Chapter II

GENERALIZED LAPLACE TRANSFORMATION

2.1 Introduction:-

In this chapter we study some classical results, lemmas regarding to the generalized Laplace transformation defined as

$$F(s) = \int_0^\infty (st)^\lambda \, e^{-st} \, L_k^a(st) f(t) \, dt \qquad .....(2.1.1)$$

where, L_k^a is the Laguerre polynomial and is defined as

$$L_k^a(st) = \sum_{n=0}^{k} \frac{(-1)^n (1+a)_k (st)^n}{n! \, (k-n)! \, (1+a)_n}$$

$$= \frac{(1+a)_k}{k!} \, {}_1F_1(-k;\ 1+a;\ st)$$

$$= \frac{(-1)^k}{k!} \, U(-k;\ 1+a;\ st)$$

and when $\lambda = 0$ and k = 0 this transformation reduces to the well known one sided Laplace transform, defined as

$$F(s) = \int_0^\infty e^{-st} f(t) \, dt$$

In the last section we simplify some lemmas and results given in [8 , P. 22, 23] which will required to study inversion theorem for distributional generalized Laplace transformation.

2.2 Some classical results of generalized Laplace transforms

Before to prove the classical inversion therorem of generalized Laplace transformation we simplify some lemmas and theorems, which are further used in inversion theorem.

Lemma 2.2.1:

If

(i) $a < a + \xi < b$

(ii) $\gamma(t) \in c^2, \gamma'(a) = 0, \gamma''(a) < 0, a \le t \le a + \xi$

$\gamma(t)$ is non-increasing in $a \le t \le b$,

then,

$$\int_a^b t^{\lambda+n} e^{k\gamma(t)} dt \sim a^{\lambda+n} e^{k\gamma(a)} \left[\frac{-\pi}{2k\gamma''(a)}\right]^{+\frac{1}{2}}; \quad k \to \infty \quad [8, P. 17]$$

Lemma 2.2.2:

If

(i) $a < a + \xi < b$

(ii) $\gamma(t) \in c^2, \gamma'(a) = 0, \gamma''(a) < 0, a \le t \le a + \xi$

$\gamma(t)$ is non-increasing in $a \le t \le b$,

(iii) $f(t) \in L, a \le t \le b; f(a) \neq 0$

(iv) $a(t) = \int_a^t [f(x) - f(a)] dx = o(t-a); \quad t \to a^+$

Then,

$$\int_a^b f(t)\, t^{\lambda+n} e^{k\gamma(t)} dt = f(a)\, a^{\lambda+n} e^{k\gamma(a)} \left[\frac{-\pi}{2k\gamma''(a)}\right]^{+\frac{1}{2}}; \quad k \to \infty \; [8, P. 17]$$

Lemma 2.2.3:

If

(i) $a < a + \xi < b$

(ii) $\gamma(t) \in c^2, \gamma'(a) = 0, \gamma''(a) < 0, a \le t \le a + \xi$,

$\gamma(t)$ is non-increasing in $a \le t \le b$

(iii) $f(t) \in L, a \le t \le b; \quad f(b) \neq 0$

(iv) $a(t) = \int_t^b [f(x) - f(b)]\, dx = o(b-t)\,; \quad t \to b^-$

then,

$$\int_a^b f(t)\, t^{\lambda+n}\, e^{k\gamma(t)}\, dt \backsim f(b)\, b^{\lambda+n}\, e^{k\gamma(b)} \left[\frac{-\pi}{2k\,\gamma''(b)}\right]^{+\frac{1}{2}}; \quad k \to \infty \quad [8, P. 18]$$

Theorem 2.2.1:

If

(i) f(t) $\in$ L, $(0 < x \le t \le R)$ for fixed x and large R

(ii) $\int_x^\infty e^{-ct} f(t)\, dt$ converges for a fixed positive c.

(iii) $\int_x^\infty [f(y) - f(x)]\, dy = o(t-x); \quad t \to x^+$

then,

$$\lim_{k \to \infty} \frac{k^{\lambda+n+k+1}}{\lceil \lambda+n+k+1} \int_x^\infty \frac{t^{\lambda+n+k}}{x^{\lambda+n+k+1}}\, e^{-\left(\frac{k}{x}\right)t} f(t)\, dt = \frac{f(x)}{2}, \quad [8, P. 18]$$

λ and k be the non-negative integers.

