CHAPTER - I

AN EXPOSITION

OF

ABSTRACT IDEAL THEORY OF COMMUTATIVE RINGS.

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§. 1. INTRODUCTION

The theory of multiplicative lattices is classical and has been an inseparable part of any text book of lattice theory and universal algebra. In 1939, Ward and Dilworth [10], have initiated the abstract ideal theory of commutative rings and defined multiplicative lattice and since then, there has been a steady development of the theory of multiplicative lattices. For such multiplicative lattices, abstract analogues of the Noether Decomposition Theorems for commutative rings and various ideal theoretical results were formulated and proved by Ward and Dilworth [10] and Dilworth [7].

Let R be a commutative ring with unit element. On ideals I, J of R, defines the following operations :

i) $I \wedge J = I \cap J$

ii) $I \vee J = [I \cup J]$ where [I] denotes the ideal of R generated by I.

It is well known that I(R), the set of ideals of R, forms a complete modular lattice under the above two operations. On I(R), we define operation multiplication as follows ;

$IJ = \{ \Sigma_{finite}\, a_r b_r \;/\; a_r \in I,\ b_r \in J \}$ for $I, J \in I(R)$.

This product IJ of two ideals I and J of I(R), is the ideal generated by all the product ab with $a \in I$, $b \in J$.

One also notes that the multiplication distributes over join in the sense $I(J \vee K) = IJ \vee IK$. Where $I, J, K \in I(R)$. Moreover $IJ \subseteq I \cap J$.

Thus $I(R)$ with these binary operations becomes an algebraic structure called multiplicative lattices.

Let us now build up the necessary apparatus which needs to understand the core of this dissertation.

<u>**Definition**</u> <u>**1.1**</u> :

A multiplicative lattice L is a complete lattice provided with a commutative, associative and join - distributive multiplication for which the greatest element 1 acts as the identity for multiplication.

Indeed, explicitly, by a multiplicative lattice we mean a lattice $\langle L; \wedge, \vee \rangle$ with a binary operation of multiplication satisfying the conditions :

(i) $\langle L; \wedge, \vee \rangle$ is a complete lattice,

(ii) $ab = ba$; $a(bc) = (ab)c$;

$a(\vee_j b_j) = \vee_j (ab_j)$,

(iii) $a.1 = a$. where $a, b, c, b_j \in L$.

The product of two lattice element a and b in L is ab.

Using this concept, Ward and Dilworth [10] could abstract the notions of prime ideal, primary ideal, residuation etc. However after several years Dilworth [7] succeeded to introduce a weak concept of " Principal element" by obtaining satisfactory abstract version of Krull's Principal Ideal theorem. We shall discuss this briefly.

Definition 1.2 :

Let L be a multiplicative lattice. Let a and b be two elements of L. The residual of a by b is defined as the element $\bigvee \{ z \in L / zb \leq a \}$ and is denoted by $a : b$.

We note that whenever brackets are not introduced the residuation and multiplication operations are performed first and then the lattice operations $\wedge$ and $\vee$ are performed.

Definition 1.3 :

An element x of a multiplicative lattice L is meet principal if $(a \wedge b : x) x \geq ax \wedge b$ for all $a, b \in L$.

Definition 1.4 :

An element x of a multiplicative lattice L is join principal if $(a \vee bx) : x \leq a:x \vee b$ for all $a, b \in L$.

Definition 1.5 :

An element x of L is called principal if it is both meet and join principal.

The following is an example of principal element.

Example :

Let R be a commutative ring with unity. Consider L as the lattice of ideals of R. Let $X = (x)$ be a principal ideal of R. Denote by I(R), the set of ideals of R, let $A, B \in I(R)$. If $z \in AX \cap B$ then $z \in AX$ and $z \in B$ so that $z = ax$ for some $a \in A$. We known that

$$A : B = \{ r \in R / rB \subseteq A \}$$

Since $ax = z \in B$, where $x \in X$ implies that $a \in B : X$ and hence that $a \in A \cap B : X$. Thus $z = ax \in (A \cap B : X)$.

$$\Rightarrow AX \cap B \subseteq (A \cap B : X).$$

Accordingly, X is meet principal, And,

Let $\beta \in (A + BX) : X$. Then $\beta z \in (A + BX)$ for all $z \in X = (x)$. In particular $\beta x \in (A + BX)$ and hence $\beta x = a + bx$, where $a \in A$, $b \in B$.

[note that : + denotes the join of ideals].

Thus $\beta x = a + bx \Rightarrow a = (\beta - b)x$

$$\Rightarrow (\beta - b) \in A : X$$

and hence $\beta = (\beta - b) + b \in A : X + B$

$$\Rightarrow (A + BX) : X \subseteq (A : X) + B$$

Accordingly, X is join principal. Hence in lattice of ideal of R every principal ideal is principal element.

§. 2. PRELIMINARY CONCEPTS OF MULTIPLICATIVE LATTICES.

We begin with the concept of residuation in a multiplicative lattice L. All the nice properties of residuation are quite well known in the context of theory of multiplicative lattices.

Proposition 2.1 :

Let a, b and c be elements of L. $a \geq (a : b)\, b$.

Proof :

We known that $(a : b) = \bigvee \{ z \in L \,/\, zb \leq a \}$.

Let $x \in \{ z \in L \,/\, z\, b \leq a \} \Rightarrow xb \leq a$

$\Rightarrow [\bigvee \{ z \in L \,/\, zb \leq a \}]\, b \leq a$

$\Rightarrow (a : b)\, b \leq a$.

Therefore, $a \geq (a : b)\, b$. ∎

Proposition 2.2 :

$a \geq xb \Leftrightarrow a : b \geq x$.

Proof :

Let $x \in L$ such that $xb \leq a$.

Then $x \in \{ z \in L \,/\, zb \leq a \} \Rightarrow x \leq \bigvee \{ z \in L \,/\, zb \leq a \}$

$\Rightarrow x \leq a : b$.

Therefore, $a \geq xb$ implies $a : b \geq x$. The converse is obvious. ∎

Proposition 2.3 :

$a \geq b$ if and only if $a : b = 1$.

Proof :

First, suppose $a \geq b$ Then $xb \leq b \leq a$, for all $x \in L$

$\Rightarrow x \in \{ z \in L \,/\, zb \leq a \}$. for all $x \in L$.

i.e. $\{ x \,/\, x \in L \} \subseteq \{ z \in L \,/\, zb \leq a \}$.

$\Rightarrow \bigvee \{ x \,/\, x \in L \} \leq \bigvee \{ z \in L \,/\, zb \leq a \}$.

Therefore we have $1 \le a : b$. But $a : b \le 1$ implies $a : b = 1$. Conversely, suppose $a : b = 1$. To prove $b \le a$ By proposition (2.1) we have $a \ge (a : b) b = 1.b \Rightarrow a \ge b$. ∎

Proposition 2.4 :

$$(a \wedge b) : c = (a : c) \wedge (b : c).$$

Proof :

We known that $(a \wedge b) : c = \bigvee \{ z \in L / zc \le a \wedge b \}$.

Let $x \in \{ z \in L / zc \le a \wedge b \} \Rightarrow xc \le a \wedge b$

$\Rightarrow xc \le a$ and $xc \le b$

$\Rightarrow x \le (a : c) \wedge (b : c)$.

Therefore, $\bigvee \{ x \in L / xc \le a \wedge b \} \le (a : c) \wedge (b : c)$.

$\Rightarrow (a \wedge b) : c \le (a : c) \wedge (b : c)$.

Conversely,

Suppose $x \in \{ z \in L / zc \le a \} \cap \{ z \in L / zc \le b \}$

$\Rightarrow xc \le a$ and $xc \le b$

$\Rightarrow xc \le a \wedge b$

$\Rightarrow x \le (a \wedge b) : c$

$\Rightarrow \bigvee \{ x \in L / xc \le a \} \le a \wedge b : c$ and

$\bigvee \{ x \in L / xc \in b \} \le (a \wedge b) : c$.

$\Rightarrow (a : c) \le (a \wedge b) : c$ and

$(b : c) \le (a \wedge b) : c$

$\Rightarrow (a : c) \wedge (b : c) \le (a \wedge b) : c$

Therefore $(a \wedge b) : c = (a : c) \wedge (b : c)$. ∎

Proposition 2.5 :

$$a : (bc) = (a : b) : c$$

Proof :

We know that $a : (bc) = \bigvee \{ z \in L / z (bc) \leq a \}$.

