbb{R}$ continuous at Zero such that $e^{i\theta(w)}$ is 2π periodic.

Let $\phi(t)$ be an orthonormal scaling function satisfying (1.3-2)

then, $$\hat{\phi}_\theta(w) = e^{i\theta(w)}\hat{\phi}(w) \qquad (2.2\text{-}1)$$

is the Fourier Transform of an orthonormal scaling function, $\phi_\theta(t)$ satisfying

1. $\sum_{k=-\infty}^{\infty} \left|\hat{\phi}_\theta\left(w + 2\pi k\right)\right|^2 = 1$
2. $\hat{\phi}_\theta(w) = m(w/2)\hat{\phi}_\theta(w/2)$
3. $\hat{\phi}_\theta$ *is continous at* 0 *and* $\hat{\phi}_\theta(0) = 1$

Proof :

1. As $\hat{\phi}_\theta(w) = e^{i\theta(w)}\hat{\phi}(w)$

$$\left|\hat{\phi}_\theta(w)\right| = \hat{\phi}(w)$$

$$\therefore \sum_{k=-\infty}^{\infty}\left|\hat{\phi}_\theta(w+2\pi k)\right|^2 = \sum_{k=-\infty}^{\infty}\left|\hat{\phi}(w+2\pi k)\right|^2 = 1$$

2. Again consider $\hat{\phi}_\theta(w) = e^{i\theta(w)}\hat{\phi}(w)$

$$\therefore \hat{\phi}_\theta(w) = e^{i\theta(w)} m\left(\frac{w}{2}\right)\hat{\phi}(\frac{w}{2}) \qquad (\because \text{ by 2.2-1,2})$$

$$= e^{i\theta(w)} m\left(\frac{w}{2}\right) e^{-i\theta\left(\frac{w}{2}\right)}\hat{\phi}_\theta(\frac{w}{2}) \qquad (\because \text{ by 2.2-2})$$

$$= e^{i\left[\theta(w)-\theta\left(\frac{w}{2}\right)\right]} m\left(\frac{w}{2}\right)\hat{\phi}_\theta(\frac{w}{2})$$

$$= m_\theta\left(\frac{w}{2}\right)\hat{\phi}_\theta(\frac{w}{2})$$

3. Since $e^{i\left[\theta(w)-\theta\left(\frac{w}{2}\right)\right]}$ is 4π periodic and

$$\left|m_\theta(w)\right|^2 + \left|m_\theta(w+\pi)\right|^2 = 1 \ a.e.$$
$$iff \ \left|m(w)\right|^2 + \left|m(w+\pi)\right|^2 = 1 \ a.e.$$

Clearly $\hat{\phi}_\theta(w)$ is continuous at w = 0 and $\hat{\phi}_\theta(0) = 1$.

Since $\theta(w)$ being odd and continuous must have value zero at zero.

To construct Mother wavelet ψ_θ we use relation

$$\hat{\psi}_\theta(w) = e^{i\left(\frac{w}{2}\right)} \overline{m}_\theta\left(\frac{w}{2}+\pi\right)\hat{\phi}_\theta\left(\frac{w}{2}\right)$$

$$= e^{i\left(\frac{w}{2}-\theta(w+2\pi)+\theta(w+2\pi)\right)} \overline{m}\left(\frac{w}{2}+\pi\right)\hat{\phi}_\theta\left(\frac{w}{2}\right)$$

$$\therefore \hat{\psi}_\theta(w) = e^{i\left(\frac{w}{2}-\theta(w+2\pi)+\theta(w+2\pi)\right)} e^{i\theta\left(\frac{w}{2}\right)} \overline{m}\left(\frac{w}{2}-\pi\right)\hat{\phi}\left(\frac{w}{2}\right) \qquad (2.2\text{-}2)$$

we can now construct many new examples based on the three we already know, the sinc function and the two raised cosine wavelets (2.1-5) and (2.1-7).

The simplest case is when $\phi(t)$ is sinc function i.e. $\hat{\phi}(w)$ is the characteristic function of the interval $[-\pi\ \pi]$ and $\theta(w)$ is piecewise constant which does not satisfy continuity hypothesis at the origin.

From this we get the following number of examples.

For fixed nonnegative integer n,

Let $I_0 = \left[0\ \frac{\pi}{2^n}\right)$ and $I_k = \left[\frac{\pi}{2^{n-k+1}}\ \frac{\pi}{2^{n-k}}\right)$ k = 1,2,3,.........,n. (2.2-3)

Let us define $\theta_1(w)$ on $[0\ \pi)$ as $\theta_1(w) = \epsilon_k \ on\ I_k$ k = 0,1,2,3,.........,n.

Where ϵ_k are any real numbers.

Let $\theta(w)$ be odd extension of $\theta_1(w)$ to $(-\pi\ \pi)$ extended periodically to whole real line.

Therefore $\theta(w)$ is of the form in proposition (2.2-1) except for continuity at zero. The resulting $\{\phi_\theta(t-n)\}_{n=-\infty}^{\infty}$ is an orthonormal set in $L^2(\mathbb{R})$ and Satisfies the dilation equation for all w .We can ensure that $\hat{\phi}_\theta$ is real by taking $\in_k$ to be multiplies of π in particular 0 or π .

It is easy to see that in this case the function $m_\theta\left(\frac{w}{2}\right)$ is 4π periodic extension of the function

$$\overline{m}_\theta(w)=\begin{cases}1 & I_0\\ e^{i(\in_k-\in_{k-1})} & on\ I_k\ \ k=1,2,....,n.\\ 0 & on[\pi\ 2\pi)\end{cases}$$

The Fourier Transform of mother wavelet can be found by formula

$\hat{\psi}(w)=e^{i\left(\frac{w}{2}\right)}\overline{m}\left(\frac{w}{2}+\pi\right)\hat{\phi}\left(\frac{w}{2}\right)$ and it is

$$e^{-i\left(\frac{w}{2}\right)}\hat{\psi}(w)=\begin{cases}e^{\in_n}a_k\cdot\ if\ \ \pi\left(2-\frac{1}{2^{n-k}}\right)<|w|<\pi\left(2-\frac{1}{2^{n-k+1}}\right)\\ e^{\in_n}a_0\ \ if\ \ \pi\left(2-\frac{1}{2^n}\right)<|w|<2\pi\\ 0\ \ oterwise\end{cases}$$

where $a_0=1$ and $a_k=e^{i(\in_k-\in_{k-1})}$ k = 1,2,3…,n.

Therefore by inverse Fourier Transform we get

$$\psi(t)=\frac{-1}{\pi\left(t-\frac{1}{2}\right)}\left\{\sin 2\pi\left(t-\frac{1}{2}\right)+\sum_{k=1}^{n}(a_k-a_{k-1})\sin\alpha_k\left(t-\frac{1}{2}\right)-a_n\sin\alpha_{n+1}\left(t-\frac{1}{2}\right)\right\}$$

where $\alpha_k=\pi\left(\frac{2^{n-k+2}-1}{2^{n-k+1}}\right)$ where k = 1,2,3,….,n+1 (2.2-4)

Example 2.2-1 :

For n = 1

$I_0=\left[0\ \frac{\pi}{2}\right)$ and $I_1=\left[\frac{\pi}{2}\ \pi\right)$

$\alpha_1=\pi\left(\frac{2^2-1}{2}\right)=\frac{3\pi}{2}$, $\alpha_2=\pi\left(\frac{2-1}{1}\right)=\pi$

Let $\epsilon_0=0$

Case I : $\epsilon_1=0$

$\therefore a_0=1$ *and* $a_1=1$

$$\therefore \psi(t)=\frac{-1}{\pi\left(t-\frac{1}{2}\right)}\left\{\sin 2\pi\left(t-\frac{1}{2}\right)-\sin\pi\left(t-\frac{1}{2}\right)\right\}$$

which is the sinc function as shown in figure below

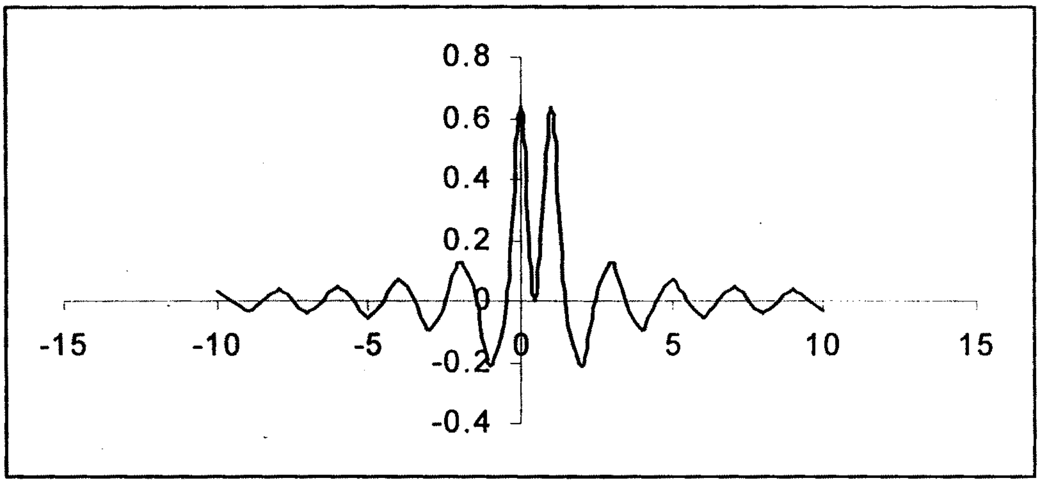

Fig. 2.2-1

Case II : $\in_1 = \pi$

then $a_0 = 1$ *and* $a_1 = -1$.