Proof :-

Choose any positive number δ and let

$$a(y) = \int_{x+\delta}^{y} e^{-ct} f(t)\, dt; \quad y \ge x+\delta \qquad .....(2.2.1)$$

Then by the condition (ii), there exist a constant M such that

$$|\, a(y)\,| \le M, \qquad x \le y < \delta$$

Set

$$I_k = A \int_x^{x+\delta} \frac{t^{\lambda+n+k}}{x^{\lambda+n+k+1}}\, e^{-\left(\frac{k}{x}\right)t} f(t)\, dt + A \int_{x+\delta}^{\infty} \frac{t^{\lambda+n+k}}{x^{\lambda+n+k+1}}\, e^{-\left(\frac{k}{x}\right)t} f(t)\, dt$$

$$= I_k' + I_k'' \qquad .....(2.2.2)$$

where $A = \frac{k^{\lambda+n+k+1}}{\lfloor \lambda+n+k+1}$

from (2.2.1)

$$f(t) = e^{ct}\, \alpha'(t)$$

and integrating I_k'' by parts we get

$$I_k'' = \frac{A}{x^{\lambda+n+k+1}} \int_{x+\delta}^{\infty} \left[t^{\lambda+n+k}\, e^{-(k-cx)\frac{t}{x}}\right] \alpha'(t)\, dt$$

$$= \frac{A}{x^{\lambda+n+k+1}} \left\{ \left[t^{\lambda+n+k}\, e^{-(k-cx)\frac{t}{x}}\alpha(t)\right]_{x+\delta}^{\infty} - \int_{x+\delta}^{\infty} \alpha(t)\, d\left[t^{\lambda+n+k}\, e^{-(k-cx)\frac{t}{x}}\right] dt \right\}$$

$$= -\frac{A}{x^{\lambda+n+k+1}} \left\{ \int_{x+\delta}^{\infty} \alpha(t)\, d\left[t^{\lambda+n+k}\, e^{-(k-cx)\frac{t}{x}}\right] dt \right\}$$

$\because$ The integrated part is vanishes for $k > cx$ and $\alpha(x + \delta) = 0$

Let us consider the function,

$$X(t) = t^{\lambda+n+k}\, e^{-(k-cx)\frac{t}{x}}$$

for sufficiently large $k > k_0$, $X(t)$ is maximum at

$$t = (\lambda+n+k)\frac{x}{k-cx}$$

$$= x\left[1+\frac{\lambda+n}{k}\right]\left[1-\frac{cx}{x}\right]^{-1}$$

Thus, $X(t)$ is decreasing in $x + \delta \le t \le \infty$, when $k > k_0$.

Hence,

$$I_k'' \le \frac{A}{x} M\left(\frac{x+\delta}{x}\right)^{\lambda+n+k} e^{-(k-cx)\left(1+\frac{\delta}{x}\right)}$$

$$\le \frac{A}{x} M\left(1+\frac{\delta}{x}\right)^{\lambda+n+k} e^{-(k-cx)\left(1+\frac{\delta}{x}\right)}$$

$$= J_k \quad (say)$$

Now,

$$\frac{J_{k+1}}{J_k} = \frac{\frac{A}{x} M\left(1+\frac{\delta}{x}\right)^{\lambda+n+k+1} e^{-(k+1-cx)\left(1+\frac{\delta}{x}\right)}}{\frac{A}{x} M\left(1+\frac{\delta}{x}\right)^{\lambda+n+k} e^{-(k-cx)\left(1+\frac{\delta}{x}\right)}}$$

$$= \left(1+\frac{\delta}{x}\right) e^{-1-\frac{\delta}{x}}$$

$$\therefore \lim_{k\to\infty} I_k'' = \lim_{k\to\infty} J_k = 0 \quad \{\because \left(1+\tfrac{\delta}{x}\right) e^{-1-\frac{\delta}{x}} < 1 \qquad .....(2.2.3)$$

$$I_k' = \frac{A}{x^{\lambda+n+k+1}} \int_x^{x+\delta} \left[t^{\lambda+n+k} e^{-(k-cx)\frac{t}{x}}\right] f(t)\, dt \qquad .....(2.2.4)$$

Take $\gamma(t) = \log t - \frac{t}{x}$ in lemma (2.2.2)

$$\gamma'(t) = \frac{1}{t} - \frac{1}{x} < 0 \qquad \{\because x < t \quad \therefore \frac{1}{x} > \frac{1}{t}$$

$$\gamma'(x) = 0$$

$$\gamma''(x) = -\frac{1}{x^2} < 0$$

Hence the hypothesis of lemma (2.2.2) is satisfied.