Let $x \leq (a : bc)$. Then $x \leq \bigvee \{ z \in L / z (bc) \leq a \}$

$\iff xbc \leq a$ (by proposition 2.2)

$\iff xc \leq a : b$

$\iff x \leq (a : b) : c$.

This shows that $(a : bc) \leq (a : b) : c$. ∎

Proposition 2.6 :

$$a \leq a : b.$$

Proof :

Let $x \leq a$. Then $xb \leq a$ and hence we have $x \leq \bigvee \{ z \in L / zb \leq a \} = a : b$

Therefore we have $a \leq a : b$. ∎

Proposition 2.7 :

$$a : 1 = a$$

Proof :

We know that $a : 1 = \bigvee \{ z \in L / z1 \leq a \text{ i.e. } z \leq a \}$

Since $a = \bigvee \{ z \in L / z \leq a \}$, obviously $a : 1 = a$. ∎

Proposition 2.8 :

$$a \leq (ab) : b$$

Proof :

We know that $(ab) : b = \bigvee \{ z \in L / zb \leq ab \}$

Let $x \leq a$. Then $xb \leq ab$ and hence $x \in \{ z \in L / zb \leq ab \}$.

Thus we have $x \leq \bigvee \{ z \in L \ / \ zb \leq ab \} = (ab) : b$.

Therefore $a \leq (ab) : b$. ∎

Proposition 2.9 :

$$(a : c) \vee (b : c) \leq (a \vee b) : c$$

Proof :

Let $x \in \{ z \in L \ / \ zc \leq a \} \cup \{ z \in L \ / \ zc \leq b \}$

$\Rightarrow xc \leq a$ or $xc \leq b$

$\Rightarrow xc \leq a \vee b$ and hence $x \leq (a \vee b) : c$

$\Rightarrow [\bigvee \{ z \in L \ / \ zc \leq a \}] \vee [\bigvee \{ z \in L \ / \ zc \leq b \}]$

$\leq (a \vee b) : c$

i.e. $(a : c) \vee (b : c) \leq (a \vee b) : c$. ∎

Proposition 2.10 :

$$(a \wedge b) c \leq (ac) \wedge (bc).$$

Proof :

Let $x \leq (a \wedge b) c$. Then $x \leq ac$ and $x \leq bc$

$\Rightarrow x \leq (ac) \wedge (bc)$.

Therefore $(a \wedge b) c \leq (ac) \wedge (bc)$. ∎

Proposition 2.11 :

$$(a \wedge b) : b = a : b.$$

Proof :

Let $x \in \{ z \ / \ zb \leq a \wedge b \}$. Then $xb \leq a \wedge b \leq a$.

$\Rightarrow x \leq a : b$ i.e. $x \in \{ z \in L \ / \ zb \leq a \}$.

$\Rightarrow \bigvee \{ z \in L \ / \ zb \leq a \wedge b \} \leq \bigvee \{ z \in L \ / \ zb \leq a \}$.

Hence $(a \wedge b) : b \leq a : b$.

For the reserve inequality, let $x \in \{ z \in L / zb \le a \}$. Then $xb \le a$. Also in L, $xb \le b$ and hence $xb \le a \wedge b$ implies $x \in \{ z \in L / zb \le a \wedge b \}$

$\Rightarrow \bigvee \{ z \in L / zb \le a \} \le \bigvee \{ z \in L / zb \le a \wedge b \}$

$\Rightarrow a : b \le (a \wedge b) : b$.

Therefore $(a \wedge b) : b = a : b$. ∎

Proposition 2.12 :

$$a : (a \vee b) = a : b.$$

Proof :

Let $x \in \{ z \in L / z (a \vee b) \le a \}$

$\Rightarrow x (a \vee b) \le a$

$\Rightarrow xa \vee xb \le a$

$\Rightarrow xb \le a$ and hence $x \in \{ z \in L / zb \le a \}$.

$\Rightarrow a : (a \vee b) \le a : b$.

For the reverse inequality, let $x \in \{ z \in L / zb \le a \}$. Then $xb \le a$. As in L, $xa \le a$ we have $xa \vee xb \le a$ i.e $x (a \vee b) \le a$. This gives $x \in \{ z \in L / z (a \vee b) \le a \}$ and we have $a : b \le a : (a \vee b)$. Therefore $a : (a \vee b) = a : b$. ∎

Proposition 2.13 :

$$a : (b \vee c) = (a : b) \wedge (a : c).$$

Proof :

Take $x \in \{ z \in L / z (b \vee c) \le a \}$. This gives $x (b \vee c) \le a$, which again implies $xb \le a$ and $xc \le a$. Hence $x \in \{ z \in L / zb \le a \}$ and $x \in \{ z \in L / zc \le a \}$ leading to $x \le (a : b) \wedge (a : c)$. Therefore $a : (b \vee c) \le (a : b) \wedge (a : c)$.

For the reverse inequality, let $x \in \{ z \in L / zb \leq a \}$ and $x \in \{ z \in L / zc \leq a \}$. Then $xb \leq a$ and $xc \leq a$. This gives $xb \vee xc \leq a$ i.e. $x (b \vee c) \leq a$ which again implies $x \in \{ z \in L / z (b \vee c) \leq a \}$ and we have

$(a : b) \wedge (a : c) \leq a : (b \vee c)$.

Therefore $a : (b \vee c) = (a : b) \wedge (a : c)$. ∎

By straight forward arguments from elementary concept in multiplicative lattice L we report the following important properties.

Proposition 2.14 :

If $a \vee c = b \vee c = 1$ then $ab \vee c = 1$.

Proof :

Suppose $a \vee c = b \vee c = 1$. Since $b \vee c = 1$, we have $a (b \vee c) = a 1 = a$ i.e. $a (b \vee c) = a$.

Also $a \vee c \leq a (b \vee c)$ and $a \vee c = 1$ establishes $1 \leq a (b \vee c)$. Thus $1 \leq a (b \vee c)$ implies $a = 1$.

Now $a (b \vee c) = ab \vee ac \Rightarrow 1 = ab \vee 1 c$.

Therefore we have $ab \vee c = 1$. ∎

Proposition 2.15 :

If $a \vee c = 1$ then $(a \wedge b) \vee c = b \vee c$.

Proof :

Suppose $a \vee c = 1$. Clearly $(a \wedge b) \vee c \leq b \vee c$.

For reverse inequality, let $x \leq b \vee c$.

Then since $x \leq b \vee c$ and $x \leq 1 = a \vee c$, it follows that $x \leq (b \vee c) (a \vee c)$ which gives the inequality

$x \leq ab \vee ac \vee cb \vee c$, $\Rightarrow b \vee c \leq (a \wedge b) \vee c$.

Therefore $(a \wedge b) \vee c = b \vee c$. ∎

We note the following result.

Proposition 2.16 :

$(a_1 \vee a_2 \vee \ldots \vee a_n)^{k1+k2+\ldots+kn} \leq a_1^{k1} \vee a_2^{k2} \vee \ldots \vee a_n^{kn}$. Where $a_i \in L$ and ki are integers. $(1 \leq i \leq n)$.

The concept of associated primes are very important in the theory of decompositions for lattices, especially in the investigations of primary decompositions as stated in Dilworth [7]. We recall the definition of prime element and primary element.

Definition 2.17 :

Let L be a multiplicative lattice satisfying the ascending chain condition. An element $p \in L$ is called prime if $ab \leq p \Rightarrow a \leq p$ or $b \leq p$ for all $a, b \in L$.

One notes that, the element 1 is a prime element and "prime" will normally refer to prime element other than 1. The prime ideal of I(R) is a prime element.

Definition 2.18 :

Let L be a multiplicative lattice satisfying the ascending chain condition. An element $q \in L$ is called primary if $ab \leq q \Rightarrow a \leq q$ or $b^k \leq q$ for some positive integer k.

It can be readily seen that the above definitions are used for establishing many fruitful results relating prime and primary ideals in a commutative ring to multiplicative lattices. We shall assume that the given multiplicative lattice L always satisfies the ascending chain condition.

<u>Definition</u> <u>2.19</u> :

If q is a primary element of L then $\bigvee \{ x \in L \,/\, x^s \leq q,$ for some integer s $\}$ is a minimal prime containing q and is called the <u>prime element associated with q</u>, which is denoted by p_q or $\sqrt{q}$.

We note the simple properties without proof of p_q, prime associated with q given by Dilworth [7].

(2.19:1) $p_q^k \leq q \leq p_q$ for some integer k.