$$\therefore \psi(t) = \frac{-1}{\pi\left(t-\frac{1}{2}\right)}\left\{\sin 2\pi\left(t-\frac{1}{2}\right)+\left(a_1-a_0\right)\sin\alpha_1\left(t-\frac{1}{2}\right)-a_1\sin\alpha_2\left(t-\frac{1}{2}\right)\right\}$$

$$\therefore \psi(t) = \frac{-1}{\pi\left(t-\frac{1}{2}\right)}\left\{\sin 2\pi\left(t-\frac{1}{2}\right)-2\sin\frac{3\pi}{2}\left(t-\frac{1}{2}\right)+\sin\pi\left(t-\frac{1}{2}\right)\right\}$$

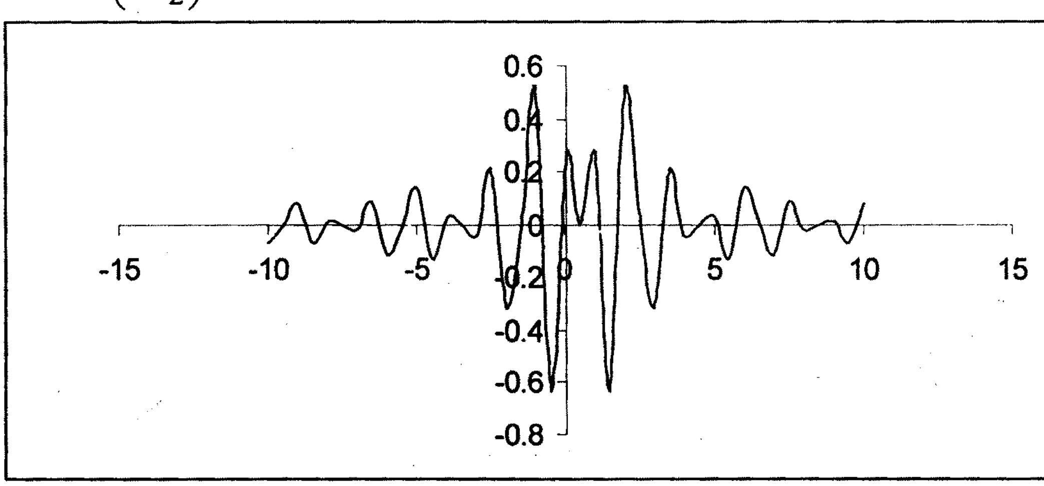

Fig. 2.2-2

Example 2.2-2 :

For n = 2

$$I_0=\left[0\ \frac{\pi}{4}\right),I_1=\left[\frac{\pi}{4}\ \frac{\pi}{2}\right)\ \text{and}\ \ I_2=\left[\frac{\pi}{2}\ \pi\right)$$

$$\therefore \alpha_1=\frac{7\pi}{4}\ ,\alpha_2=\frac{3\pi}{2}\ ,\alpha_3=\pi$$

Let $\epsilon_0=0$ then we have three possible cases

Case I : $\epsilon_1=0\ ,\epsilon_2=\pi$

then $a_0=1,\ a_1=1\ and\ a_2=-1$

$$\therefore \psi(t)=\frac{-1}{\pi\left(t-\frac{1}{2}\right)}\left\{\sin 2\pi\left(t-\frac{1}{2}\right)-2\sin\frac{3\pi}{2}\left(t-\frac{1}{2}\right)+\sin\pi\left(t-\frac{1}{2}\right)\right\}$$

which is same as example 2.2-1, case II .

Case II : $\epsilon_1=\pi\ ,\epsilon_2=\pi$

then $a_0=1,\ a_1=-1\ and\ a_2=1$

$$\therefore \psi(t)=\frac{-1}{\pi\left(t-\frac{1}{2}\right)}\left\{\sin 2\pi\left(t-\frac{1}{2}\right)-2\sin\frac{7\pi}{4}\left(t-\frac{1}{2}\right)+2\sin\frac{3\pi}{2}\left(t-\frac{1}{2}\right)-\sin\pi\left(t-\frac{1}{2}\right)\right\}$$

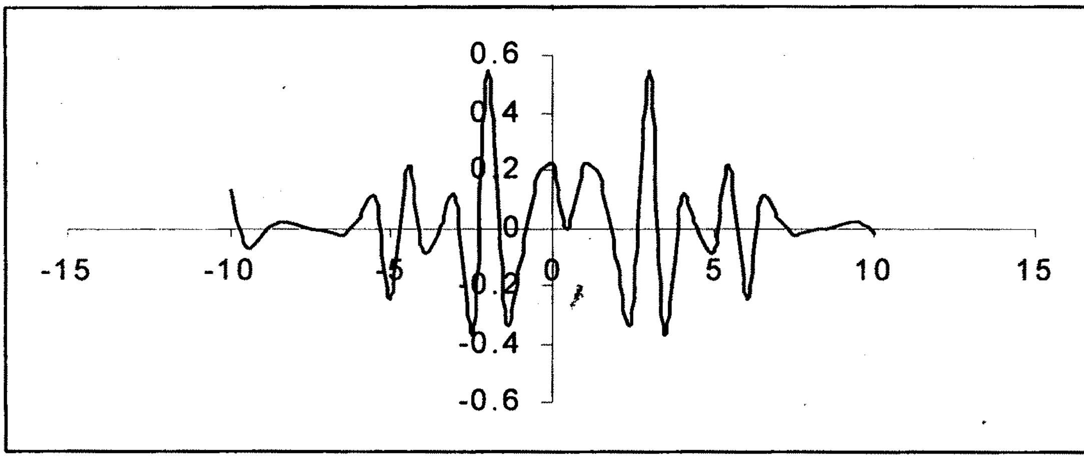

Fig. 2.2-3

Case III : $\in_1 = \pi , \in_2 = 0$

$\therefore a_0 = 1 , a_1 = -1 \, and \;\, a_2 = -1$

$$\therefore \psi(t) = \frac{-1}{\pi\left(t-\frac{1}{2}\right)}\left\{\sin 2\pi\left(t-\frac{1}{2}\right) - 2\sin\frac{7\pi}{4}\left(t-\frac{1}{2}\right) + \sin\pi\left(t-\frac{1}{2}\right)\right\}$$

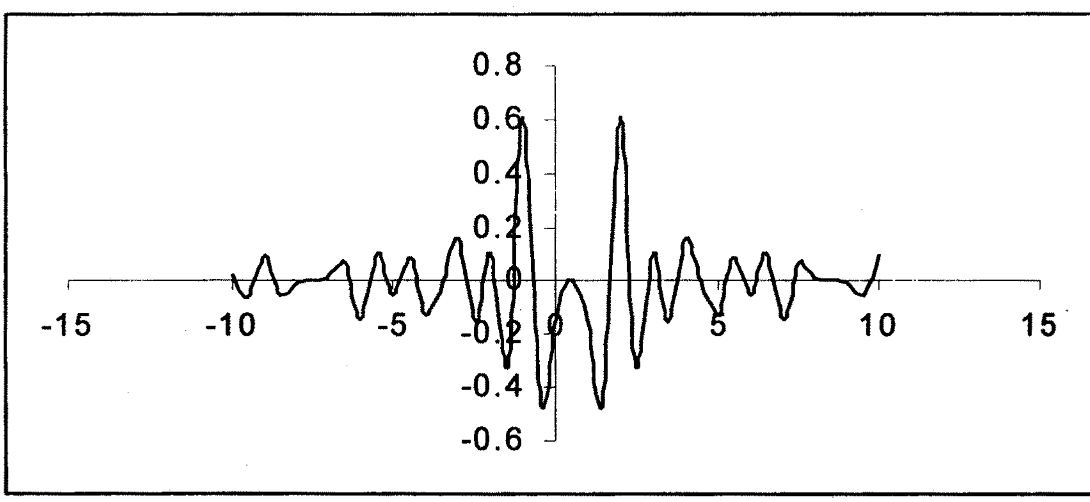

Fig. 2.2-4

Example 2.2-3 :

For n = 3

$$I_0=\left[0\ \frac{\pi}{8}\right), I_1=\left[\frac{\pi}{8}\ \frac{\pi}{4}\right) I_2=\left[\frac{\pi}{4}\ \frac{\pi}{2}\right) \text{ and } I_3=\left[\frac{\pi}{2}\ \pi\right)$$

$$\alpha_1=\frac{15\pi}{8}, \alpha_2=\frac{7\pi}{4}, \alpha_3=\frac{3\pi}{2} \quad \text{and } \alpha_4=\pi$$

Let $\in_0=0$ then we have seven possible cases

Case I $:\in_1=0, \in_2=0$ and $\in_3=\pi$

$\therefore a_0=1, a_1=1, a_2=1 \, and \, a_3=-1$

$$\therefore \psi(t)=\frac{-1}{\pi\left(t-\frac{1}{2}\right)}\left\{\sin 2\pi\left(t-\frac{1}{2}\right)-2\sin\frac{3\pi}{2}\left(t-\frac{1}{2}\right)+\sin\pi\left(t-\frac{1}{2}\right)\right\}$$

which is same as example 2.2-1, case II .