$\therefore$ by the lemma (2.2.2),

$$\int_x^{x+\delta} f(t)\, t^{\lambda+n} e^{k\gamma(t)}\, dt = \int_x^{x+\delta} f(t)\, t^{\lambda+n} e^{k\left(\log t - \frac{t}{x}\right)}\, dt$$

$$= \int_x^{x+\delta} f(t)\, t^{\lambda+n+k} e^{-\left(\frac{k}{x}\right)t}\, dt$$

$$= f(x)\, x^{\lambda+n} e^{k\gamma(x)} \left[-\frac{\pi}{2k\,\gamma''(x)}\right]^{+\frac{1}{2}}, \qquad k\to\infty$$

$$= f(x)\, x^{\lambda+n} e^{k(\log x - 1)} \left[+\frac{\pi}{2k} x^2\right]^{\frac{1}{2}}, \qquad k\to\infty$$

$$= f(x)\, x^{\lambda+n+k+1} e^{-k} \left[\frac{\pi}{2k}\right]^{\frac{1}{2}}, \qquad k\to\infty$$

Putting in (2.2.4), we get

$$I_k' = \frac{k^{\lambda+n+k+1}}{\lfloor \lambda+n+k+1} f(x)\, e^{-k} \left[\frac{\pi}{2k}\right]^{\frac{1}{2}}, \qquad k\to\infty$$

By the Sterling's formula, we have

$$\overline{|\lambda+n+k+1} = (\lambda+n+k)! \backsim \sqrt{2\pi(\lambda+n+k)} \cdot (\lambda+n+k)^{\lambda+n+k} e^{-(\lambda+n+k)}$$

$$\therefore\ I'_k \backsim \frac{k^{\lambda+n+k+1} f(x)\, e^{-k}\left[\frac{\pi}{2k}\right]^{\frac{1}{2}}}{\sqrt{2\pi(\lambda+n+k)} \cdot (\lambda+n+k)^{\lambda+n+k}\, e^{-(\lambda+n+k)}}$$

$$= \frac{f(x)}{2}\left[\frac{k^{\lambda+n+k+\frac{1}{2}} \cdot e^{\lambda+n}}{(\lambda+n+k)^{\lambda+n+k+\frac{1}{2}}}\right]; \qquad k \to \infty \qquad .....(2.2.5)$$

Hence putting the values of (2.2.3) and (2.2.5) in (2.2.2), we get

$$I_k \backsim \frac{f(x)}{2}; \qquad k \to \infty$$

Hence the proof.

Theorem 2.2.2:

If

(i) $f(t) \in L, (0 < \in \leq t < x)$ for fixed x and small positive $\in$.

(ii) $\int_{0^+}^{x} t^r f(t)\, dt$ converges for a fixed constant r.

(iii) $\int_{x}^{y} [f(t) - f(x)]\, dt = o(x-y); \quad y \to x^-$

then,

$$\lim_{k\to\infty} \frac{k^{\lambda+n+k+1}}{\overline{|\lambda+n+k+1}} \int_{0^+}^{x} \frac{t^{\lambda+n+k}}{x^{\lambda+n+k+1}}\, e^{-\left(\frac{k}{x}\right)t} f(t)\, dt = \frac{f(x)}{2} \text{ [8, P. 19]}$$

