CHAPTER-1

PROPERTIES OF 0-DISTRIBUTIVE LATTICES

Chapter 1

Properties of 0-Distributive Lattices

As 0-distributive lattice are the generalization of distributive lattice and pesudo-complemented lattice they have many interesting properties. In this chapter we discuss some properties of 0-distributive lattices.

§ 1 Examples

Example of 0-distributive lattice.

e.g Consider the lattice L = { 0 , a , b , c, 1 } whose diagrammatic representation is as shown in Fig. 1.1.

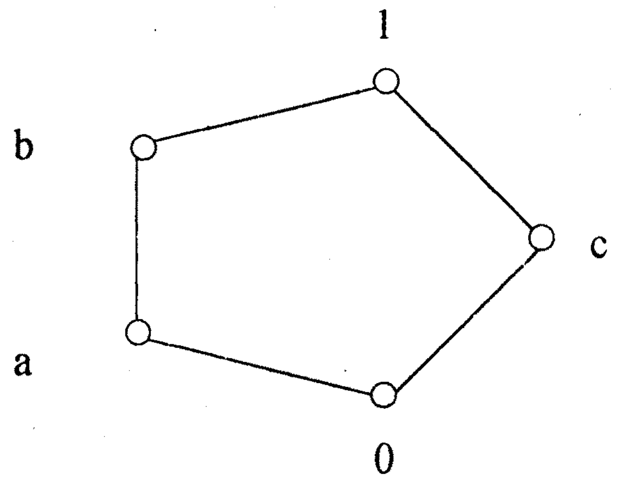

Fig.1.1

This Lattice is 0-distributivelattice.

Example of not 0-distributive lattice.

e.g Consider the lattice L = { 0 , a , b , c , 1 } whose diagrammatic representation is as shown in Fig. 1.2.

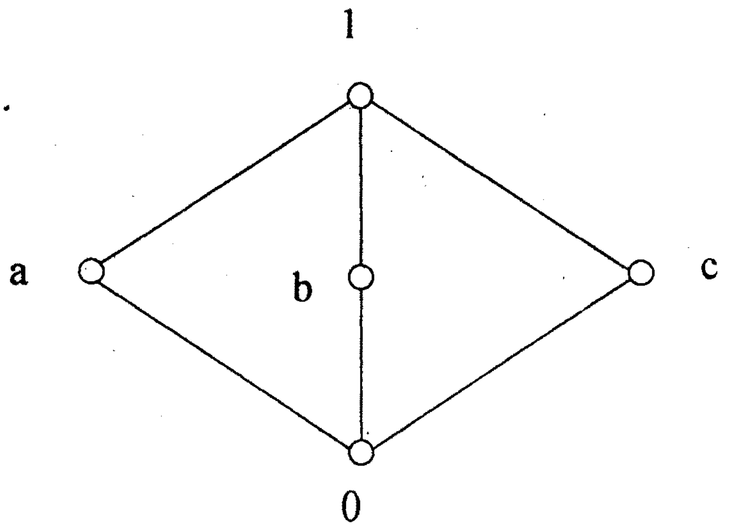

Fig.1.2

This lattice is not 0-distributive, because $a \wedge b = 0$, $a \wedge c = 0$ but $a \wedge (b \vee c) \neq 0$.

Example of 0-distributive and 1-distributive lattice.

e.g Consider the lattice L = { 0 , a , b , c , d , e , f , 1 } whose diagrammatic representation is as shown in Fig. 1.3.

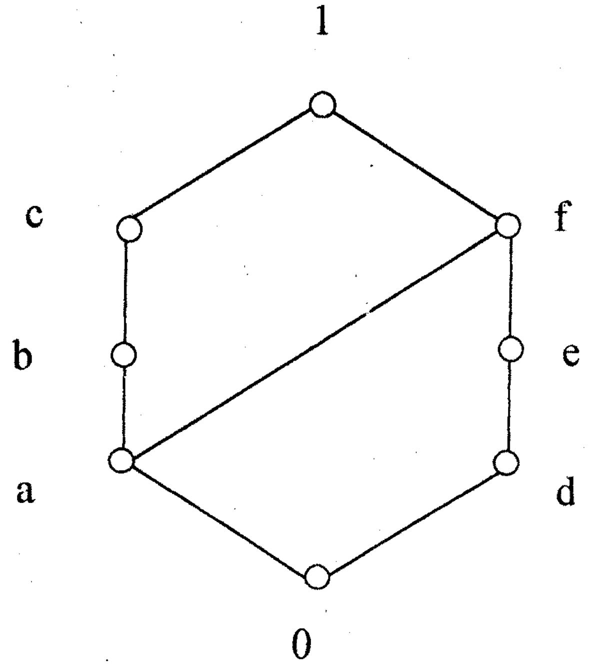

Fig.1.3

This lattice is 0-1distributive lattice but not complement lattice as the element a in L has no complement.