(2.19:2) $ab \leq q \Rightarrow a \leq q$ or $b \leq p_q$.

<u>Remark</u> <u>(2.20)</u> : Let q be primary element in L. Then $a \nleq p_q$ implies $q : a = q$.

Proof :

Always $q : a \geq q$ (by proposition 2.6).

Let $x \leq (q : a)$. Then $xa \leq q$, $a \nleq p_q = \sqrt{q}$ and q is primary imply that $x \leq q$.

Hence $q : a \leq q$. Therefore $q : a = q$. ∎

<u>Remark</u> <u>(2.21)</u> :

In L, the meet of primary elements with the associated prime p is also primary element with the same associated prime element p.

§. 3. NORMAL PRIMARY DECOMPOSITION OF MULTIPLICATIVE LATTICES

In 1956, BEHRENS [4] gave the necessary and sufficient condition for a non associative ring to have a Noetherian ideal theory. KURATA [13] has continued such study of ideal theory. This theory has been strengthened by LESIEUR [14] and McCARTHY [16] in their study of primary decomposition in multiplicative lattices. In [2], the concepts of ideals are abstracted to multiplicative lattice and they investigate adequate and fruitful results to obtain a necessary condition in the context of primary decomposition of an element of Lattice L. For this, we recall the definition of irreducible element (see,[9]).

Definition 3.1 :

Let L be a lattice. An element $q \in L$ is called meet irreducible if $q = x \wedge y$ implies $q = x$ or $q = y$ for $x, y \in L$.

Definition 3.2 :

In L, a representation $a = q_1 \wedge q_2 \wedge \ldots \wedge q_n$ (where $q_1, q_2, \ldots, q_n$ are irreducible) is called a finite decomposition of q.

According to Dilworth [8], we report the concept of normal primary decomposition. We say that an element a has a primary decomposition if there exist primary element q_1, $q_2, \ldots, q_n$ such that $q = q_1 \wedge q_2 \wedge \ldots \wedge q_n$.

By deleting superfluous primary elements q_i and combining the primaries associated with the same prime, the original primary decomposition can be refined to primary decomposition in which district primary elements are associated with district prime elements. Such a primary decomposition is called a normal primary decomposition.

The fundamental theorem on normal primary decomposition states that :

" In a lattice L, any two normal decompositions of an element a have the same number of components and the same set of associated primes. "

To establish the proof of some fundamental results on isolated component, first, we need few facts concerning " isolated component of a ".

Definition 3.3 :

Let $a = q_1 \wedge q_2 \wedge \ldots \wedge q_m$ be a normal decomposition of a, and $p_1, p_2, \ldots, p_n$ denote the associated prime elements of a. A subset C of $\{p_1, p_2, \ldots, p_n\}$ is said to be isolated set if it satisfies the condition,

$p_i \in C \Rightarrow p_j \in C$ whenever $p_j \leq p_i$.

Definition 3.4 :

Let C be an isolated set of associated primes of a. An element $a_C = \wedge \{ q_i \mid p_i \in C \}$ is called an isolated component of a.

In light of this we have the theorem, in which the relationship between isolated component of a and normal primary decomposition is discussed in a lattice.

Theorem : 3.5 :

The isolated component a_C of a is dependent on a and c but not on any particular normal decomposition of a.

Proof :

Let $a = q_1 \wedge q_2 \wedge \ldots \wedge q_n$ be a normal decomposition of a and suppose $p_1, p_2, \ldots, p_n$ are the associated primes of a (where $p_i = \sqrt{q_i}$.)

Let $a = q_1' \wedge q_2' \wedge \ldots \wedge q_n'$ be second normal decomposition. Also $\{ p_1, p_2, \ldots, p_n \}$ is the set of associated primes of a with respect to second primary decomposiion

$a = q_1' \wedge q_2' \wedge \ldots \wedge q_n'$.

Let $a_C' = \wedge \{ q_i' \, / \, p_i \in C \}$.

Take $b' = \wedge \{ q_j' \, / \, p_j \notin C \}$.

Then $q_i \geq a = a_C' \wedge b \geq a_C' b'$ ....(3.5.1)

If $p_i \in C$ then $b' \nleq p_i$ because $b' \leq p_i \Rightarrow p_j^k \leq q_j' \leq p_i$ and hence $p_j \leq p_i$ since p_i is prime. This implies $p_j \in C$ which is a contradiction to $p_j \notin C$.

Thus, if $p_i \in C$ then $b' \nleq p_i$

Also from (3.5.1) $a_C' \, b' \leq q_i$ and $b' \leq p_i$.

(where q_i is primary and $p_i \in C$) implies that $a_C' \leq q_i$ for all i such that $p_i \in C$.

Therefore $a_C' \leq \wedge \{ q_i \, / \, p_i \in C \} = a_C$.

Similarly $a_C \leq a_C'$ and hence $a_C = a_C'$.

This complete the proof. ∎

§. 4. NOETHER LATTICES

INTRODUCTION :

Noether lattices were introduced by Dilworth R.P. [8]. Noether lattices constitute a natural abstraction of the lattice of ideals of a Noetherian commutative ring.

In [10], Ward and Dilworth extended the Noether decomposition theory to suitable defined multiplicative lattices. In [7] Dilworth defined a principal element and extended the Krull Intersection Theorem and Principal Ideal Theorem to what they called a Noether lattice. The theory of Noether lattices is also developed in papers [2], [6], [12].

Let us report the concept of Noether lattice.

Definition 4.1 : (Noether Lattice)

" A Noether lattice is modular, multiplicative lattice satisfying the ascending chain condition in which every element is the join of principal elements. "

It is known that every element of a lattice satisfying the ascending chain condition has a decomposition into meet irreducible elements. In some multiplicative lattices satisfying ascending chain condition, elements do not have primary decomposition (see ward and Dilworth [10]). A meet irreducible element can have a primary decomposition only if it is itself primary, so the elements of L will have the primary decomposition if and only if every meet irreducible element is primary.

Several properties of principal element play important role in the theory of Noether lattice. For that we report the following lemmas :

Lemma 4.2 :

An element m is meet principal if and only if $(a \wedge b : m)\, m = am \wedge b$ for all a, b $\in$ L.

Proof :

Suppose m $\in$ L is meet principal element. Then by definition 1.4, we have $(a \wedge b : m)\, m \geq am \wedge b$, for all a, b $\in$ L. By proposition 2.10 and proposition 2.1, it follows that $(a \wedge b : m)\, m \leq am \wedge (b : m)\, m \leq am \wedge b$ and therefore $(a \wedge b : m)\, m = am \wedge b$, for all a, b $\in$ L. Clearly, the conversion is obvious. ∎

Lemma 4.3 :

An element m is join principal if and only if $(a \vee bm) : m = a : m \vee b$ for all a, b $\in$ L.

Proof :

Suppose m $\in$ L is join principal element. Then by definition 1.5, we have $(a \vee bm) : m \leq a : m \vee b$ for all a, b $\in$ L. By proposition 2.9 and 2.8, it follows that $(a \vee bm) : m \geq a : m \vee (bm) : m \geq a : m \vee b$. Therefore $(a \vee bm) : m = a : m \vee b$, for all a, b $\in$ L. The converse is obvious. ∎

Lemma 4.4 :

If m is meet principal element of L then

$(b : m) m = (b \wedge m)$ for all $b \in L$.

Proof :

Since m is meet principal element of L by Lemma 4.2 it follows that $(a \wedge b : m) m = am \wedge b$ for all a, b in L. Putting a = 1, we have $(b : m) m = m \wedge b = b \wedge m$ for all b in L. ∎

Lemma 4.5 :

If m is join principal element of L then

$(bm) : m = b \vee (0 : m)$ for all b in L.

Proof :

Since m is join principal element of L, by Lemma 4.3, it follows that $(a \vee bm) : m = a : m \vee b$ for ll a, b in L. Putting a = 0, we have $(bm) : m = (0 : m) \vee b$

$= b \vee (0 : m)$ for all b in L. ∎

By glueing together Lemma 4.4 and Lemma 4.5, we have an easy consequences as given below.

Remark 4.6: If m is meet principal then

$b \leq m \Rightarrow (b : m) m = b$.

Proof :

Since m is meet principal, by Lemma 4.4, we have

$(b : m) m = b \wedge m$ for all $b \in L$.

If $b \leq m$, $(b : m) m = b \wedge m = b$ ∎

Remark 4.7 : If m is join principal then

$(a : m) \leq b \Rightarrow (bm : m) = b$.