Case II : $\in_1=0, \in_2=\pi$ and $\in_3=0$

then $a_0=1, a_1=1, a_2=-1 \, and \, a_3=-1$

$$\therefore \psi(t)=\frac{-1}{\pi\left(t-\frac{1}{2}\right)}\left\{\sin 2\pi\left(t-\frac{1}{2}\right)-2\sin\frac{7\pi}{4}\left(t-\frac{1}{2}\right)+\sin\pi\left(t-\frac{1}{2}\right)\right\}$$

which is same as example 2.2-2, case III.

Case III : $\in_1=0, \in_2=\pi$ and $\in_3=\pi$

then $a_0 = 1$, $a_1 = 1, a_2 = -1$ *and* $a_3 = 1$

$$\therefore \psi(t) = \frac{-1}{\pi\left(t-\frac{1}{2}\right)}\left\{\sin 2\pi\left(t-\frac{1}{2}\right) - 2\sin\frac{7\pi}{4}\left(t-\frac{1}{2}\right) + 2\sin\frac{3\pi}{2}\left(t-\frac{1}{2}\right) - \sin\pi\left(t-\frac{1}{2}\right)\right\}$$

which is same as example 2.2-2, case II .

Case IV : $\in_1 = \pi, \in_2 = 0, \in_3 = 0$

$\therefore a_0 = 1$, $a_1 = -1, a_2 = -1$ *and* $a_3 = 1$

$$\therefore \psi(t) = \frac{-1}{\pi\left(t-\frac{1}{2}\right)}\left\{\sin 2\pi\left(t-\frac{1}{2}\right) - 2\sin\frac{15\pi}{8}\left(t-\frac{1}{2}\right) + 2\sin\frac{3\pi}{2}\left(t-\frac{1}{2}\right) - \sin\pi\left(t-\frac{1}{2}\right)\right\}$$

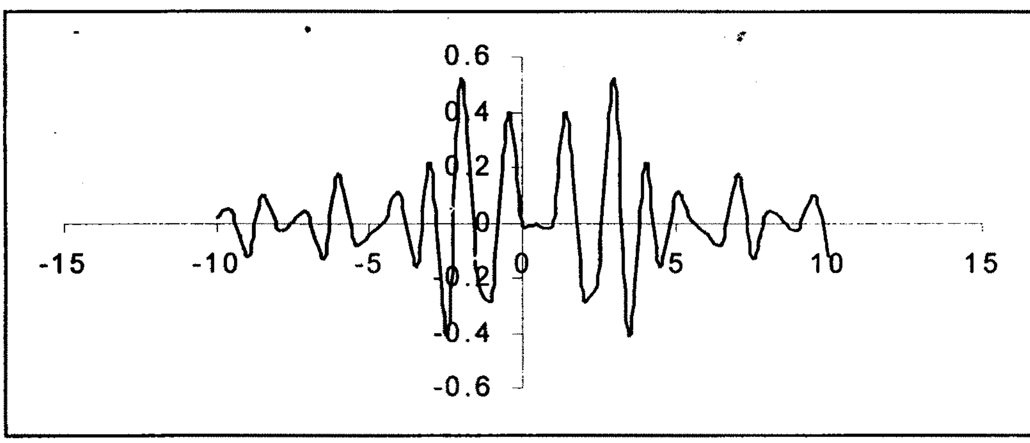

Fig. 2.2-5

Case V : $\in_1 = \pi, \in_2 = 0$ and $\in_3 = \pi$

$\therefore a_0 = 1$, $a_1 = -1, a_2 = -1$ *and* $a_3 = -1$

$$\therefore \psi(t) = \frac{-1}{\pi\left(t-\frac{1}{2}\right)}\left\{\sin 2\pi\left(t-\frac{1}{2}\right) - 2\sin\frac{15\pi}{8}\left(t-\frac{1}{2}\right) + \sin\pi\left(t-\frac{1}{2}\right)\right\}$$

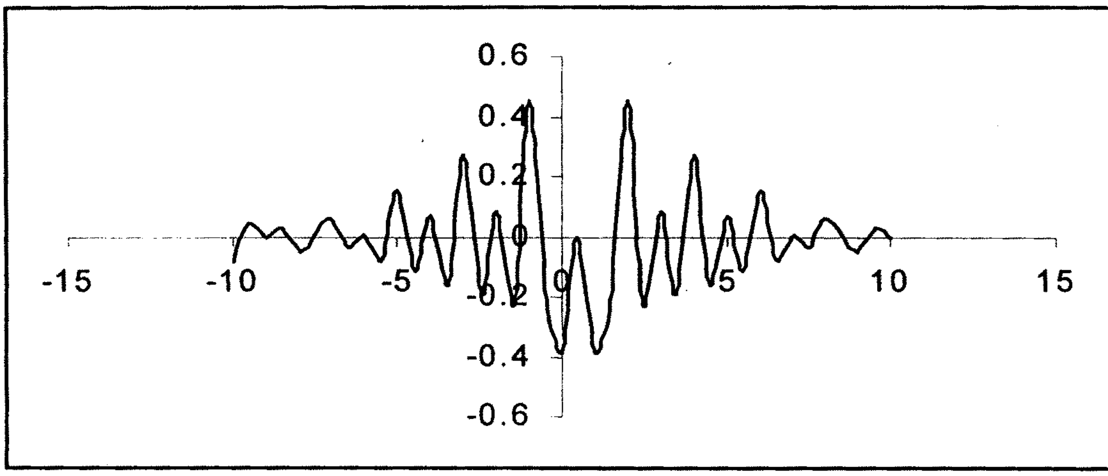

Fig. 2.2-6

Case VI : $\in_1 = \pi,\ \in_2 = \pi$ and $\in_3 = 0$

$\therefore a_0 = 1,\ a_1 = -1, a_2 = 1\, and\ a_3 = -1$

$$\therefore \psi(t) = \frac{-1}{\pi\left(t-\frac{1}{2}\right)}\left\{\sin 2\pi\left(t-\frac{1}{2}\right) - 2\sin\frac{15\pi}{8}\left(t-\frac{1}{2}\right) + 2\sin\frac{7\pi}{4}\left(t-\frac{1}{2}\right) - 2\sin\frac{3\pi}{2}\left(t-\frac{1}{2}\right) + \sin\pi\left(t-\frac{1}{2}\right)\right\}$$

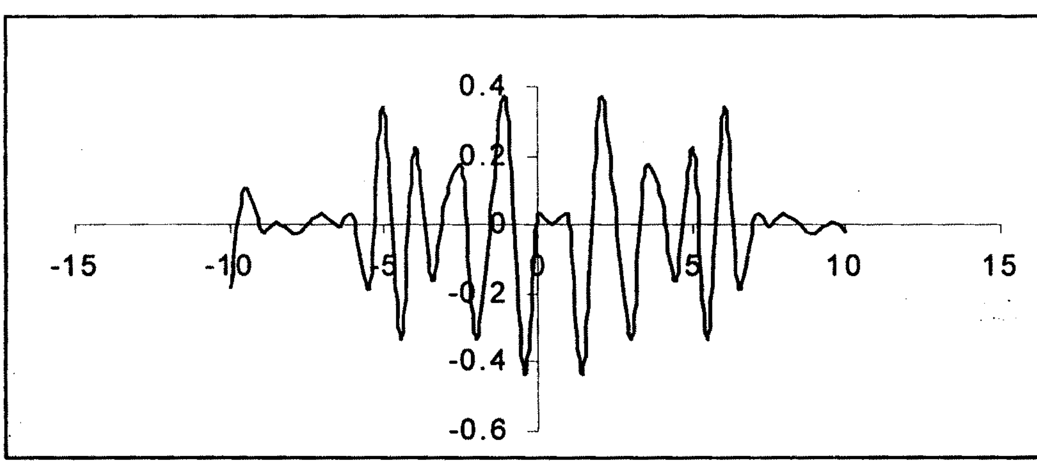

Fig. 2.2-7

Case VII : $\epsilon_1 = \epsilon_2 = \epsilon_3 = \pi$

$\therefore a_0 = 1,\ a_1 = -1,\ a_2 = 1 \, and \ a_3 = 1$

$$\therefore \psi(t) = \frac{-1}{\pi\left(t-\frac{1}{2}\right)}\left\{\sin 2\pi\left(t-\frac{1}{2}\right) - 2\sin\frac{15\pi}{8}\left(t-\frac{1}{2}\right) + 2\sin\frac{7\pi}{4}\left(t-\frac{1}{2}\right) - \sin\pi\left(t-\frac{1}{2}\right)\right\}$$