Example of 0-distributive but not 1- distributive lattice.

e.g Consider the lattice L = { 0 , a , b , c , d , e ,1 } whose diagrammatic representation is as shown in Fig. 1.4.

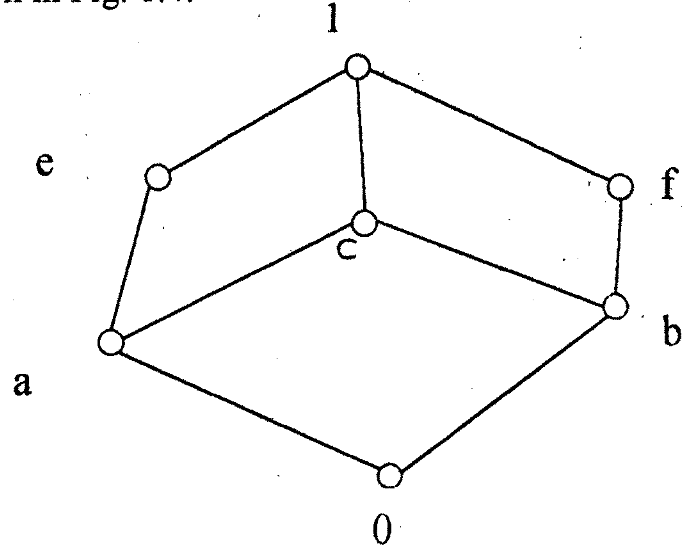

Fig.1.4

This lattice is 0-distributive lattice but not 1- distributive lattice.

Example of 0-distributive but not distributive lattice.

e.g Consider the lattice L = { 0 , a , b , c , d , 1 } whose diagrammatic representation is as shown in Fig. 1.5.

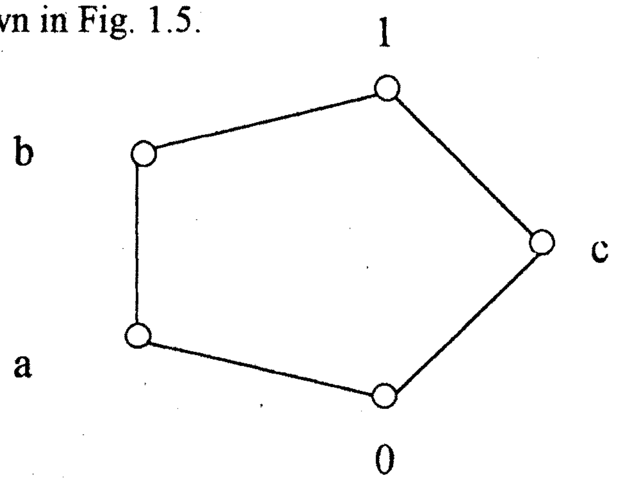

Fig.1.5

This lattice is 0-distributive lattice but distributive lattice.

Example of 1-distributive but not 0- distributive lattice.

e.g Consider the lattice L = { 0 , a , b , c , d , e , f ,1 } whose diagrammatic representation is as shown in Fig. 1.6.

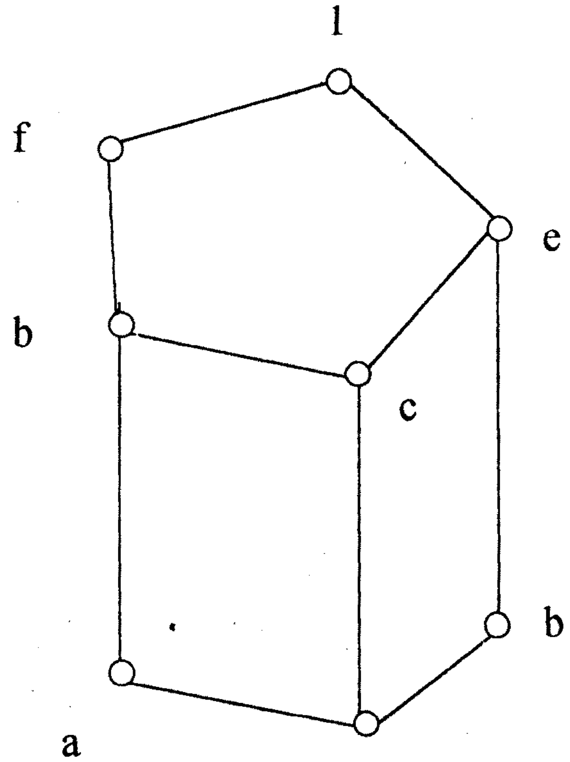

Fig.1.6

This lattice is 1-distributive lattice but 0-distributive lattice.