Proof : Since m is join principal, by Lemma 4.5, we have

$(bm) : m = b \vee (0 : m)$ for all $b \in L$.

Let $(0 : m) \leq b$. Then $(bm) : m = b$. ∎

Lemma 4.8 :

The product $m_1 m_2$ of meet principal elements m_1 and m_2 is also meet principal.

Proof :

Using proposition 2.5 we have

$[a \wedge b : (m_1 m_2)] m_1 m_2$

$= ([a \wedge (b : m_2) : m_1] m_1) m_2$.

By using Lemma 4.2 and for that setting $b : m_2$ as b and m as m_1 we conclude that $([a \wedge (b : m_2) : m_1] m_1) m_2$

$= [am_1 \wedge (b : m_2)] m_2$. Again by using Lemma 4.2 this yields

$[a \wedge b : (m_1 m_2)] m_1 m_2 = am_1 m_2 \wedge b$ for all a, b in L.

Therefore $m_1 m_2$ is meet principal. ∎

Lemma 4.9 :

The product $m_1 m_2$ of join principal elements m_1 and m_2 is also join principal.

Proof :

By proposition 2.5, we have $(a : bc) = (a : b) : c$. Using this, it follows that

$(a \vee b m_1 m_2) : m_1 m_2 = [(a \vee bm_1 m_2) : m_2] : m_1$.

By using Lemma 4.3, we conclude that

$(a \vee bm_1 m_2) : m_1 m_2 = (a : m_2 \vee bm_1) : m_1$. Again by using Lemma 4.3, we have $(a : m_2 \vee bm_1) : m_1 = (a : m_2) : m_1 \vee b$. And hence by applying proposition 2.5 this yields

$(a \vee bm_1 m_2) : m_1 m_2 = (a : m_1 m_2) \vee b$ for all a, b in L.

Therefore $m_1 m_2$ is join principal. ∎

We shall see that the above Lemma 4.8 and Lemma 4.9 lead us to a pleasant result as follows :

Remark 4.10 : If m_1 and m_2 are principal then m_1m_2 is also principal.

For a modular, multiplicative lattice L satisfying the ascending chain condition we discuss the suitable condition which insures that the meet irreducible element is primary. First, we recall the definition of modular lattice.

Definition 4.11 :

In a lattice L, a pair of elements a, b is called modular when the following condition holds :

$(c \vee a) \wedge b = c \vee (a \wedge b)$ for every $c \leq b$.

A modular pair (a, b) is denoted by (a, b)M.

A lattice L is called modular if and only if (a, b)M for all a, b $\in$ L.

These concepts are very important in the lattice theory especially in the investigations of symmetric lattices as reported in Maeda and Maeda [1970], (see [15]).

Theorem 4.12 :

Let L be a modular, multiplicative lattice satisfying the ascending chain condition. If every element of L is a join of meet principal elements then every meet irreducible element of L is primary.

Proof :

Let q be a meet irreducible element of L suppose $am \leq q$ when $a \not\leq q$ and m is meet principal.

By using proposition (2.13), we have

$(a \vee q) : m \leq (a \vee q) : m^2 \leq \ldots \leq (a \vee q) : m^k \leq \ldots$.

As the ascending chain condition holds in L, there exists k such that $(a \vee q) : m^k = (a \vee q) : m^{k+1} = \ldots$ (4.12.1)

Let $c = (a \vee q) \wedge (m^{k+1} \vee q)$. Then

$c : m^{k+1} = [(a \vee q) \wedge (m^{k+1} \vee q)] : m^{k+1}$

By using proposition 2.4, it follows that

$c : m^{k+1} = [(a \vee q) : m^{k+1}] \wedge [(m^{k+1} \vee q) : m^{k+1}]$

We use the proposition 2.3 i.e. $b \leq a$ if and only if $a : b = 1$.

and therefore $m^{k+1} \leq (m^{k+1} \vee q) \Rightarrow (m^{k+1} \vee q) : m^{k+1} = 1$

Hence $c : m^{k+1} = [(a \vee q) : m^{k+1}] \wedge 1$ implies

$c : m^{k+1} = (a \vee q) : m^{k+1}$. By using 4.12.1 we have

$c : m^{k+1} = (a \vee q) : m^k$ ....(4.12.2)

Also by Lemma 4.8 it follows that m^k and m^{k+1} are meet principal elements and hence by using Lemma 4.4 we conclude that

$(c : m^{k+1}) m^{k+1} = c \wedge m^{k+1}$ and

$[(a \vee q) : m^k] m^k = (a \vee q) \wedge m^k$ ...(4.12.3)

Also $q \leq c$ because we have $q \leq a \vee q$ and $q \leq m^{k+1} \vee q$ implies $q \leq (a \vee q) \wedge (m^{k+1} \vee q) = c$.

By the modularity condition of L, we have

$(c \wedge m^{k+1}) \vee q = c \wedge (m^{k+1} \vee q)$ as $c \geq q$.

Also we have $c \wedge (m^{k+1} \vee q) \leq c$ and

$c = (a \vee q) \wedge (m^{k+1} \vee q)$ implies that $c \leq m^{k+1} \vee q$.

Therefore $c = c \wedge (m^{k+1} \vee q) = (c \wedge m^{k+1}) \vee q$.

By using Lemma 4.2, it follows that $c = q \vee (c : m^{k+1}) m^{k+1}$

$\Rightarrow c = q \vee [(a \vee q) : m^k] m^{k+1}$ (by using 4.12.2).

Hence $c = q \vee [(a \vee q) \wedge m^k] m$ (by using 4.12.3)

Therefore $c \leq q \vee (a \vee q) m = q \vee am \vee qm \leq q \leq c$.

This shows $q = c = (a \vee q) \wedge (m^{k+1} \vee q)$.

Since q is meet irreducible element and $a \nleq q$ implies $q \neq a \vee q$, it follows that $q = m^{k+1} \vee q$, which gives $m^{k+1} \leq q$. Thus we have shown that if q is meet irreducible element and m is meet principal then $am \leq q$, where $a \nleq q$ implies $m^{k+1} \leq q$ for some positive integer k. Let $ab \leq q$ where $a \nleq q$. As every element of L is a join of meet principal elements, we have

$b = m_1 \vee m_2 \vee \ldots \vee m_r$, where m_i is meet principal

$(1 \leq i \leq r)$. Since $ab \leq q$, we have

$a(m_1 \vee m_2 \vee \ldots \vee m_r) \leq q \Rightarrow am_i \leq q$ for each $i = 1,2,\ldots r$

But $a \nleq q$, so there exists k_i such that $m_i^{ki} \leq q$.

Let $k = k1 + k2 + \ldots + kr$. By using proposition 2.16.

$$b^k = (m_1 \vee m_2 \vee \ldots \vee m_r)^k$$
$$= (m_1 \vee m_2 \vee \ldots \vee m_r)^{k1+k2+\ldots+kr}$$
$$\leq m_1^{k1} \vee m_2^{k2} \vee \ldots \vee m_r^{kr} \leq q.$$

Therefore $ab \leq q$ and $a \nleq q$ implies $b^k \leq q$ for some positive integer k. This conclude that every meet irreducible element is primary. ∎

This helps us to understand the following conclusion.

Remark 4.13 : Every meet irreducible element of a Noether lattice is primary.

Using the propositions of § 2, we can state a nice corollary that :

Every element of a Noether lattice has a normal primary decomposition.

The next result is useful for abstract version of inter section theorem.

Lemma 4.14 :

Let L be a Noether lattice.

Let $a = q_1 \wedge q_2 \wedge \ldots \wedge q_n$ be a normal decomposition a and let $\{ p_1, p_2, \ldots, p_n \}$ be the set of associated primes. If b is an arbitrary element of L then the set $C = \{ p_i \mid p_i \vee b \neq 1 \}$ is isolated set of primes.

proof :

Suppose $p_i \in C = \{ p_i \mid p_i \vee b \neq 1 \}$ and $p_j \leq p_i$. Then $p_i \vee b \neq 1 \Rightarrow p_j \vee b \neq 1$. Otherwise, $p_j \leq p_i$ $\Rightarrow p_j \vee b \leq p_i \vee b$. And hence $p_i \vee b \neq 1$ implies $p_j \in C$ when $p_i \geq p_j$.

Therefore C is an isolated set of primes. ∎

If $b \in L$ and $C = \{ p_i \mid p_i \vee b \neq 1 \}$ is isolates set of primes of a, we denote the corresponding isolated component of a by symbol a_b. So that $a_b = \wedge \{ q_i \mid p_i \in C \}$

$= \wedge \{ q_i \mid p_i \vee b \neq 1 \}$

where C is isolated set of prime. We now prove an abstract form of intersection theorem.