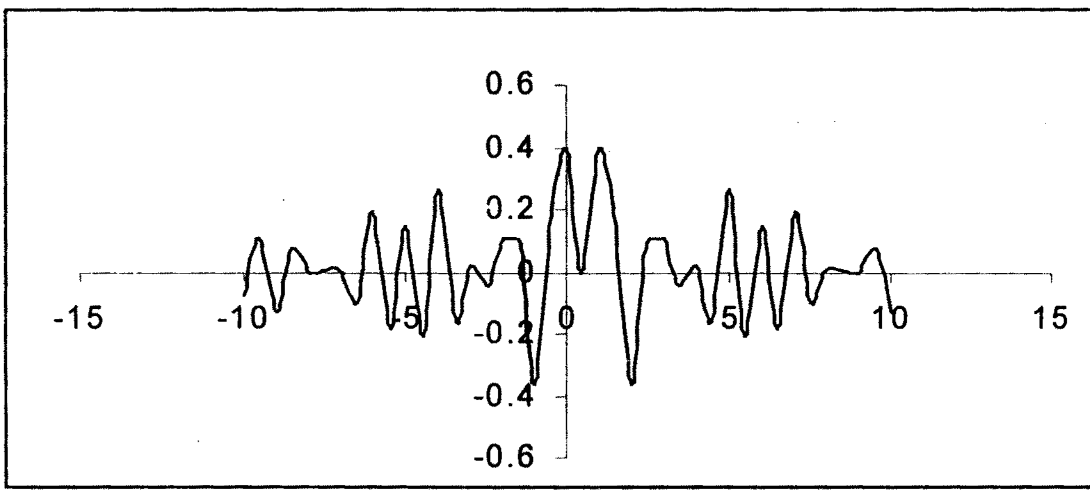

Fig. 2.2-8

Example 2.2-4 :

For n = 4

$$I_0 = \left[0\ \frac{\pi}{16}\right), I_1 = \left[\frac{\pi}{16}\ \frac{\pi}{8}\right) I_2 = \left[\frac{\pi}{8}\ \frac{\pi}{4}\right) I_3 = \left[\frac{\pi}{4}\ \frac{\pi}{2}\right), I_4 = \left[\frac{\pi}{2}\ \pi\right)$$

$$\therefore \alpha_1 = \frac{31\pi}{16}, \alpha_2 = \frac{15\pi}{8}, \alpha_3 = \frac{7\pi}{4}, \alpha_4 = \frac{3\pi}{2}, \alpha_5 = \pi$$

Let $\epsilon_0 = 0$ then we have, fifteen possible cases .

Case I : $\in_1 = 0, \in_2 = 0, \in_3 = 0, \in_4 = \pi$

$\therefore a_0 = 1, a_1 = 1, a_2 = 1, a_3 = 1$ *and* $a_4 = -1$

$$\therefore \psi(t) = \frac{-1}{\pi\left(t-\frac{1}{2}\right)}\left\{\sin 2\pi\left(t-\frac{1}{2}\right) - 2\sin\frac{3\pi}{2}\left(t-\frac{1}{2}\right) + \sin\pi\left(t-\frac{1}{2}\right)\right\}$$

which is same as example 2.2-1, case II .

Case II : $\in_1 = 0, \in_2 = 0, \in_3 = \pi, \in_4 = 0$

$\therefore a_0 = 1, a_1 = 1, a_2 = 1, a_3 = -1$ *and* $a_4 = -1$

$$\therefore \psi(t) = \frac{-1}{\pi\left(t-\frac{1}{2}\right)}\left\{\sin 2\pi\left(t-\frac{1}{2}\right) - 2\sin\frac{7\pi}{4}\left(t-\frac{1}{2}\right) + \sin\pi\left(t-\frac{1}{2}\right)\right\}$$

which is same as example 2.2-2, case III.

Case III : $\in_1 = 0, \in_2 = 0, \in_3 = \pi, \in_4 = \pi$

$\therefore a_0 = 1, a_1 = 1, a_2 = 1, a_3 = -1$ *and* $a_4 = 1$

$$\therefore \psi(t) = \frac{-1}{\pi\left(t-\frac{1}{2}\right)}\left\{\sin 2\pi\left(t-\frac{1}{2}\right) - 2\sin\frac{7\pi}{4}\left(t-\frac{1}{2}\right) + 2\sin\frac{3\pi}{2}\left(t-\frac{1}{2}\right) - \sin\pi\left(t-\frac{1}{2}\right)\right\}$$

which is same as example 2.2-2, case II .

Case IV : $\in_1 = 0, \in_2 = \pi, \in_3 = 0, \in_4 = 0$

$\therefore a_0 = 1, a_1 = 1, a_2 = -1, a_3 = -1$ *and* $a_4 = 1$

$$\therefore \psi(t) = \frac{-1}{\pi\left(t-\frac{1}{2}\right)}\left\{\sin 2\pi\left(t-\frac{1}{2}\right) - 2\sin\frac{15\pi}{8}\left(t-\frac{1}{2}\right) + 2\sin\frac{3\pi}{2}\left(t-\frac{1}{2}\right) - \sin\pi\left(t-\frac{1}{2}\right)\right\}$$

which is same as example 2.2-3, case IV.

Case V : $\in_1 = 0, \in_2 = \pi, \in_3 = 0, \in_4 = \pi$

$\therefore a_0 = 1, a_1 = 1, a_2 = -1, a_3 = -1$ *and* $a_4 = -1$

$$\therefore \psi(t) = \frac{-1}{\pi\left(t-\frac{1}{2}\right)}\left\{\sin 2\pi\left(t-\frac{1}{2}\right) - 2\sin\frac{15\pi}{8}\left(t-\frac{1}{2}\right) + \sin\pi\left(t-\frac{1}{2}\right)\right\}$$

which is same as example 2.2-3, case V

Case VI : $\in_1 = 0, \in_2 = \pi, \in_3 = \pi, \in_4 = 0$

$\therefore a_0 = 1, a_1 = 1, a_2 = -1, a_3 = 1$ *and* $a_4 = -1$

$$\therefore \psi(t) = \frac{-1}{\pi\left(t-\frac{1}{2}\right)}\left\{\sin 2\pi\left(t-\frac{1}{2}\right) - 2\sin\frac{15\pi}{8}\left(t-\frac{1}{2}\right) + \right.$$

$$\left. 2\sin\frac{7\pi}{4}\left(t-\frac{1}{2}\right) - 2\sin\frac{3\pi}{2}\left(t-\frac{1}{2}\right) + \sin\pi\left(t-\frac{1}{2}\right)\right\}$$

which is same as example 2.2-3, case VI

Case VII : $\in_1 = 0, \in_2 = \pi, \in_3 = \pi, \in_4 = \pi$

$\therefore a_0 = 1, a_1 = 1, a_2 = -1, a_3 = 1$ *and* $a_4 = 1$

$$\therefore \psi(t) = \frac{-1}{\pi\left(t-\frac{1}{2}\right)}\left\{\sin 2\pi\left(t-\frac{1}{2}\right) - 2\sin\frac{15\pi}{8}\left(t-\frac{1}{2}\right) + 2\sin\frac{7\pi}{4}\left(t-\frac{1}{2}\right) - \sin\pi\left(t-\frac{1}{2}\right)\right\}$$

which is same as example 2.2-3, case VII.

Case VIII : $\in_1 = \pi, \in_2 = 0, \in_3 = 0, \in_4 = 0$

$\therefore a_0 = 1, a_1 = -1, a_2 = -1, a_3 = 1$ *and* $a_4 = 1$

$$\therefore \psi(t)=\frac{-1}{\pi\left(t-\frac{1}{2}\right)}\left\{\sin 2\pi\left(t-\frac{1}{2}\right)-2\sin\frac{31\pi}{16}\left(t-\frac{1}{2}\right)+2\sin\frac{7\pi}{4}\left(t-\frac{1}{2}\right)-\sin\pi\left(t-\frac{1}{2}\right)\right\}$$

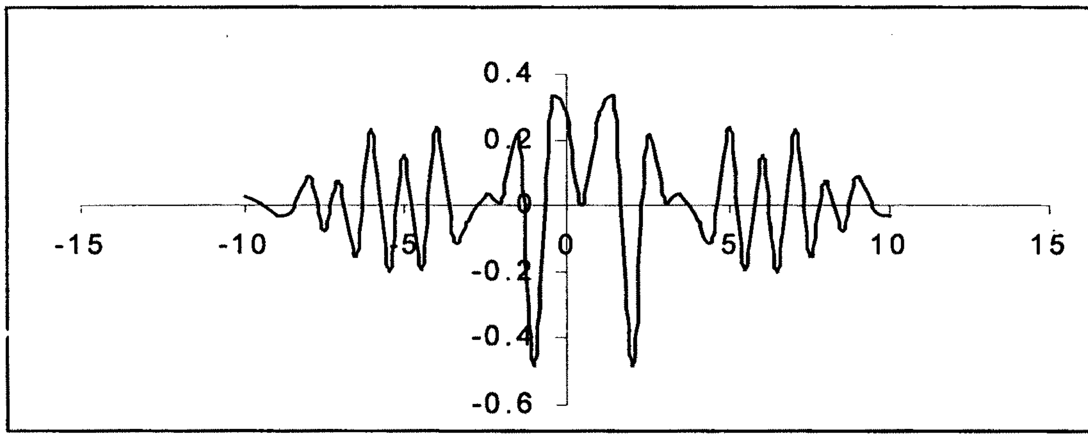

Fig. 2.2-9

Case IX : $\in_1=\pi\,,\in_2=0,\in_3=0\,,\in_4=\pi$

$\therefore a_0=1\,,a_1=-1,a_2=-1,a_3=1\;\; and\;\; a_4=-1$

$$\therefore \psi(t)=\frac{-1}{\pi\left(t-\frac{1}{2}\right)}\left\{\sin 2\pi\left(t-\frac{1}{2}\right)-2\sin\frac{31\pi}{16}\left(t-\frac{1}{2}\right)+\right.$$

$$\left.2\sin\frac{7\pi}{4}\left(t-\frac{1}{2}\right)-2\sin\frac{3\pi}{2}\left(t-\frac{1}{2}\right)+\sin\pi\left(t-\frac{1}{2}\right)\right\}$$