Example of 0-distributive but not pseudo-complemented lattice.

e.g Let $L = C_1 \cup C_2 \cup \{0, 1\}$ where C_1 and C_2 are two infinite chains without least and greatest elements .Define an ordering as follows . For all $c_1 \in C_1$ and $c_2 \in C_2$, $c_1 \parallel c_2$, $0 < c_1 < 1$ and $0 < c_2 < 1$. Clearly L is 0 and 1 distributive lattice. But neither distributive not pseduo-complemented.

§ 2 Some Properties of 0-Distridutive Lattices.

In this article we prove some properties of 0-distributive lattices which will be used in characterizing them.

Result 1.2.1 - Let $a, a_1, \ldots, a_n$ be elements of 0-distributive lattice L such that $a \wedge a_1 = \ldots = a \wedge a_n = 0$ .Then $a \wedge (a_1 \vee \ldots \vee a_n) = 0$.

Proof:- since L is 0-distributive $a \wedge a_1 = 0, a \wedge a_2 = 0$ implies

$a \wedge (a_1 \vee a_2) = 0$.

Assume $a \wedge (a_1 \vee \ldots \vee a_{k-1}) = 0$ for $2 < k < n$.

Then $a \wedge [(a_1 \vee \ldots \vee a_{k-1}) \vee a_k] = a \wedge (b \vee a_k)$

where $b = a_1 \vee \ldots \vee a_{k-1}$.

By induction hypothesis $a \wedge b = 0$. Also $a \wedge a_k = 0$.

Hence $a \wedge (b \vee a_k) = 0$ as L is 0-distributive.

i.e. $a \wedge [(a_1 \vee \ldots \vee a_{k-1}) \vee a_k] = 0$. Thus theorem is follows by induction.

♦♦♦

Relation between 0-distributve and pseud-ocomplemented lattice is given in the following theorem [9].

Result 1.2.2- Any pseudo-complemented lattice is 0-distributive.

Proof:- Let L be a pseudocomplemented lattice and a , b , c $\in$L such that $a \wedge b = 0$ and $a \wedge c = 0$.

Then $b \leq a^*$ and $c \leq a^*$. Hence $b \vee c \leq a^*$.

It follows that $a \wedge (b \vee c) = 0$. Thus L is 0-distributive.

♦♦♦

Though every 0-distributive lattice need not be pesudo-complemented. (See example in chapter 1) [9].

In particular we have

Result 1.2.3- Any finite 0-distributive lattice pseudocomplemented.

Proof:- Let L be a finite 0-distributive lattice and $a \in L$. Since L is finite. Hence $(a)^*$ is finite. Let $(a)^* = \{ a_1 \ldots a_n \}$ and $b = (a_1 \vee \ldots \vee a_n)$. By theorem 1.2 1 b is the pseudocomplement of a. This shows that L is pesudocomplemented lattice.

♦♦♦

Remark: Dualizing Theorems 1.2.1, 1.2.2 and 1.2.3 we get corresponding theorems for 1-distributive [9].

Result 1.2.4- Let L be a 0-distributive lattice. Then F° is an ideal for any filter F in L.

Proof:-Let L be a 0-distributive lattice.

We have to prove F° is an ideal for any filter F in L.

(i) As $0 \wedge f = 0$, for each $f \in F$ we get $0 \in F^\circ$.

Hence $F^\circ \neq \phi$.

(ii) Let $x, y \in F^\circ$. To prove $x \vee y \in F^\circ$.

As $x, y \in F^\circ$ we get

$x \wedge f_1 = 0$ for some $f_1 \in F$ and $x \wedge f_2 = 0$, for some $f_2 \in F$.

We get $x \wedge (f_1 \wedge f_2) = 0$ and $y \wedge (f_1 \wedge f_2) = 0$ for $(f_1 \wedge f_2) \in F$.

(Since F is a filter .$f_1, f_2 \in F$ imply $f_1 \wedge f_2 \in F$).

But as L is 0-distributive. We get $(x \vee y) \wedge (f_1 \wedge f_2) = 0$.

This shows that $x \vee y \in F^\circ$.

(iii) As $x \leq y$, $x \in L$ and $y \in F^\circ$.

We prove that $x \in F^\circ$.

As $y \in F^\circ$ implies that $y \wedge f = 0$ for some $f \in F$.

$x \leq y$ imply $x \wedge f \leq y \wedge f$.

Then $x \wedge f = 0$ for some $f \in F$.

It implies that $x \in F^\circ$.

From (i), (ii) and (iii) we get

F° is an ideal for any filter F in L.

♦♦♦

Result 1.2.4- In 0-distrtributive lattice, F° is an semi-prime ideal for any filter F in L[3].