Theorem 4.15 : [Intersection Theorem]

Let L be a Noether lattice and $a, b \in L$.

Then $\wedge_k (a \vee b^k) = a_b$.

Proof :

Let $a = q_1 \wedge q_2 \wedge \ldots \wedge q_n$ be any normal decomposition of a. Let $p_1, p_2, \ldots, p_n$ denote the set of associated primes of a.

If $b \in L$ then $\{ p_i / p_i \vee b \neq 1 \} = C$, by Lemma 4.13 is clearly an isolated set of primes.

Then $a_b = \wedge \{ q_i / p_i \in C \} = \wedge \{ q_i / p_i \vee b \neq 1 \}$.

Is an isolated component of a.

Let $a_b^* = \wedge \{ q_j / p_j \vee b = 1 \}$.

Now $p_j \vee b = 1 \Rightarrow p_j^k \vee b = 1$ for all integers k. As p_j is associated prime of q_j, there exists a positive integer k such that $p_j^k \le q_j \le p_j$. Hence $p_j^k \vee b = 1 \le q_j \vee b$ $\Rightarrow q_j \vee b = 1$. But then $a_b^* = \wedge \{ q_j / p_j \vee b = 1 \}$

$$\ge \pi \{ q_j / p_i \vee b = 1 \}.$$

Hence $q_j \vee b = 1$ for all q_j such that $p_j \vee b = 1$ implies by (proposition 2.14) that $[\pi \{ q_j / p_j \vee b = 1 \}] \vee b = 1$. This shows $a_b^* \vee b = 1$. By using proposition 2.14, we have $a_b^* \vee b^k = 1$ for all k. Therefore $a \vee b^k = (a_b \wedge a_b^*) \vee b^k = a_b \vee b^k$ for all k, by (proposition 2.15) and hence $a \vee b^k \ge a_b$ for all k and it implies that

$$a_b \le \wedge_k (a \vee b^k) \qquad .....(4.14.1)$$

For the reverse inequality, let $m \le \wedge_k (a \vee b^k)$ where m is principal elements. Let $a \vee bm = r_1 \wedge r_2 \wedge ... \wedge r_t$ be a normal decomposition of $a \vee bm$ with associated primes p_1', $p_2', ..., p_t'$. Then $a \vee bm \le r_1, r_2, ..., r_t$ ....(4.14.2)

Hence $bm \le r_i$ for all i = 1, 2,..., t where r_i is primary.

$\Rightarrow m \le r_i$ or $b \le p_i'$ (by using definitions 2.17 and 2.18).

If $b \le p_i'$ then $b^{ki} \le p_i'^{ki} \le ri$ for some integer ki ...(4.14.3)

Now $a \le r_i$ and $b^{ki} \le r_i$ and by using (4.14.2) and (4.14.3), it follows that $m \le \wedge_k (a \vee b^k) \le a \vee b^{ki} \le r_i$.

In either case, we get $m \le r_i$ and hence

$a \vee bm = r_1 \wedge r_2 \wedge ... \wedge r_t \ge m$ i.e $m \le a \vee bm$.

Also $a \leq a \vee bm$, $m \leq a \vee bm$ imply $a \vee m \leq a \vee bm$. Clearly, $a \vee bm \leq a \vee m$ and hence $a \vee bm = a \vee m$. Since m is principal element by lemma 4.3 we have

$a : m \vee b = (a \vee bm) : m \Rightarrow a : m \vee b = (a \vee m) : m = 1$

[because $a \vee bm = a \vee m$ and $m \leq a \vee b \Rightarrow (a \vee b): m = 1$]

Now $p_i \vee b \neq 1$ implies $a : m \nleq P_i$. Otherwise

$a : m \leq P_i \Rightarrow (a : m) \vee b = 1 \leq p_i \vee b$ which gives $p_i \vee b = 1$, a contradiction. Since $(a : m)m \leq a \leq q_i$ with $a:m \nleq P_i$ it follows that $m \leq q_i$ for which $p_i \vee b \neq 1$. This yields that $m \leq \wedge \{ q_i / p_i \vee b \neq 1 \}$.

i.e. $m \leq a_b$. This conclude if $m \leq \wedge_k (a \vee b^k)$ then $m \leq a_b$. Since L is a Noether lattice, the element $\wedge_k (a \vee b^k)$ is a join of principal elements and hence $\wedge_k (a \vee b_k) = \vee m_i$ (where m_i is principal). But m is principal and $m \leq \wedge_k (a \vee b^k) \Rightarrow m \leq a_b$. This shows that $\wedge_k (a \vee b^k) \leq a_b$. Hence $a_b = \wedge_k (a \vee b^k)$. ∎

This paves the way for a simple and sharp formulation as follows :

Corollary 4.16 :

For an element b of a Noether lattice L, $0_b = \wedge_k b^k$.

Proof :

In the above theorem 4.15 take $a = 0$. We have $0_b = \wedge_k (0 \vee b^k) = \wedge_k b^k$. ∎

Remark 4.17 : If $a \neq 1$ is an element of a Noether lattice L then there exists a maximal element $p \neq 1$ such that $a \leq p$. By Definition 2.17 and proposition 2.14, it follows that such p is prime element, moreover p is a maximal prime element of L.

Let us recall the definition of local Noether lattice, which will help us in discussing a crucial properties.

Definition 4.18 :

A Noether lattice L is called local if it contains precisely one maximal prime.

Properties 4.19 :

(1) If L is a local Noether lattice and $b \neq 1$ is an element of L then $\bigwedge_k (a \vee b^k) = a$ for each $a \in L$.

(2) If L is a local Noether lattice and $b \neq 1$ is an element of L then $\bigwedge_k b^k = 0$.

Proof (1) :

Let L be a local Noether lattice by definition 4.18, it follows that L contains maximal prime p_o. If $p_i \neq 1$ and $b \neq 1$ then $p_o \geq p_i \vee b$ and hence $p_i \vee b \neq 1$. Thus $a_b = \bigwedge \{ q_i \ / \ p_i \vee b \neq 1 \} = a$. But $a_b = \bigwedge_k (a \vee b^k)$ implies that $a = \bigwedge_k (a \vee b^k)$. ∎

Proof (2) :

In above Property 4.19 (1) putting $a = 0$. We get $\bigwedge_k b^k = 0$. ∎

§. 5. QUOTIENT LATTICES

The concept of quotient lattice is well stretched in lattice theory. It has been discussed in the context of abstract commutative ideal theory by Dilworth [7]. To study such concepts, we first define quotient lattice.

Let L be a Noether lattice and $d \in L$ be any element. Then clearly the set $L_{/d} = \{ a \in L \,/\, a \geq d \}$ is a sublattice of L. Define an operation $\star$ on $L_{/d}$ as follows :

For all $a, b \in L_{/d}$, $\quad a \star b = ab \vee d$.

With respect to this operation $\star$, the lattice $L_{/d}$ is multiplicative and 1 is the largest element of $L_{/d}$. Such a multiplicative lattice $L_{/d}$ is called as quotient lattice.

As an immediate consequence of the above definition, we have the following remarks.

Remark 5.1 : The quotient lattice $L_{/d}$ is closed with respect to residuation and this residuation is also the residuation associated with $a \star b$, since $a \geq x \star b$ if and only if $a \geq xb$.

Lemma 5.2 :

The quotient lattice $L_{/d}$ is a Noether lattice, if L is Noether. Also $m \vee d$ is a principal element of $L_{/d}$ whenever m is principal element of L.

Proof :

Suppose L is a Noether lattice. Since L is a modular lattice and satisfies the ascending chain condition, then obviously it follows that $L_{/d}$ is also modular and satisfies the ascending chain condition.

Next, let m be a principal element of L. Let a, b $\in L_{/d}$.
Then $[a \vee b \circ (m \vee d)] : (m \vee d) = [a \vee b(m \vee d) \vee d] : (m \vee d)$

$$= (a \vee bm \vee bd \vee d) : (m \vee d)$$

$$= (a \vee bm \vee d) : (m \vee d)$$

$$= [(a \vee bm \vee d) : m] \wedge [(a \vee bm \vee d) : d]$$

(by proposition 2.13)

$$= [(a \vee bm \vee d) : m] \wedge 1 \qquad \text{(by proposition 2.3)}$$

$$= (a \vee bm \vee d) : m$$

$$= (a \vee bm) : m \qquad (\text{since } a \in L / d \Rightarrow a \geq d)$$

$= a : m \vee b$. Hence, we have

$$[a \vee b \circ (m \vee d)] : (m \vee d) = a : m \vee b$$

$= a : (m \vee d) \vee b$ for all a, b $\in L_{/d}$. This shows the element $m \vee d$ is a join principal element of $L_{/d}$.