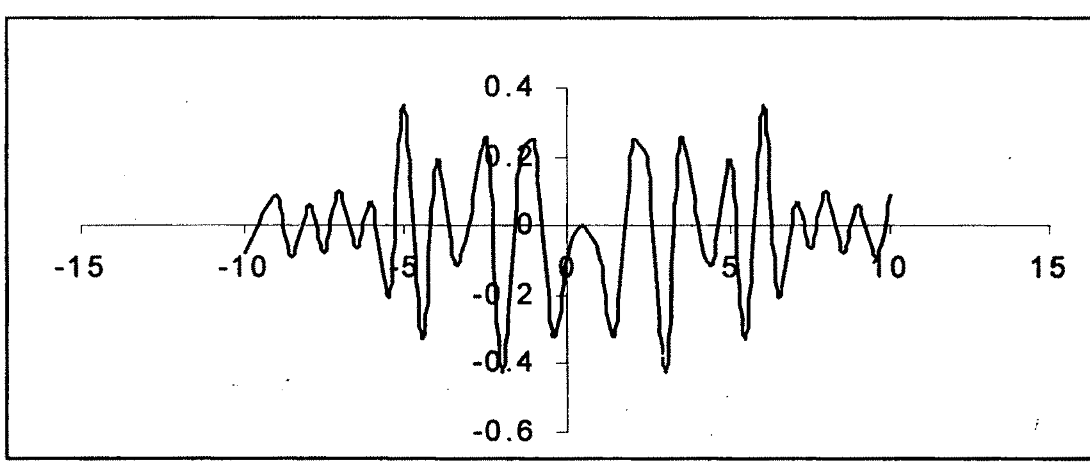

Fig. 2.2-10

Case X : $\epsilon_1 = \pi, \epsilon_2 = 0, \epsilon_3 = \pi, \epsilon_4 = 0$

$\therefore a_0 = 1, a_1 = -1, a_2 = -1, a_3 = -1$ *and* $a_4 = -1$

$$\therefore \psi(t) = \frac{-1}{\pi\left(t-\frac{1}{2}\right)}\left\{\sin 2\pi\left(t-\frac{1}{2}\right) - 2\sin\frac{31\pi}{16}\left(t-\frac{1}{2}\right) + \sin\pi\left(t-\frac{1}{2}\right)\right\}$$

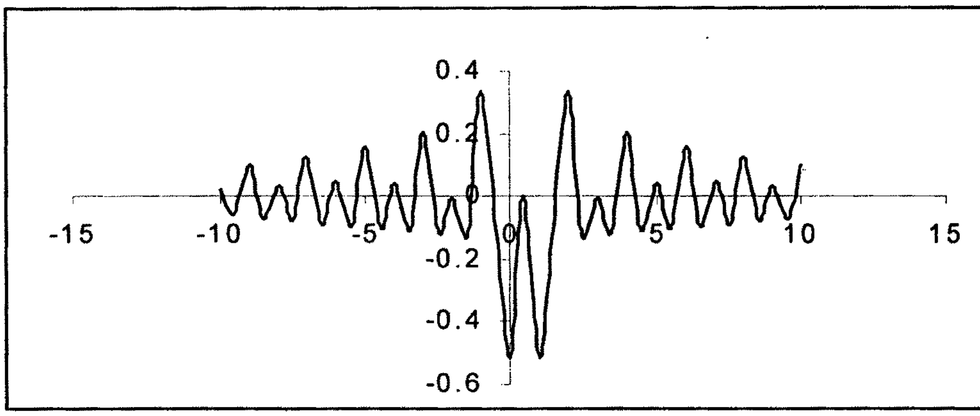

Fig. 2.2-11

Case XI : $\epsilon_1 = \pi, \epsilon_2 = 0, \epsilon_3 = \pi, \epsilon_4 = \pi$

$\therefore a_0 = 1, a_1 = -1, a_2 = -1, a_3 = -1$ *and* $a_4 = 1$

$$\therefore \psi(t) = \frac{-1}{\pi\left(t-\frac{1}{2}\right)}\left\{\sin 2\pi\left(t-\frac{1}{2}\right) - 2\sin\frac{31\pi}{16}\left(t-\frac{1}{2}\right) + 2\sin\frac{3\pi}{2}\left(t-\frac{1}{2}\right) - \sin\pi\left(t-\frac{1}{2}\right)\right\}$$

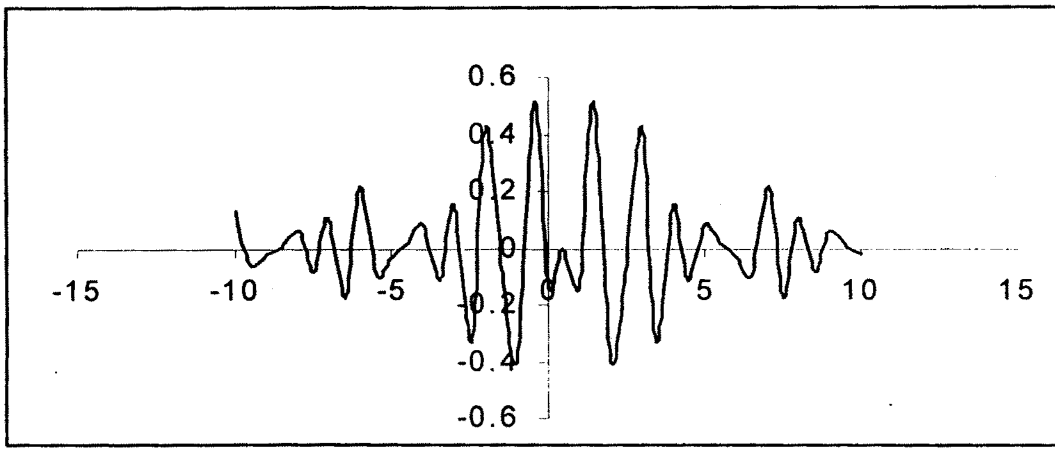

Fig. 2.2-12

Case XII : $\epsilon_1 = \pi, \epsilon_2 = \pi, \epsilon_3 = 0, \epsilon_4 = 0$

$\therefore a_0 = 1, a_1 = -1, a_2 = 1, a_3 = -1$ *and* $a_4 = 1$

$$\therefore \psi(t) = \frac{-1}{\pi\left(t-\frac{1}{2}\right)}\left\{\sin 2\pi\left(t-\frac{1}{2}\right) - 2\sin\frac{31\pi}{16}\left(t-\frac{1}{2}\right) + 2\sin\frac{15\pi}{8}\left(t-\frac{1}{2}\right) - \right.$$

$$\left. 2\sin\frac{7\pi}{4}\left(t-\frac{1}{2}\right) + 2\sin\frac{3\pi}{2}\left(t-\frac{1}{2}\right) - \sin\pi\left(t-\frac{1}{2}\right)\right\}$$

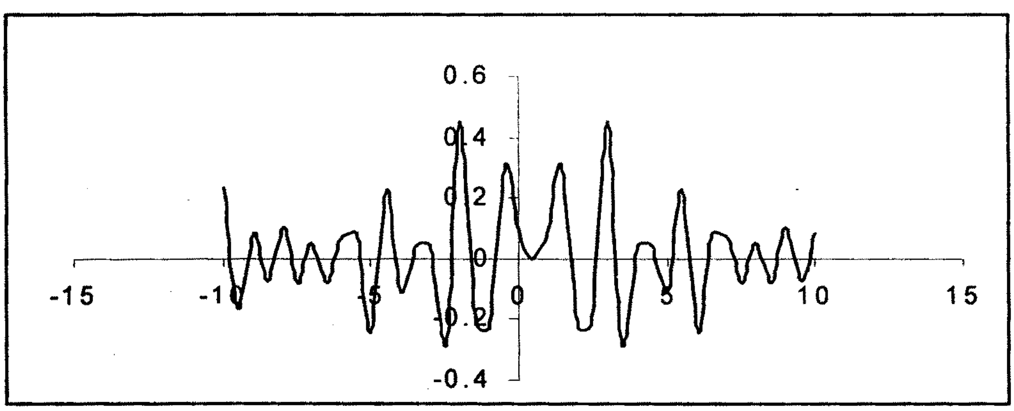

Fig. 2.2-13

Case XIII : $\epsilon_1 = \pi, \epsilon_2 = \pi, \epsilon_3 = 0, \epsilon_4 = \pi$

$\therefore a_0 = 1, a_1 = -1, a_2 = 1, a_3 = -1$ *and* $a_4 = -1$

$$\therefore \psi(t) = \frac{-1}{\pi\left(t-\frac{1}{2}\right)}\left\{\sin 2\pi\left(t-\frac{1}{2}\right) - 2\sin\frac{31\pi}{16}\left(t-\frac{1}{2}\right) + 2\sin\frac{15\pi}{8}\left(t-\frac{1}{2}\right) - 2\sin\frac{7\pi}{4}\left(t-\frac{1}{2}\right) + \sin\pi\left(t-\frac{1}{2}\right)\right\}$$