Proof:- Let L be a 0-distributive lattice.

We have to prove F° is a semi-prime ideal for any filter F in L.

By Result 1.2.3, F° is an ideal for any filter F in L.

We have to prove F° is semi-primness.

Let $a \wedge b \in F°$ and $a \wedge c \in F°$ for a, b, c $\in$ L.

Then $a \wedge b \wedge x = 0$ and $a \wedge c \wedge y = 0$ for some x, y in L.

But then $a \wedge (x \wedge y) \wedge b = 0$ and $a \wedge (x \wedge y) \wedge c = 0$.

As L is 0-distributive we get $(a \wedge x \wedge y) \wedge (b \vee c) = 0$.

This shows that $a \wedge (b \vee c) \in F°$.

Hence F° is a semi-prime ideal in L.

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Result 1.2.5- L is a 0-distributive lattice iff every maximal filter in L is prime[6].

Proof:- Let L be 0-distributive lattice.

Let M be a maximal filter of L.

We have to prove that M is a prime filter.

Assume $x \vee y \in M$ with $x \notin M$ and $y \notin M$.

Then $0 \in M \vee [x)$ implies $x \wedge m_1 = 0$ for some $m_1 \in M$ and $y \notin M$ then $0 \in M \vee [y)$ implies $y \wedge m_2 = 0$ for some $m_2 \in M$.

But then $x \wedge (m_1 \wedge m_2) = 0$ and $y \wedge (m_1 \wedge m_2) = 0$.

Thus we get $(x \vee y) \wedge (m_1 \wedge m_2) = 0$ as L is 0-distributive.

As $m_1 \wedge m_2 \in M$ and $x \vee y \in M$ we get

$(x \vee y) \wedge (m_1 \wedge m_2) \in M$.

i.e. $0 \in M$; a contradiction.

Hence $x \vee y \in M$ implies $x \in M$ or $y \in M$.

Thus M is a prime filter.

Conversely, assume that every maximal filter in L is prime.

We have to prove L is a 0-distributive lattice.

As $x \wedge y = 0$ and $x \wedge z = 0$.

Let if possible $x \wedge (y \vee z) \neq 0$.

Consider $[x \wedge (y \vee z))$.

This is a proper filter of L.

By Result 0.2.1, there exists maximal filter M such that $x \wedge (y \vee z) \in M$.

By assumption , M is a prime filter.

$x \wedge (y \vee z) \in M$ implies $x \in M$ and $y \vee z \in M$. (since F is filter iff $x \wedge y \in F \Leftrightarrow x \in M$ and $y \in M$.

Then $x \in M$ and ($y \in M$ or $z \in M$). (since M is prime).

i.e.$x \in M$ and $y \in M$ or $x \in M$ and $z \in M$.

we get $x \wedge y \in M$ or $x \wedge z \in M$.

Thus $0 \in M$; a contradiction.

Therefore $x \wedge y = 0$, $x \wedge z = 0$ implies $x \wedge (y \vee z) = 0$.

Hence L is 0-distributive lattice.

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Result 1.2.6- L is a 0-distributive lattice iff $\{x^*\}$ is an ideal for any x in L.

Proof:- Only if part

Let L be a 0-distributive lattice.

We have to prove $\{x^*\}$ is an ideal for any x in L.

(i) As $0 \wedge x = 0$, for each $x \in L$ we get, $0 \in \{x^*\}$.

Hence $\{x^*\} \neq \phi$.

(ii) Let $y, z \in \{x^*\}$. To prove $y \vee z \in \{x^*\}$.

As $y, z \in \{x^*\}$ we get $x \wedge y = 0$ and $x \wedge z = 0$.

But as L is 0-distributive.

We get $x \wedge (y \vee z) = 0$.

This shows that $y \vee z \in \{x^*\}$.

(iii) As $y \leq z$, $y \in L$ and $z \in \{x^*\}$.

We prove that $y \in \{x^*\}$.

As $z \in \{x^*\}$ implies that $x \wedge z = 0$.

$y \le z$ imply $x \wedge y \le x \wedge z$.

Then $x \wedge y = 0$.

This implies that $y \in \{x^*\}$.

From (i), (ii) and (iii) we get $\{x^*\}$ is an ideal for any x in L.

If part

Let $\{x^*\}$ be an ideal for any x in L.

We have to prove L is 0-distributive lattice.

As $x \wedge y = 0$ and $x \wedge z = 0$.

This implies $y, z \in \{x^*\}$.

As $\{x^*\}$ is an ideal for any x in L we get $y \vee z \in \{x^*\}$.

Therefore $x \wedge (y \vee z) = 0$.

Hence L is 0-distributive lattice.

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