On the other hand, to show $m \vee d$ is a meet principal element of $L_{/d}$, we have to show,

$$[a \wedge b : (m \vee d)] \circ (m \vee d) = a \circ (m \vee d) \wedge b.$$

Now $[a \wedge b : (m \vee d)] \circ (m \vee d)$

$$= [a \wedge (b : m) \wedge (b : d)] \circ (m \vee d).$$

Also by using Proposition 2.3, it follows that $b \geq d \Rightarrow b:d = 1$. and hence we have, $[a \wedge b : (m \vee d)] \circ (m \vee d)$

$$= [a \wedge (b : m) \wedge 1] \circ (m \vee d)$$

$= [a \wedge (b : m)] \circ (m \vee d)$. Using $a \circ b = ab \vee d$ we have, $[a \wedge b : (m \vee d)] \circ (m \vee d)$

$$= [a \wedge (b : m)](m \vee d) \vee d.$$

$$= [a \wedge (b : m)] m \vee (a \wedge b : m) d \vee d.$$

Again as m is meet principal, we have $(a \wedge b : m) m = am \wedge b$ so we get

$[a \wedge b : (m \vee d)]$ ¤ $(m \vee d) = (am \wedge b) \vee [(a \wedge b : m)d] \vee d$.

We have $(a \wedge b : m)d \leq d \Rightarrow [a \wedge b : (m \vee d)]$ ¤ $(m \vee d)$

$= (am \wedge b) \vee d = b \wedge (am \vee d)$.(Since L is modular).

Therefore

$[a \wedge b : (m \vee d)]$ ¤ $(m \vee d) = (am \vee d) \wedge b$

$= [a$ ¤ $(m \vee d)] \wedge b$.

It follows that $m \vee d$ is both meet and join principal. So the element $m \vee d$ is principal element. Now if $a \in L_{/d}$ then $a \geq d \Rightarrow a \vee d = a$. Since every element of L is a join of principal elements, $a \in L \Rightarrow a = m_1 \vee m_2 \vee \ldots \vee m_k$ where each m_i is principal.

Therefore $a \vee d = a = (m_1 \vee m_2 \vee \ldots \vee m_k) \vee d$.

Hence $a = (m_1 \vee d) \vee (m_2 \vee d) \vee \ldots \vee (m_k \vee d)$. Where each $m_i \vee d$ is a principal element in $L_{/d}$. i.e. $a \in L_{/d}$ is the join of principal elements in $L_{/d}$. Thus $L_{/d}$ is a modular multiplicative lattice satisfying the ascending chain condition in which each element is the join of principal elements. Hence $L_{/d}$ is a Noether lattice. ■

Theorem 5.3 :

An element is prime in L if and only if it is prime in $L_{/d}$.

Proof :

Let $a, b, c \in L_{/d}$. So that $a, b, c \geq d$. We note that $a \geq bc \Leftrightarrow a \geq b$ ¤ c for all $a, b, c \in L_{/d}$. Because $a \geq bc$, $a \geq d \Rightarrow a \geq bc \vee d = b$ ¤ c and $a \geq b$ ¤ $c \Rightarrow a \geq bc \vee d$

Thus $a \geq bc$. Hence $a \geq bc \Leftrightarrow a \geq b$ ¤ c .... (5.3.1.)

Suppose a is prime in L. To prove a is prime in $L_{/d}$.

Let $b \ast c \leq a \Rightarrow bc \leq a$ [by using (5.3.1)].

This implies $b \leq a$ or $c \leq a$ and hence a is prime in $L/_{d}$.

Conversely, Let a be prime in $L/_{d}$. Let $bc \leq a$ by (5.3.1)

$b \ast c \leq a \Rightarrow b \leq a$ or $c \leq a$ i.e. a is prime in L. ∎

Theorem 5.4 :

An element is primary in L if and only if it is primary in $L/_{d}$.

Proof :

Let $a, b, c \in L/_{d}$. So that $a, b, c \geq d$.

Suppose a is primary in $L/_{d}$ and $bc \leq a$. Then $b \ast c \leq a \Rightarrow b \leq a$ or $c^n \leq a$, $n \in z_+$. Hence a is primary in L.

Conversely, suppose a is primary in L and let $b \ast c \leq a$ This implies $bc \leq a$. Hence we have $b \leq a$ or $c^n \leq a$ for some integer n. Therefore a is primary in $L/_{d}$. ∎

Theorem 5.5 :

If L is a local Noether lattice and $d \neq 1$ then $L/_{d}$ is a local Noether lattice.

Proof :

Suppose L is a local Noether lattice and let p be the only maximal element of L. We have $L/_{d} = \{ x \in L \,/\, x \leq d \}$ As p is a maximal element of L and $L/_{d} \subseteq L$, it follows that $p \in L/_{d}$ and is the only maximal element of $L/_{d}$. We know that if L is Noether, $L/_{d}$ is also Noether lattice.

Hence $L/_{d}$ is a local Noether lattice. ∎

§. 6. CONGRUENCE LATTICES

In this last section we study the nation of congruence relations in the context of Dilworth's work concerning multiplicative lattices. Let us recall the definition of a congruence relation on lattice L (see [12]).

Definition 6.1 :

An equivalence relation θ on a lattice L is called a congruence relation on L if

$a_0 \equiv b_0(\theta)$ and $a_1 \equiv b_1(\theta)$ imply that

$a_0 \vee a_1 \equiv b_0 \vee b_1\ (\theta)$, $a_0 \wedge a_1 \equiv b_0 \wedge b_1\ (\theta)$.

for $a_o, b_o, a_1, b_1 \in L$.

The following Lemmas 5.1 to 5.4 shows that congruence mod d is a congruence relation on L preserving meet, join, multiplication and residuation. The set $[a]_\theta = \{ x \in L / x \equiv a\ (\theta) \}$ is the congruence class containing a. Suppose L is a Noether lattice and let d be an arbitrary element of L. If $p_1, p_2, \ldots,$ p_n are the primes associated with normal decomposition $a = q_1 \wedge q_2 \wedge \ldots \wedge q_n$ then $\{ p_i \in L / p_i \leq d \}$ is an isolated set of primes and hence determines an isolated component a_d of a.

For a given normal decomposition

$a = q_1 \wedge q_2 \wedge \ldots \wedge q_n$. we have $a_d = \wedge \{ q_i / d \geq p_i \}$.

Let $a_d' = \wedge \{ q_j / d \ngeq p_j \}$. Then $a = a_d \wedge a_d'$.

We define $a \equiv b\ (\ d\ )$ iff $a_d = b_d$.

Lemma 6.2 : Let L be a Noether lattice and $d \in L$

If $a \geq b$ then $a_d \geq b_d$.

Proof : Let $a = q_1 \wedge q_2 \wedge \ldots \wedge q_n$ be a normal decomposition of a. Let $p_1, p_2, \ldots, p_n$ be the associated

primes then the set $\{ p_i / p_i \geq d \}$ is an isolated set of primes of a. This determines an isolated component $a_d = \wedge \{ q_i / p_i \leq d \}$ of a. Let $a_d' = \wedge \{ q_j / p_j \nleq d \}$.
If $d \geq p_i$ then $q_i \geq a \geq b = b_d \wedge b_d' \geq b_d b_d'$.
where $b = q_1' \wedge q_2' \wedge \ldots \wedge q_m'$ is a normal primary decomposition of b.

Let $b_d = \wedge \{ q_j' / d \geq p_j' \}$, $b_d' = \wedge \{ q_j' / d \ngeq p_j' \}$ and suppose $p_1', p_2', \ldots, p_m'$ are associated primes of $q_1', q_2', \ldots, q_m'$.
If $p_i \geq b_d' = \wedge \{ q_j' / d \ngeq p_j' \}$. and hence $p_i \geq \prod \{ q_j' / d \ngeq p_j' \}$ then as p_i is prime, $\prod \{ q_j' / d \ngeq p_j \} \leq p_i$ implies that $q_j' \leq p_i$ for some j, for which $d \ngeq p_j'$.
Hence $q_j' \leq p_i \Rightarrow p_j' \leq p_i$ where $d \ngeq p_j'$. It follows that $d \ngeq p_i$ which contrary to the fact that $d \geq p_i$ and hence $p_i \ngeq b_d'$.