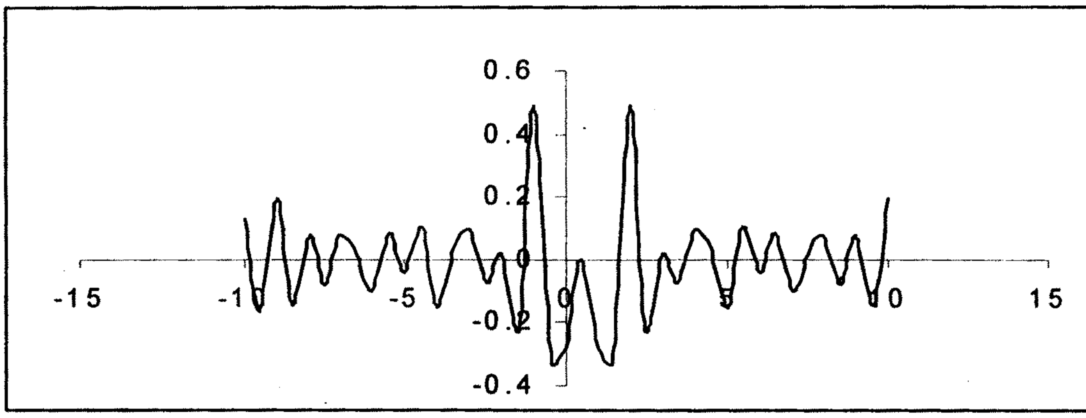

Fig. 2.2-14

Case XIV : $\epsilon_1 = \pi, \epsilon_2 = \pi, \epsilon_3 = \pi, \epsilon_4 = 0$

$\therefore a_0 = 1, a_1 = -1, a_2 = 1, a_3 = 1$ *and* $a_4 = -1$

$$\therefore \psi(t) = \frac{-1}{\pi\left(t-\frac{1}{2}\right)}\left\{\sin 2\pi\left(t-\frac{1}{2}\right) - 2\sin\frac{31\pi}{16}\left(t-\frac{1}{2}\right) + 2\sin\frac{15\pi}{8}\left(t-\frac{1}{2}\right) - 2\sin\frac{3\pi}{2}\left(t-\frac{1}{2}\right) + \sin\pi\left(t-\frac{1}{2}\right)\right\}$$

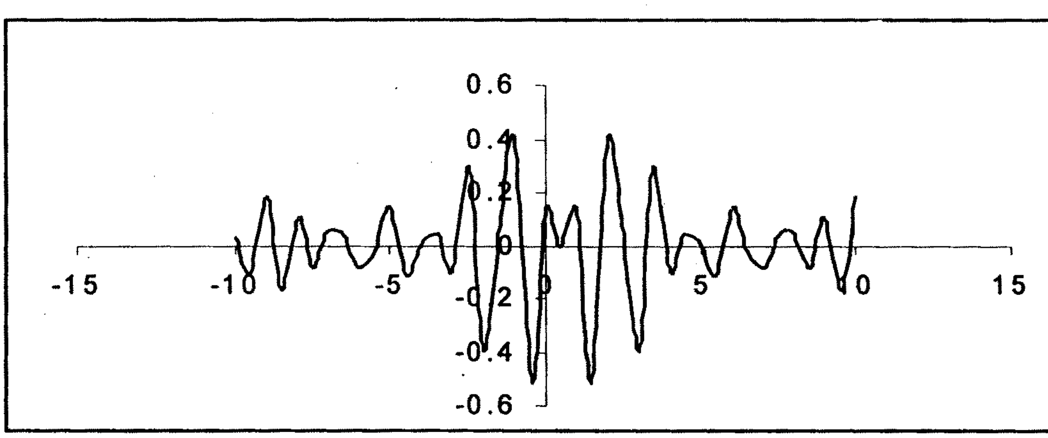

Fig. 2.2-15

Case XV : $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = \pi$

$\therefore a_0 = 1,\ a_1 = -1, a_2 = 1, a_3 = 1 \ \ and \ \ a_4 = 1$

$$\therefore \psi(t) = \frac{-1}{\pi\left(t-\frac{1}{2}\right)}\left\{\sin 2\pi\left(t-\frac{1}{2}\right) - 2\sin\frac{31\pi}{16}\left(t-\frac{1}{2}\right) + 2\sin\frac{15\pi}{8}\left(t-\frac{1}{2}\right) - \sin\pi\left(t-\frac{1}{2}\right)\right\}$$

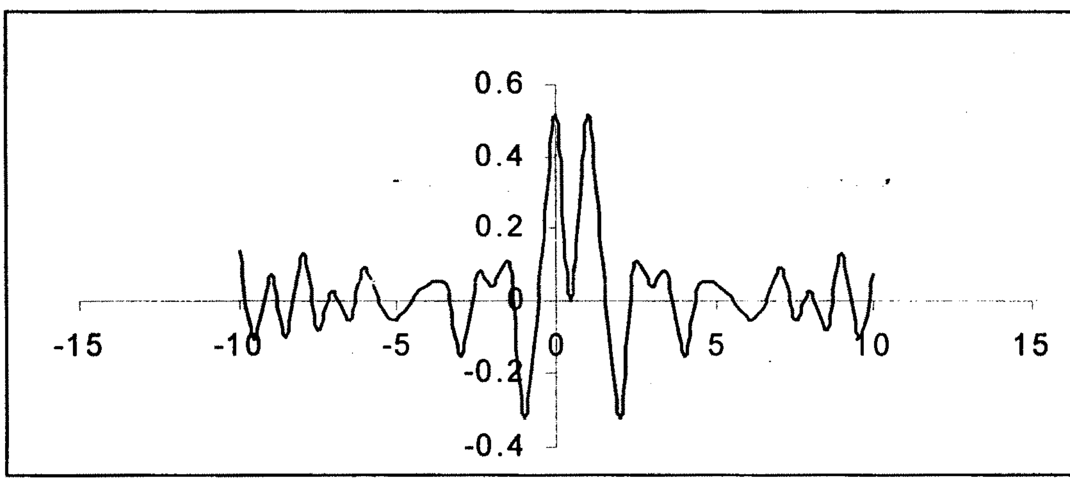

Fig. 2.2-16

Many other examples of this type are possible, but all share the shortcoming of the Shannon wavelet, that they have very poor time localization.

To improve the time localization, we begin with the raised-cosine Scaling function and require that $\theta(w)$ be continuous.

Example 2.2-5 : Let $\theta(w)$ be 2π periodic extension of the function

$$\bar{\theta}(w)=\begin{cases}\dfrac{w}{1-\beta} & |w|<\pi(1-\beta)\\ \pi\operatorname{sgn}(w) & \pi(1-\beta)<|w|<\pi \quad \textit{where } 0<\beta\le\dfrac{1}{3}\end{cases}$$

Let $\phi(t)$ be the scaling function arising from the raised cosine wavelet (2.1-4) or in other words

$$\text{Let } \hat{\phi}(w)=\begin{cases}1 & 0\le|w|\le\pi(1-\beta)\\ \cos\left[\dfrac{|w|-\pi(1-\beta)}{4\beta}\right] & \pi(1-\beta)<|w|\le\pi(1+\beta)\\ 0 & \pi(1+\beta)\le|w|\end{cases}$$

As by (2.2-2) $\hat{\phi}_\theta(w)=e^{i\theta(w)}\hat{\phi}(w)$

$$\hat{\phi}_\theta(w)=\begin{cases}e^{i\left(\frac{w}{1-\beta}\right)} & |w|\le\pi(1-\beta)\\ -\cos\left[\dfrac{|w|-\pi(1-\beta)}{4\beta}\right] & \pi(1-\beta)<|w|\le\pi(1+\beta)\\ 0 & \textit{otherwise}\end{cases}$$

Let $\alpha=\dfrac{1}{1-\beta}$

For the time domain, taking inverse Fourier Transform again we get

$$\phi_\theta(t)=\frac{1}{2\pi}\int_{-\pi(1-\beta)}^{\pi(1-\beta)}e^{iw(\alpha-t)}\,dw-\frac{1}{\pi}\int_{\pi(1-\beta)}^{\pi(1+\beta)}\cos\left[\frac{w-\pi(1-\beta)}{4\beta}\right]\cos(wt)\,dw$$

$$=\frac{1}{2\pi}\left[\frac{e^{iw(\alpha-t)}}{i(\alpha-t)}\right]_{-\pi(1-\beta)}^{\pi(1-\beta)}-\frac{1}{\pi}\int_{\pi(1-\beta)}^{\pi(1+\beta)}\frac{1}{2}\left\{\cos\left[w\left(t+\frac{1}{4\beta}\right)-\frac{\pi(1-\beta)}{4\beta}\right]+\right.$$
$$\left.+\cos\left[w\left(t-\frac{1}{4\beta}\right)+\frac{\pi(1-\beta)}{4\beta}\right]\right\}dw$$

$$=\frac{1}{\pi(\alpha-t)}\left[\frac{e^{i\pi(1-\beta)(\alpha-t)}-e^{-i\pi(1-\beta)(\alpha-t)}}{2i}\right]+\frac{1}{2\pi}\left[\frac{\sin\left[w\left(t+\frac{1}{4\beta}\right)-\frac{\pi(1-\beta)}{4\beta}\right]}{\left(t+\frac{1}{4\beta}\right)}+\right.$$
$$\left.+\frac{\sin\left[w\left(t-\frac{1}{4\beta}\right)+\frac{\pi(1-\beta)}{4\beta}\right]}{\left(t-\frac{1}{4\beta}\right)}\right]_{\pi(1-\beta)}^{\pi(1+\beta)}dw$$