Proof :-

Let us choose a positive δ less than x and set

$$a(y) = \int_{y}^{x-\delta} t^r f(t)\, dt \qquad .....(2.2.6)$$

then by condition (ii), there exist a constant M such that

$$|a(y)| < M; \quad 0 < y \leq x-\delta$$

Set

$$I_k = A\int_0^{x-\delta} \frac{t^{\lambda+n+k}}{x^{\lambda+n+k+1}} e^{-\left(\frac{k}{x}\right)t} f(t)\,dt + A\int_{x-\delta}^{x} \frac{t^{\lambda+n+k}}{x^{\lambda+n+k+1}} e^{-\left(\frac{k}{x}\right)t} f(t)\,dt$$

$$= I_k' + I_k'' \qquad \text{.....(2.2.7)}$$

where $A = \frac{k^{\lambda+n+k+1}}{|\lambda+n+k+1}$

Integrating I_k' by parts, we get

$$I_k' = A\int_0^{x-\delta} \frac{t^{\lambda+n+k}}{x^{\lambda+n+k+1}} e^{-\left(\frac{k}{x}\right)t} f(t)\,dt$$

$$= \frac{A}{x^{\lambda+n+k+1}} \int_0^{x-\delta} \left[t^{\lambda+n+k-r}\; e^{-\left(\frac{k}{x}\right)t}\right] \alpha'(t)\,dt$$

$$= \frac{A}{x^{\lambda+n+k+1}} \left\{\left[t^{\lambda+n+k-r} e^{\left(-\frac{k}{x}\right)t} \alpha(t)\right]_0^{x-\delta} - \int_{0^+}^{x-\delta} \alpha(t)\,d\left[t^{\lambda+n+k-r}\; e^{\left(-\frac{k}{x}\right)t}\right] dt\right\}$$

$$= -\frac{A}{x^{\lambda+n+k+1}} \left\{\int_0^{x-\delta} \alpha(t)\,d\left[t^{\lambda+n+k-r}\; e^{\left(-\frac{k}{x}\right)t}\right] dt\right\}$$

$\because$ The integrated part is vanishes for $\lambda + n + k - r > 0$ and $\alpha\,(x - \delta) = 0$

Now consider the function,

$$H(t) = t^{\lambda+n+k-r}\; e^{\left(-\frac{k}{x}\right)t}$$

H(t) is maximum at $t = x\left[1 + \frac{(\lambda+n-r)}{k}\right] > x - \delta, \qquad \textit{for } k > k_0$

Thus, H(t) is non-decreasing in $0 \le t \le x - \delta$,

Hence,

$$|\,I_k'\,| \le \frac{A}{x} M\left(1 - \frac{\delta}{x}\right)^{\lambda+n+k-r} e^{-k\left(1-\frac{\delta}{x}\right)}$$

$$= J_k \qquad (say)$$

$$\therefore \lim_{k\to\infty} I_k' = \lim_{k\to\infty} J_k$$

$$= \lim_{k\to\infty}\left[\frac{A}{x} M\left(1 - \frac{\delta}{x}\right)^{\lambda+n+k-r} e^{-k\left(1-\frac{\delta}{x}\right)}\right]$$

$$\therefore \lim_{k \to \infty} I'_k = 0 \qquad .....(2.2.8)$$

$$\therefore I''_k = A \int_{x-\delta}^{x} \frac{t^{\lambda+n+k}}{x^{\lambda+n+k+1}} e^{-\left(\frac{k}{x}\right)t} f(t)\, dt \qquad .....(2.2.9)$$

Take $\gamma(t) = log t - \frac{t}{x}$ in lemma (2.2.3)

$$\gamma'(t) = \frac{1}{t} - \frac{1}{x} < 0$$

$$\gamma'(x) = 0$$

$$\gamma''(x) = -\frac{1}{x^2} < 0$$

Hence the hypothesis of lemma (2.2.3) is satisfied.

$$\int_{x-\delta}^{x} f(t)\, t^{\lambda+n} e^{k\gamma(t)} dt = \int_{x-\delta}^{x} f(t)\, t^{\lambda+n} e^{k\left(\log t - \frac{t}{x}\right)} dt$$

$$= \int_{x-\delta}^{x} f(t)\, t^{\lambda+n+k} e^{-\left(\frac{k}{x}\right)t} dt$$

$$= f(x)\, x^{\lambda+n} e^{k\gamma(x)} \left[-\frac{\pi}{2k\gamma''(x)}\right]^{\frac{1}{2}}, \qquad k \to \infty$$

$$= f(x)\, x^{\lambda+n} e^{k(\log x - 1)} \left[+\frac{\pi}{2k} x^2\right]^{\frac{1}{2}}$$

$$= f(x)\, x^{\lambda+n+k+1} e^{-k} \left[\frac{\pi}{2k}\right]^{\frac{1}{2}}, \qquad k \to \infty$$