Now $b_d b_d' \leq q_i$, where q_i primary,
and $b_d' \ngeq p_i$ implies that $b_d \leq q_i$

$\Rightarrow b_d \leq \wedge \{ q_i / d \geq p_i \}$ i.e. $b_d \leq a_d$

Thus $a \leq b \Rightarrow a_d \geq b_d$. ∎

<u>Lemma</u> <u>6.3</u> :

Let L be a Noether lattice.

If $a \equiv b \ (d)$ then $a \wedge c \equiv b \wedge c \ (d)$.

Proof :

Let $a \wedge c = q_1 \wedge q_2 \wedge \ldots \wedge q_n$ be a normal decomposition of $a \wedge c$ with the associated primes $p_1, p_2, \ldots, p_n$.

Let $a = q_1' \wedge q_2' \wedge \ldots \wedge q_m'$ be a normal primary decomposition of a and $p_1', p_2', \ldots, p_m'$ be the associated primes of $q_1', q_2', \ldots q_m'$. If d is an arbitrary element of L then $\{ p_i' / p_i' \leq d \}$ is an isolated set of primes and determines an isolated component $a_d = /\backslash \{ q_i' / d \geq p_i' \}$ of a.

Let $a_d' = /\backslash \{ q_i' / d \not\geq p_j' \}$. Then $a = a_d /\backslash a_d'$.
If $d \geq p_i$, we have $p_i \geq q_i \geq a /\backslash c = a_d /\backslash a_d' /\backslash c$ and hence $p_i \geq (a_d /\backslash c) a_d'$.

Suppose $a_d' \leq p_i$. Then $p_i \geq a_d'$. Thus we have
$p_i \geq /\backslash \{ q_j' / d \not\geq p'_j \} \geq \prod \{ q_j / d \not\geq p_j' \}$
As p_i is prime, $\prod \{ q_j' / d \not\geq p_j' \} \leq p_i$ implies that $q_j' \leq p_i$. for some j and where $d \not\geq p_j'$.
As p_j' is the smallest prime containing q_j' and $q_j' \leq p_i$, it follows that $p_j' \leq p_i$. Hence $d \not\geq p_j'$ implies $d \not\geq p_i$. Contrary to $d \geq p_i$. Therefore $a_d' \not\leq p_i$.

Now $(a_d /\backslash c) a_d' \leq q_i$, where q_i is primary and $a_d' \not\leq p_i$ implies that $a_d /\backslash c \leq q_i$. Now $a \equiv b (d) \Leftrightarrow a_d = b_d$
Thus $q_i \geq a_d /\backslash c = b_d /\backslash c \geq b /\backslash c$. We have shown that if $d \geq p_i$ then $q_i \geq b /\backslash c$. Hence $b /\backslash c \leq /\backslash \{ q_i / d \geq p_i \}$.
Therefore $b /\backslash c \leq /\backslash \{ q_i / d \geq p_i \} = (a /\backslash c)_d$.
By Lemma 6.2 we have $(a /\backslash c)_d = [(a /\backslash c)_d]_d \geq (b /\backslash c)_d$.
Similarly we can show that $(b /\backslash c)_d \geq (a /\backslash c)_d$.
Therefore $(a /\backslash c)_d = (b /\backslash c)_d \Rightarrow a /\backslash c \equiv b /\backslash c (d)$ ∎

<u>**Lemma**</u> <u>**6.4**</u> :

Let L be a Noether lattice.

If $a \equiv b (d)$ then $a \backslash/ c \equiv b \backslash/ c (d)$.

Proof :

Let $a \backslash/ c = q_1 /\backslash q_2 /\backslash \ldots /\backslash q_n$

be a normal decomposition of $a \backslash/ c$ with the associated primes $p_1, p_2, \ldots , p_n$.

Let $a = q_1' \wedge q_2' \wedge \ldots \wedge q_m'$ be a normal decomposition of a and $p_1', p_2', \ldots, p_m'$ be the associated primes of a.

If d is an arbitrary element of L, we have

$a = a_d \wedge a_d'$ where $a_d = \wedge \{ q_i' / d \geq p_i' \}$ and

$a_d' = \wedge \{ q_j' / d \ngeq p_j' \}$

If $d \geq p_i$ then $p_i \geq q_i \geq a \vee c \geq a = a_d \wedge a_d'$ (6.4.1)

and $a \vee c \leq q_i$, for all i so $q_i \geq c$.

We have $p_i \geq a_d'$. Suppose $p_i \geq a_d' = \wedge \{ q_j' / d \ngeq p_j' \}$.

Then $p_i \geq \prod \{ q_j' / d \ngeq p_j' \}$. Thus we have

$q_j' \leq p_i$ for some i as p_i is prime, where $d \ngeq p_j'$.

Now p_j' is the smallest prime containing q_j'. Hence we have

$q_j' \leq p_i \Rightarrow p_j' \leq p_i$.

Now $d \ngeq p_j' \Rightarrow d \geq p_j$, a contradiction. Hence $a_d' \nleq p_i$.

Now $a \equiv b\ (d)$ iff $a_d = b_d$. From (6.4.1) we have

$q_i \geq a_d \wedge a_d' \geq a_d . a_d'$ and q_i is primary, where

$a_d' \nleq p_i \Rightarrow a_d \leq q_i$. This gives $q_i \geq a_d = b_d \geq b$.

Also $q_i \geq c \Rightarrow q_i \geq b \vee c$ (for all i, where $d \geq p_i$)

$\Rightarrow b \vee c \leq \wedge \{ q_i / d \geq p_i \} = (a \vee c)_d$

By Lemma 6.2, we have $(b \vee c)_d \leq [(a \vee c)_d]_d = (a \vee c)_d$

similarly $(a \vee c)_d \leq (b \vee c)_d \Rightarrow (a \vee c)_d = (b \vee c)_d$

Therefore $a \vee c \equiv b \vee c\ (d)$. ∎

<u>Lemma 6.5</u> :

If $a \equiv b\ (d)$ then $ac \equiv bc\ (d)$.

Proof :

Let $ac = q_1 \wedge q_2 \wedge \ldots \wedge q_n$ be a normal decomposition of ac with the associated primes $p_1, p_2, \ldots, p_n$.

Let $a = q_1' \wedge q_2' \wedge \ldots \wedge q_m'$ be a normal decomposition of aLet

d be an arbitrary element of L. Then $a_d = \wedge \{ q_i' \mid d \geq p_i' \}$ is an isolated component of a. Let $a_d' = \wedge \{ q_j' \mid d \geq p_j' \}$ So that $a = a_d \wedge a_d'$. If $d \geq p_i$ we have

$q_i \geq ac = (a_d \wedge a_d') c$

$\Rightarrow q_i \geq a_d a_d' c = (a_d c) a_d'$

$\Rightarrow p_i \ngeq a_d'$.

Because $p_i \geq a_d' = \wedge \{ q_j' \mid d \ngeq p_j' \} \geq \pi \{ q_j' \mid d \ngeq p_j' \}$

$\Rightarrow q_j' \leq p_i$, for at least one j for which $d \ngeq p_j'$.

Now p_j' is the smallest prime containing q_j' and $q_j' \leq p_i \Rightarrow p_j' \leq p_i$. Hence $d \ngeq p_j' \Rightarrow d \ngeq p_i$, a contradiction which proves that $p_i \ngeq a_d'$.

Now $q_i \geq (a_d c) a_d'$ and $a_d' \nleq p_i \Rightarrow a_d c \leq q_i$.

Also $a_d = b_d$ as $a \equiv b (d)$. So $q_i \geq a_d c = b_d c \geq bc$ where $d \geq p_i$. Hence $bc \leq \wedge \{ q_i \mid d \geq p_i \} = (ac)_d$.

By Lemma 6.2, $(bc)_d \leq [(ac)_d]_d = (ac)_d$.

Similarly $(ac)_d \leq (bc)_d \Rightarrow (ac)_d = (bc)_d$.

Therefore $ac \equiv bc (d)$.

<u>Lemma 6.6</u> :

If $a \equiv b (d)$ then $a : c \equiv b : c (d)$ and $c : a \equiv c : b (d)$.

Proof :

Let $x : y = q_1 \wedge q_2 \wedge \ldots \wedge q_n$ be a normal decomposition of $x : y$ and $p_1, p_2, \ldots, p_n$ be the associated primes of $q_1, q_2, \ldots, q_n$.