$$=\frac{\sin[\pi(1-\beta)(t-\alpha)]}{\pi(t-\alpha)}+\frac{4\beta}{2\pi}\left\{\frac{\sin\left[\pi(1+\beta)\left(t+\frac{1}{4\beta}\right)-\frac{\pi(1-\beta)}{4\beta}\right]-\sin\left[\pi(1-\beta)\left(t+\frac{1}{4\beta}\right)-\frac{\pi(1-\beta)}{4\beta}\right]}{4\beta t+1}\right.$$
$$\left.\frac{\sin\left[\pi(1+\beta)\left(t-\frac{1}{4\beta}\right)+\frac{\pi(1-\beta)}{4\beta}\right]-\sin\left[\pi(1-\beta)\left(t-\frac{1}{4\beta}\right)+\frac{\pi(1-\beta)}{4\beta}\right]}{4\beta t-1}\right\}$$

i.e. $$\phi_\theta(t)=\frac{\sin[\pi(1-\beta)t]}{\pi}\left[\frac{1-\alpha t(4\beta)^2}{(t-\alpha)\left[(4\beta t)^2-1\right]}\right]+\frac{4\beta\cos[\pi(1+\beta)t]}{\pi\left[(4\beta t)^2-1\right]}$$

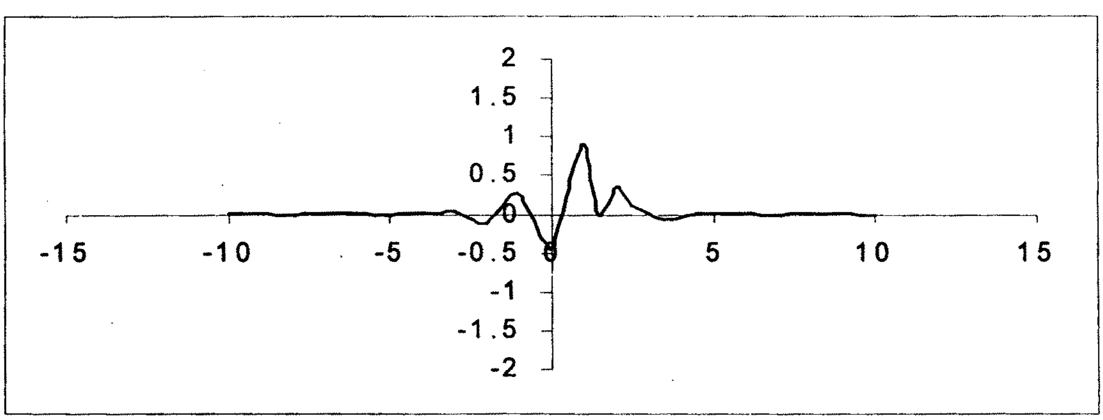

Fig. 2.2-17 $\left(\phi_\theta(t)\ \textit{of example}\ 2.2-5\ \textit{with}\ \beta=1/3\right)$

The mother wavelet can similarly be calculated but becomes quite complex and therefore looses the advantages of a closed form expression.

This Mother wavelet has better time localization than the ones given in the previous examples.

Lemma (2.2-1) :

Let $F(a;b;z)$ denote the confluent hypergeometric function

${}_1F_1(a;b;z)$ defined by $F(a;b;z)=\sum_{k=0}^{\infty}\frac{(a)_k z^k}{(b)_k k!}$ where $(a)_k=\frac{\Gamma(a+k)}{\Gamma(a)}$

with $b \neq -n$, n = 0,1,2,......... then

$$F(1;v;iz)+F(1;v;-iz)=2\Gamma(v)\frac{Y_{v-1}(z)}{z^{v-1}} \quad and$$
$$F(1;v;iz)-F(1;v;-iz)=2i\Gamma(v)\frac{Y_v(z)}{z^{v-1}}$$

where Y_v is the Young's function .

Proof : By definition

$$\begin{aligned} F(1;v;iz)+F(1;v;-iz) &= \sum_{k=0}^{\infty}\frac{(iz)^k}{(v)_k}\left[1+(-1)^k\right] \\ &= 2\sum_{k=0}^{\infty}\frac{(iz)^k}{(v)_{2k}} \\ &= 2\Gamma(v)\sum_{k=0}^{\infty}\frac{(-1)^k z^{2k}}{\Gamma(v+2k)} \\ &= 2\Gamma(v)\frac{Y_{v-1}(z)}{z^{v-1}} \end{aligned}$$

Thus $F(1;\nu;iz)+F(1;\nu;-iz)=2\Gamma\nu\dfrac{Y_{\nu-1}(z)}{z^{\nu-1}}$ (2.2-5)

Similarly

$$\begin{aligned}F(1;\nu;iz)-F(1;\nu;-iz) &= \sum_{k=0}^{\infty}\frac{(iz)^k}{(\nu)_k}\left[1-(-1)^k\right]\\ &= 2i\Gamma\nu\sum_{k=0}^{\infty}\frac{(-1)^k z^{2k+1}}{(\nu)_{2k+1}}\\ &= 2i\Gamma\nu\sum_{k=0}^{\infty}\frac{(-1)^k z^{2k+1}}{\Gamma(\nu+2k+1)}\\ &= 2i\Gamma\nu\frac{Y_\nu(z)}{z^{\nu-1}}\end{aligned}$$

$$\Rightarrow F(1;\nu;iz)-F(1;\nu;-iz)=2i\Gamma\nu\frac{Y_\nu(z)}{z^{\nu-1}} \tag{2.2-6}$$

Now we obtain an orthonormal wavelet basis using the procedure described in Lemmas (1.3-1) and (2.1-1) and Theorem (2.1-1).

Example 2.2-6 :

The function

$$\phi(t)=\frac{\sin\left(\frac{2\pi t}{3}\right)}{\pi t}+\frac{\sqrt{3}}{\pi(2t)^{\frac{3}{2}}}\left[Y_{3/2}\left(\frac{2\pi t}{3}\right)\cos\left(\frac{2\pi t}{3}\right)-Y_{5/2}\left(\frac{2\pi t}{3}\right)\sin\left(\frac{2\pi t}{3}\right)\right] \tag{2.2-7}$$

is an orthogonal scaling function and the set $\left\{\psi_{m,n}(t)=2^{m/2}\psi\left(2^m t-n\right)\right\}_{m,n=-\infty}^{\infty}$ is an orthonormal wavelet basis of $L^2(\mathbb{R})$

where,

$$\psi\left(t-\frac{1}{2}\right)=\frac{\sqrt{3/2}}{(\pi t)^{3/2}}\left\{\left[\cos\left(\frac{2\pi t}{3}\right)I_{0,\frac{1}{2}}\left(\frac{2\pi t}{3}\right)-\sin\left(\frac{2\pi t}{3}\right)I_{1,\frac{1}{2}}\left(\frac{2\pi t}{3}\right)\right]+\right.$$
$$\left.\sqrt{\frac{\pi}{8}}\left[\cos\left(\frac{4\pi t}{3}\right)Y_{3/2}\left(\frac{4\pi t}{3}\right)-\sin\left(\frac{4\pi t}{3}\right)Y_{5/2}\left(\frac{4\pi t}{3}\right)\right]\right\} \tag{2.2-8}$$

Proof :

Let $h(t)=\frac{3}{2\pi}X_{\left[\frac{-\pi}{3}\ \frac{\pi}{3}\right]}(t)$ and

$$g(w)=\int_{w-\pi}^{w+\pi}h(t)dt$$

Therefore by Lemma (2.1-1)

$$g(w)=\begin{cases}0 & if\ |w|\geq\frac{4\pi}{3}\\ \left(\frac{3}{2\pi}\right)w+2 & if\ -\frac{4\pi}{3}\leq w\leq\frac{-2\pi}{3}\\ 1 & if\ |w|\leq\frac{2\pi}{3}\\ -\left(\frac{3}{2\pi}\right)w+2 & if\ \frac{2\pi}{3}\leq w\leq\frac{4\pi}{3}\end{cases}$$

Then g(t) is a sampling function and a scaling function of MRA .

Since $\hat{g}(w)$ is non-negative, by Lemma (1.3-1)

We define $\hat{\phi}(w)=\sqrt{g(w)}$

Moreover since

$$\sum_{k=-\infty}^{\infty}\left|\hat{\phi}(w+2k\pi)\right|^2 = \sum_{k=-\infty}^{\infty} g(w+2k\pi)$$

$$= \sum_{k=-\infty}^{\infty}\int_{w+2k\pi-\pi}^{w+2k\pi+\pi} h(t)\,dt$$

$$= \int_{-\infty}^{\infty} h(t)\,dt$$

$$= 1$$

$\{\phi(t-n)\}$ is an orthonormal in $L^2(\mathbb{R})$.

Let $m\left(\frac{w}{2}\right)$ be the 4π periodic extension of $\hat{\phi}(w)$ then $\hat{\phi}(w)$ satisfies the

dilation equation $\hat{\phi}(w) = m\left(\frac{w}{2}\right)\hat{\phi}\left(\frac{w}{2}\right)$

Therefore $\phi(t)$ is an orthogonal scaling function of MRA.