Putting in (2.2.9), we get

$$I''_k = \frac{k^{\lambda+n+k+1}}{\left|\overline{\lambda+n+k+1}\right.} f(x)\, e^{-k} \left[\frac{\pi}{2k}\right]^{\frac{1}{2}}, \qquad k \to \infty$$

By the Sterling's formula, we have

$$\left|\overline{\lambda+n+k+1}\right. = (\lambda+n+k)! \backsim \sqrt{2\pi(\lambda+n+k)} \cdot (\lambda+n+k)^{\lambda+n+k} e^{-(\lambda+n+k)}$$

$$\therefore I''_k \backsim \frac{k^{\lambda+n+k+1} f(x)\, e^{-k} \left[\frac{\pi}{2k}\right]^{\frac{1}{2}}}{\sqrt{2\pi(\lambda+n+k)} \cdot (\lambda+n+k)^{\lambda+n+k} e^{-(\lambda+n+k)}}; \qquad k \to \infty$$

$$= \frac{f(x)}{2} \left[\frac{k^{\lambda+n+k+\frac{1}{2}} \cdot e^{\lambda+n}}{(\lambda+n+k)^{\lambda+n+k+\frac{1}{2}}}\right]; \qquad k \to \infty \qquad .....(2.2.10)$$

Hence putting the values of (2.2.8) and (2.2.10) in (2.2.7), we get

$$I_k \backsim \frac{f(x)}{2} \qquad as\ k \to \infty.$$

Hence the proof.

Theorem 2.2.3:

If

(i) $f(t) \in L, (1/R \le t \le R)$ for every $R > 1$.

(ii) $\int_1^\infty e^{-ct} f(t)\, dt$ converges for a fixed $c > 0$.

(iii) $\int_0^1 t^r f(t)\, dt$ converges for a fixed r.

(iv) $\int_x^y [f(t) - f(x)]\, dt = o(|y - x|); \quad y \to x$

then,

$$\lim_{k \to \infty} \frac{k^{\lambda+n+k+1}}{\lfloor \lambda+n+k+1} \int_x^\infty \frac{t^{\lambda+n+k}}{x^{\lambda+n+k+1}} e^{-\left(\frac{k}{x}\right)t} f(t)\, dt = f(x) \text{ [8, P. 20]}$$

Proof :-

We can easily prove this theorem by combining the theorems (2.2.1) and (2.2.2).

2.3 The Inversion Operator

If F(s) has derivative and integrals of all orders, an operator $M_{n,x}[F(s)]$ is defined for any positive x and any integer n by

$$M_{n,x}[F(s)] = (-1)^n \frac{s^{\lambda+n+1}}{\lfloor \lambda+n+k+1} D^n[s^{-a-k} D^{-n}\{s^k D^n(s^{a-\lambda} F(s))\}]; \quad s = \frac{n}{x} \qquad .....(2.3.1)$$

where $D = \frac{d}{ds}$ and

$$D^{-1}(s^\mu) = \int_0^s s^\mu\, ds, \qquad if\, Re\,(\mu+1) > 0 \qquad \text{[8, P. 20]}$$

$$= \int_s^\infty s^\mu\, ds, \qquad if\, Re\,(\mu+1) < 0$$

First we simplify the following lemma which will be used in classical inversion theorem.