Let $x = q_1' \wedge \ldots \wedge q_m'$ be a normal decomposition of x with associated primes $p_1', p_2', \ldots, p_m'$ and let d be an arbitrary element of L. Then $x_d = \wedge \{ q_i' \mid d \geq p_i' \}$.

Let $x_d' = \wedge \{ q_i' \mid d \ngeq p_i' \}$. Then $x = x_d \wedge x_d'$.

If $d \geq p_i$ Then $p_i \geq q_i \geq x : y = (x_d \wedge x_d') : y$

$\Rightarrow q_i \geq (x:y) \wedge (x_d':y) \geq (x_d:y)(x_d':y)$. ....(6.6.1)

By proposition 2.4, we claim that $p_i \not\geq x_d' : y$.

Suppose $p_i \geq x_d' : y \geq x_d'$ (Since by proposition 2.6)

$\Rightarrow p_i \geq \wedge \{ q_j' / d \not\geq p_j' \} \geq \prod \{ q_j' / d \not\geq p_j' \}$

$\Rightarrow q_j' \leq p_i$ for at least one j, since p_i is prime and for this j $d \not\geq p_j'$. As p_j' is the smallest prime containing q_j', $q_j' \leq p_i \Rightarrow p_j' \leq p_i$.

But $d \not\geq p_j' \Rightarrow d \not\geq p_i$, a contradiction. Hence $p_i \not\geq x_d' : y$.

from (6.6.1) $(x_d : y) (x_d' : y) \leq q_i$, where q_i is primary and $(x_d' : y) \not\leq p_i$. Hence $(x_d : y) \leq q_i$.

Now $(x_d : y) = x_d : (y_d \wedge y_d') \geq x_d : y_d$

Thus $q_i \geq x_d : y \geq x_d : y_d$ where $d \geq p_i$

$\Rightarrow \wedge \{ q_i / d \geq p_i \} \geq x_d : y_d$

$\Rightarrow (x : y)_d \geq x_d : y_d$.

On the other hand $x \geq (x : y) y$ By proposition (2.1] and hence by Lemma 6.2 and Lemma 6.5 we have

$x_d \geq [(x : y) y]_d = (x : y)_d y_d$.

Now $x_d \geq (x : y)_d y_d \Rightarrow x_d : y_d \geq (x : y)_d$.

Hence $(x : y)_d = x_d : y_d$.

Now if $a \equiv b (d)$ then $(a : c) = a_d : c_d$ implies $(a : c)_d = b_d : c_d = (b : c)_d$.

Similarly we have $c : a \equiv c : b (d)$. ∎

<u>Corollary 6.7</u> The relation $a \equiv b (d)$ iff $a_d \equiv b_d$ is congruence relation on L.

Proof :

Reflexivity : For any $a \in L$, $a_d = a_d \Rightarrow a \equiv a (d)$.

Symmetry : Let $a \equiv b$ (d). Then $a_d = b_d$ i.e. $b_d = a_d$ and hence $b \equiv a$ (d).

Transitivity : Let $a \equiv b$ (d) and $b \equiv c$ (d).

Then $a_d = b_d$ and $b_d = c_d \Rightarrow a_d = c_d$. Hence $a \equiv c$ (d).

Congruence mod (d) is equivalence relation.

Also let $p \equiv q$ (d) and $r \equiv s$ (d). By definition of congruence mod (d), we have $p_d = q_d$ and $r_d = s_d$. This implies $p_d \vee r_d = q_d \vee s_d$ and $p_d \wedge r_d = q_d \wedge s_d$.

$\Rightarrow (p \vee r)_d = (q \vee s)_d$ and $(p \wedge r)_d = (q \wedge s)_d$

$\Rightarrow p \vee r \equiv q \vee s$ (d) and $p \wedge r \equiv q \wedge s$ (d). This shows that the congruence mod (d) is congruence relation on L. ∎

In the view of above results we have the following

Theorem 6.8 :

Let L be a Noether lattice and

Let $[a] = \{ b \in L \,/\, b \equiv a \,(d) \}$ be the congruence class containing a. Then the set of congruence classes L_d of L is a Noether lattice.

The salient features of the multiplicative lattice L_d of congruence classes are (1) The primes of L_d

(2) The primaries of L_d.

Theorem 6.9 :

The primes elements and primary elements of L_d are precisely the congruence classes determined by the primes and primaries of L respectively. Moreover, the proper prime elements of L_d are the congruence classes determined by primes p such that $d \geq p$.

Proof :

Let L_d = { [a] / [a] is the congruence class containing a }, where [a] = { $x \in L$ / $x \equiv a$ (d) i.e. $x_d = a_d$}.
Let the congruence class [c] be a prime in L_d. Assume that c has a primary decomposition $c = q_1 \wedge q_2 \wedge \ldots \wedge q_m$ and suppose $p_1, p_2, \ldots, p_m$ are the associated primes of c. Let $d \in L$.
Let $c_d = \wedge \{ q_i / d \geq p_i \}$ and $q_1, q_2, \ldots, q_n$ be such that $d \geq p_i$ (i = 1, 2, ... , n). Then $c_d = q_1 \wedge q_2 \wedge \ldots \wedge q_n$ ($n \leq m$). Obviously, $c = q_1 \wedge q_2 \wedge \ldots \wedge q_m \leq q_i$, (i = 1, 2, ... , m) implies $c \leq q_1 \wedge q_2 \wedge \ldots \wedge q_n$.
By the property of an associated prime, we have
$p_i^{ki} \leq q_i \leq p_i$ for some ki (i = 1, 2, ... , m).
Now $c = q_1 \wedge q_2 \wedge \ldots \wedge q_m \geq p_1^{k1} \wedge p_2^{k2} \wedge \ldots \wedge p_m^{km}$.
$\Rightarrow c \geq p_1^{k1}.p_2^{k2}. \ldots . p_m^{km}$, where $p_1, p_2, \ldots, p_m$ are the primes of L. and $p_i \geq q_i \geq c$ for each i = 1, 2, ... , m.
Then $[c] \geq [p_1]^{k1}[p_2]^{k2}\ldots[p_n]^{kn}$.
Hence $[c] \geq [p_i] \geq [c]$ for some i. This shows $[p_i] = [c]$ where p_i is a prime of L i.e. the prime element [c] of L_d is determined by the prime p_i of L.
Conversely, suppose p is prime element of L. Then p is primary.
Let $d \in L$. Either $d \geq p$ or $d \ngeq p$. Then $p_d = \wedge \{ q_i / d \geq p \}$.
Thus $p_d = \wedge \{ q_i / d \geq p \} = p$ when $d \geq p$. and if $d \ngeq p$ then $p_d = \wedge \{ q_i / d \ngeq p \} = \wedge \{ p / d \ngeq p \} = 1$.
Let $[p] \geq [a][b] \Rightarrow [p] \geq [ab]$.

$$\Rightarrow p_d \geq a_d b_d.$$

and hence $p_d \geq a_d$ or $p_d \geq b_d$. Which yields that
$[p] \geq [a]$ or $[p] \geq [b]$.

Hence the primes of L_d are of the form [p]. Where p is a prime of L and [p] is proper <=> d ≥ p.

Next suppose that [c] be a primary element of L_d associated with the prime element [p]. Since $c_d = q_1 \wedge q_2 \wedge \dots \wedge q_m$ whenever $d \geq p_i$ and p_i is the prime associated with q_i. Let us assume that $[c] \neq [1]$ and hence we have $d \geq p$. Also $[c] \geq [p]^k$ for some k. Therefore we have $c_d \geq p_d{}^k = p^k$. But then we have $p_i \geq p$ for all i. Since $p \geq c_d$ it follows that for some j, $p_j \leq p$. And therefore $p = p_j$. Also $p_1, p_2, \dots,$ p_r are distinct prime associated with each q_i. And hence we have $p_i \nleq p$ for $i \neq j$. Since [c] is primary element in L_d, it follows that $[q_j] \leq [c]$. And then $[c] = [q_j]$ where q_j is primary element of L. Conversely assume that q is a primary element of L. We known that by definition, $q_d = q$ or $q_d = 1$ whenever $d \geq p_q$ or $d \ngeq p_q$.

Hence $[a][b] \leq [q] \Rightarrow [a] \leq [q]$ or $[b] \leq [p]$. And this shows that [q] is primary element of lattice L_d. ∎

We note that distinct primes contained in d gives rise to district congruence classes.

***** THE END *****