As $\hat{\phi}(w)$ is even function to obtain $\phi(t)$ in closed form, by inverse Fourier

Transform $\phi(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{\phi}(w)e^{-itw}\,dw$

$$= \frac{1}{\pi}\int_{0}^{4\pi/3}\hat{\phi}(w)\cos(wt)\,dw$$

$$= \frac{1}{\pi}\left\{\int_{0}^{2\pi/3}\cos(wt)\,dw + \int_{2\pi/3}^{4\pi/3}\sqrt{2-\frac{3w}{2\pi}}\cos(wt)\,dw\right\}$$

$$= \frac{\sin\frac{2\pi t}{3}}{\pi t} + \frac{1}{\pi}\int_{2\pi/3}^{4\pi/3}\sqrt{2-\frac{3w}{2\pi}}\cos(wt)\,dw$$

Let $\frac{3w}{2\pi} = x \Rightarrow dw = \frac{2\pi}{3}dx$

$\therefore$ when $w = \frac{2\pi}{3}$ *then* $x = 1$ and when $w = \frac{4\pi}{3}$ *then* $x = 2$

$$\therefore \phi(t) = \frac{\sin\frac{2\pi t}{3}}{\pi t} + \frac{2}{3}\int_1^2 \sqrt{2-x}\cos\left(\frac{2\pi tx}{3}\right)dx$$

setting w = x – 1 and $c = \frac{2\pi t}{3}$

$$\therefore \phi(t) = \frac{\sin\frac{2\pi t}{3}}{\pi t} + \frac{2}{3}\int_0^1 \sqrt{1-w}\cos\left(c\left(w+1\right)\right)dw$$

$$\therefore \phi(t) = \frac{\sin\frac{2\pi t}{3}}{\pi t} + \frac{2}{3}\left[\cos c\int_0^1 \sqrt{1-w}\cos(cw)\,dw - \sin c\int_0^1 \sqrt{1-w}\sin(cw)\,dw\right] \quad (2.2\text{-}9)$$

Using the equations (2.2-5) and (2.2-6) in the formulae (From[2] formula 11 and 12, P.425)

$$\int_0^1 \sqrt{1-w}\cos(cw)\,dw = \frac{1}{2}\beta\left(1,\frac{3}{2}\right)\left[{}_1F_1\left(1;\frac{5}{2};ic\right) + {}_1F_1\left(1;\frac{5}{2};-ic\right)\right] \quad \textit{and}$$

$$\int_0^1 \sqrt{1-w}\sin(cw)\,dw = \frac{1}{2i}\beta\left(1,\frac{3}{2}\right)\left[{}_1F_1\left(1;\frac{5}{2};ic\right) - {}_1F_1\left(1;\frac{5}{2};-ic\right)\right]$$

where, $\beta(a,b)$ stands for the Beta function ,we get

$$\int_0^1 \sqrt{1-w}\cos(cw)\,dw = \frac{\sqrt{\pi}}{2}\frac{Y_{3/2}(c)}{c^{3/2}} \quad (2.2\text{-}10)$$

and $\int_0^1 \sqrt{1-w}\sin(cw)\,dw = \frac{\sqrt{\pi}}{2}\frac{Y_{5/2}(c)}{c^{3/2}}$ (2.2-11)

Therefore substituting (2.2-10) and (2.2-11) in equation (2.2-9) we get ,

$$\psi\left(t+\frac{1}{2}\right)=\frac{\sqrt{3/2}}{(\pi t)^{3/2}}\left\{\left[\cos\left(\frac{2\pi t}{3}\right)I_{0,\frac{1}{2}}\left(\frac{2\pi t}{3}\right)-\sin\left(\frac{2\pi t}{3}\right)I_{1,\frac{1}{2}}\left(\frac{2\pi t}{3}\right)\right]+\right.$$
$$\left.\sqrt{\frac{\pi}{8}}\left[\cos\left(\frac{4\pi t}{3}\right)Y_{3/2}\left(\frac{4\pi t}{3}\right)-\sin\left(\frac{4\pi t}{3}\right)Y_{5/2}\left(\frac{4\pi t}{3}\right)\right]\right\}$$

To derive $\psi(t)$ explicitly,

Since the relations $\hat{\phi}(w-2\pi)=1-\hat{\phi}(w)$ for $\frac{2\pi}{3}\leq w\leq\frac{4\pi}{3}$ and

$\hat{\phi}(w+2\pi)=1-\hat{\phi}(w)$ for $-\frac{4\pi}{3}\leq w\leq-\frac{2\pi}{3}$ are not satisfied .we can not use

$\psi\left(t+\frac{1}{2}\right)=2\phi(2t)-\phi(t)$. Therefore we consider $e^{-i(w/2)}\hat{\psi}(w)=\hat{\phi}\left(\frac{w}{2}\right)-\hat{\phi}(w)$,

because of symmetry of $e^{-i(w/2)}\hat{\psi}(w)$ it is sufficient to consider it's restriction

to the positive real axis.

Therefore from definition of $\hat{\phi}(w)$ we have

$$\hat{\phi}(w-2\pi)=\begin{cases}0 & if\ w\leq\frac{2\pi}{3}\\ \sqrt{\left(\frac{3w}{2\pi}\right)-1} & if\ \frac{2\pi}{3}\leq w\leq\frac{4\pi}{3}\\ 1 & if\ \frac{4\pi}{3}\leq w\leq\frac{8\pi}{3}\\ \sqrt{\left(-\frac{3w}{2\pi}\right)+5} & if\ \frac{8\pi}{3}\leq w\leq\frac{10\pi}{3}\\ 0 & if\ \frac{10\pi}{3}\leq w\end{cases}$$

Therefore the restriction of $e^{-i(w/2)}\hat{\psi}(w)$ to positive real axis is given by

$$e^{-i(w/2)}\hat{\psi}(w)=\begin{cases}0 & if\, 0\le w\le \dfrac{2\pi}{3}\\ \sqrt{\left(\dfrac{3w}{2\pi}\right)-1} & if\, \dfrac{2\pi}{3}\le w\le \dfrac{4\pi}{3}\\ \sqrt{\left(-\dfrac{3w}{2\pi}\right)+2} & if\, \dfrac{4\pi}{3}\le w\le \dfrac{8\pi}{3}\\ 0 & if\, \dfrac{8\pi}{3}\le w\end{cases}$$

Therefore by taking inverse Fourier Transform of $e^{-i(w/2)}\hat{\psi}(w)$

$$\psi\left(t+\frac{1}{2}\right)=\frac{1}{\pi}\int_0^{8\pi/3} e^{-i(w/2)}\hat{\psi}(w)\cos(tw)dw$$

$$=\frac{1}{\pi}\left\{\int_{2\pi/3}^{4\pi/3}\sqrt{\left(\frac{3w}{2\pi}\right)-1}\cos(tw)dw+\int_{4\pi/3}^{8\pi/3}\sqrt{\left(-\frac{3w}{2\pi}\right)+2}\cos(tw)dw\right\}$$

$$=\frac{2}{3}\int_1^2\sqrt{\gamma-1}\cos\left(\frac{2\pi\gamma t}{3}\right)d\gamma+\frac{4}{3}\int_1^2\sqrt{2-\gamma}\cos\left(\frac{4\pi\gamma t}{3}\right)d\gamma$$

$$=\frac{2}{3}\int_0^1\sqrt{u}\cos(\alpha u+\alpha)du+\frac{4}{3}\int_0^1\sqrt{1-u}\cos(\beta u+\beta)du$$

where $\alpha=\dfrac{2\pi t}{3}, \beta=2\alpha$

$$\psi\left(t+\frac{1}{2}\right)=\frac{2}{3}\left\{\cos\alpha\int_0^1\sqrt{u}\cos\alpha u du-\sin\alpha\int_0^1\sqrt{u}\sin\alpha u du+\right.$$
$$\left.2\cos\beta\int_0^1\sqrt{1-u}\cos\beta u du-2\sin\beta\int_0^1\sqrt{1-u}\sin\beta u du\right\}$$

Therefore using (1.4-2)

$$\psi\left(t+\frac{1}{2}\right)=\frac{2}{3}\left\{\frac{1}{\alpha^{3/2}}\left[\cos\alpha\, I_{0,\frac{1}{2}}(\alpha)-\sin\alpha\, I_{1,\frac{1}{2}}(\alpha)\right]+\right.$$

$$\left.\frac{\pi}{\beta^{3/2}}\left[\cos\beta\, Y_{3/2}(\beta)-\sin(\beta) Y_{5/2}(\beta)\right]\right\}$$

Therefore

$$\psi\left(t+\frac{1}{2}\right)=\frac{\sqrt{3/2}}{(\pi t)^{3/2}}\left\{\left[\cos\left(\frac{2\pi t}{3}\right) I_{0,\frac{1}{2}}\left(\frac{2\pi t}{3}\right)-\sin\left(\frac{2\pi t}{3}\right) I_{1,\frac{1}{2}}\left(\frac{2\pi t}{3}\right)\right]+\right.$$

$$\left.\sqrt{\frac{\pi}{8}}\left[\cos\left(\frac{4\pi t}{3}\right) Y_{3/2}\left(\frac{4\pi t}{3}\right)-\sin\left(\frac{4\pi t}{3}\right) Y_{5/2}\left(\frac{4\pi t}{3}\right)\right]\right\}$$