Lemma 2.3.1:

If $f(s) = (st)^{\lambda} e^{-st} L_k^a(st)$ then prove that

$$M_{n,x}[f(s)] = \frac{s^{\lambda+n+1}}{\lfloor \lambda+n+k+1} \left[t^{\lambda+n} e^{-st} L_k^a(st)\right] \quad [8, P. 20]$$

Proof :-

Here $f(s) = (st)^{\lambda} e^{-st} L_k^a(st)$

By Slater [15], We define the Laguree polynomial $L_k^a(st)$ as

$$L_k^a(st) = \frac{(-1)^k}{k!} U(-k;\ 1+a;\ st)$$

Thus,

$$f(s) = (st)^{\lambda} e^{-st} \frac{(-1)^k}{k!} U(-k;\ 1+a;\ st)$$

Take $A = \frac{(-1)^k}{k!}$

Using the results 2.1.32, 2.1.31, 2.1.30 of Slater [15], we get

$$M_{n,x}[f(s)] = (-1)^n A\ t^{\lambda+n} e^{-st} U(-k\ ;\ 1+a\ ;\ st)$$

Transforming to Laguerre polynomials, we get the result

$$M_{n,x}[f(s)] = \frac{s^{\lambda+n+1}}{\lfloor \lambda+n+k+1} \left[t^{\lambda+n} e^{-st} L_k^a(st)\right]$$

Theorem 2.3.1 :

If f(t) $\in$ L in $0 \le t \le R$ for every positive R and is such that the integral (2.1.1) converges for some $s = s_0$ (say), then

$$\lim_{n \to \infty} M_{n,x}[F(s)] = f(x) \quad [8, P. 21]$$

for all positive x in the Lebsgue set for f(t).

Proof :-

We have the integral

$$F(s) = \int_0^\infty e^{-st} (st)^\lambda L_k^a(st) f(t)\, dt$$

converges for $s > s_0$.

If $f(s) = e^{-st} (st)^\lambda L_k^a(st)$ then

$$F(s) = \int_0^\infty f(s) f(t)\, dt \qquad \text{.....(2.3.2)}$$

We can evaluate the derivatives of (2.3.2) by differentiation under the sign of integration.

$$\therefore \; M_{n,x}[F(s)] = M_{n,x} \int_0^\infty f(s) f(t)\, dt$$

$$= \int_0^\infty M_{n,x}[f(s)] f(t)\, dt$$

by using the lemma (2.3.1), we get

$$M_{n,x}[F(s)] = \int_0^\infty \frac{s^{\lambda+n+1}}{\lfloor\lambda+n+k+1} \left[t^{\lambda+n} e^{-st} L_k^a(st)\right] f(t)\, dt$$

since $L_k^a(st) \backsim (st)^k$ as $st \to \infty$.

$$\therefore \; M_{n,x}[F(s)] \backsim \int_0^\infty \frac{s^{\lambda+n+1}}{\lfloor\lambda+n+k+1} \left[t^{\lambda+n} e^{-st} (st)^k\right] f(t)\, dt$$

$$= \int_0^\infty \frac{s^{\lambda+n+k+1}}{\lfloor\lambda+n+k+1} \left[t^{\lambda+n+k} e^{-st}\right] f(t)\, dt$$

Since $s = \frac{n}{x}$

$$M_{n,x}[F(s)] \backsim \frac{n^{\lambda+n+k+1}}{\lfloor\lambda+n+k+1} \int_0^\infty \frac{t^{\lambda+n+k}}{x^{\lambda+n+k+1}} e^{-(\frac{n}{x})t} f(t)\, dt$$

Taking limit as $n \to \infty$, we get,

$$\lim_{n\to\infty} M_{n,x}[F(s)] \backsim \lim_{n\to\infty} \frac{n^{\lambda+n+k+1}}{\lfloor\lambda+n+k+1} \int_0^\infty \frac{t^{\lambda+n+k}}{x^{\lambda+n+k+1}} e^{-(\frac{n}{x})t} f(t)\, dt$$

Hence by using the theorem (2.2.3), we get,

$$\lim_{n\to\infty} M_{n,x}[F(s)] \sim f(x)$$

Now we can verify the conditions of the theorem (2.2.3).

here $f(t) \in L$ in $0 \le t \le R$ for every positive R.

Hence r of the theorem (2.2.3) may be taken as zero. Also we may take c greater than s_0 (assumed real) because $\int_0^\infty e^{-ct} f(t)\,dt$ converges.

The Lebsgue set for f(t) is the set of numbers x_0 so that

$$\int_{x_0}^{x} |f(t) - f(x_0)|\,dx = o(|x - x_0|); \qquad x \to x_0$$

For such a set, the hypothesis (iv) of the theorem (2.2.3) is satisfied.

Thus, all the hypothesis of the theorem (2.2.3) is satisfied.

Hence the proof.

Lemma 2.3.2:

If n is a positive integer and t, y are any positive variables. Then,

$$\frac{\partial^n}{\partial y^n}[y^{n-1} e^{-\frac{n}{y}t} L_k^a(\tfrac{n}{y}t)]$$

$$= n^n t^n y^{n-1} e^{-\frac{n}{y}t} U(-k; 1+a+n; \tfrac{n}{y}t) \quad [8, P. 21]$$

Proof:-

We have by definition of Laguree polynominal,

$$y^{n-1} e^{-\frac{n}{y}t} L_k^a(\tfrac{n}{y}t) = \frac{(-1)^k}{k!} y^{n-1} e^{-\frac{n}{y}t} U(-k;\ 1+a;\ \tfrac{n}{y}t)$$

and $(\frac{y}{t})^{n-1} e^{-\frac{n}{y}t} U(-k;\ 1+a;\ \frac{n}{y}t)$ is a homogeneous function of order zero. Hence by the Euler's theorem and [15, P.16], we have

$$\frac{\partial}{\partial y}\left[\frac{y^{n-1}}{t^n} e^{-\frac{n}{y}t} U(-k;\ 1+a;\ \tfrac{n}{y}t)\right]$$

$$= (-1)\frac{\partial}{\partial t}\left[\frac{y^{n-2}}{t^{n-1}} e^{-\frac{n}{y}t} U(-k;\ 1+a;\ \tfrac{n}{y}t)\right]$$

$$\frac{\partial^2}{\partial y^2}\left[\frac{y^{n-1}}{t^n} e^{-\frac{n}{y}t} U(-k;\ 1+a;\ \tfrac{n}{y}t)\right]$$

$$= (-1)^2\frac{\partial^2}{\partial t^2}\left[\frac{y^{n-3}}{t^{n-2}} e^{-\frac{n}{y}t} U(-k;\ 1+a;\ \tfrac{n}{y}t)\right]$$

continuing in this way, we get

$$\frac{\partial^n}{\partial y^n}\left[\frac{y^{n-1}}{t^n} e^{-\frac{n}{y}t} U(-k; 1+a; \tfrac{n}{y}t)\right]$$

$$= (-1)^n\frac{\partial^n}{\partial t^n}\left[\frac{y^{n-(n+1)}}{t^{n-n}} e^{-\frac{n}{y}t} U(-k; 1+a; \tfrac{n}{y}t)\right]$$

$$= (-1)^n \frac{\partial^n}{\partial t^n}[y^{-1}\, e^{-\frac{n}{y}t}\, U(-k;\ 1+a;\ \tfrac{n}{y}t\,)]$$
$$= (-1)^n\, n^n\, y^{-1}\, e^{-\frac{n}{y}t}(-\tfrac{n}{y})^n\, U(-k;\ 1+a+n;\ \tfrac{n}{y}t)]$$
$$= n^n\, y^{-n-1}\, e^{-\frac{n}{y}t}\, U(-k;\ 1+a+n;\ \tfrac{n}{y}t)]$$

Thus,

$$\frac{\partial^n}{\partial y^n}\,[y^{n-1}\, e^{-\frac{n}{y}t}\, L_k^a(\tfrac{n}{y}t)] = n^n\, t^n\, y^{n-1}\, e^{-\frac{n}{y}t}\, U(-k;\ 1+a+n;\ \tfrac{n}{y}\,t\,)$$

Hence the proof.

Result 2.3.1:

If n is a positive integer and x, t are positive then

$$\frac{\partial^n}{\partial t^n}\left[\frac{t^{a+n+k}}{x^{a+n+k+1}}\, e^{-\frac{n}{x}t}\, U(-k;\ 1+a;\ \tfrac{n}{x}\,t\,\right]$$
$$= (-1)^n \frac{\partial^n}{\partial x^n}\left[\frac{t^{a+n+k-n}}{x^{a+n+k+1-n}}\, e^{-\frac{n}{x}t}\, U(-k;\ 1+a;\ \tfrac{n}{x}\,t\,\right]$$

Proof:-

Since $(\frac{t}{x})^{a+n+k}\, e^{-\frac{n}{x}t}\, U(-k;\ 1+a;\ \frac{n}{y}\,t\,)$ is a homogeneous function of order zero. Hence by the Euler's theorem, we get

$$\frac{\partial}{\partial t}\left[\frac{t^{a+n+k}}{x^{a+n+k+1}}\, e^{-\frac{n}{x}t}\, U(-k;\ 1+a;\ \tfrac{n}{x}\,t\,\right]$$
$$= (-1)\,\frac{\partial}{\partial x}\left[\frac{t^{a+n+k-1}}{x^{a+n+k+1-1}}\, e^{-\frac{n}{x}t}\, U(-k;\ 1+a;\ \tfrac{n}{x}\,t\,\right]$$
$$\frac{\partial^2}{\partial t^2}\left[\frac{t^{a+n+k}}{x^{a+n+k+1}}\, e^{-\frac{n}{x}t}\, U(-k;\ 1+a;\ \tfrac{n}{x}\,t\,\right]$$
$$= (-1)^2\,\frac{\partial^2}{\partial x^2}\left[\frac{t^{a+n+k-2}}{x^{a+n+k+1-2}}\, e^{-\frac{n}{x}t}\, U(-k;\ 1+a;\ \tfrac{n}{x}\,t\,\right]$$

continuing in this way, we get

$$\frac{\partial^n}{\partial t^n}\left[\frac{t^{a+n+k}}{x^{a+n+k+1}}\, e^{-\frac{n}{x}t}\, U(-k;\ 1+a;\ \tfrac{n}{x}\,t\,\right]$$
$$= (-1)^n \frac{\partial^n}{\partial x^n}\left[\frac{t^{a+n+k-n}}{x^{a+n+k+1-n}}\, e^{-\frac{n}{x}t}\, U(-k;\ 1+a;\ \tfrac{n}{x}\,t\,\right]$$

Hence the proof.

Lemma 2.3.3:

If G(s,y) = $y^{n-1}\, e^{-\frac{n}{y}s}\, L_k^a(\frac{n}{y}s)$

Then,

$$\frac{\partial^r}{\partial y^r}[G(s,y)] = 0(y^{n-k-r-1}\, e^{-\frac{n}{y}s}); \qquad y \to 0$$
$$= 0(y^{n-r-1}); \qquad y \to \infty.$$

r = 1, 2, 3, ... [8, P. 23]

Proof:-

By the definition of Laguree polynominal, we get

$$G(s,y) = y^{n-1}\, e^{-\frac{n}{y}s}\, \frac{(-1)^k}{k!}\, U(-k;\ 1+a;\ st)$$

$$\frac{\partial^2}{\partial y^2}[G(s,y)] = \frac{(1)^k}{k!}\, s^n\, \frac{\partial^2}{\partial s^2}\left[\frac{y^{n-3}}{s^{n-2}}\, e^{-\frac{n}{y}s}\, U(-k;\ 1+a;\ \tfrac{n}{y}s)\right]$$
$$= \frac{(1)^k}{k!} y^{n-3} e^{\frac{-n}{y}s}[(2-n)(1-n)U(-k;\ 1+a;\ \tfrac{n}{y}s) - 2(2-n)\tfrac{n}{y}s$$
$$U(-k;\ 2+a;\ \tfrac{n}{y}s) + \frac{n^2}{y^2}s^2 U(-k;\ 3+a;\ \tfrac{n}{y}s)]$$

using [15, P.17] we get

Now using the estimates of U(a; b; x) of Slater [15, P. 60]

We get

$$\frac{\partial^2}{\partial y^2}[G(s,y)] = o(y^{n-3-k}\, e^{-\frac{n}{y}s}) \qquad y \to 0.$$
$$= o(y^{n-3}) \qquad y \to \infty$$

Now using the theorems of Hardy and Littlewood [22, P. 193], we have

$$\frac{\partial^r}{\partial y^r}[G(s,y)] = o(y^{n-k-r-1}\, e^{-\frac{n}{y}s}) \qquad y \to 0.$$
$$= o(y^{n-r-1}) \qquad y \to \infty$$

Hence the proof.