CHAPTER - II

RADICALS

AND

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§. 1. INTRODUCTION

In the first chapter of this dissertation we have studied several stronger abstract formations for the notion of "Principal element" which play an important role in the theory of decomposition for lattice. (see Dilworth [1962] [7]).

The concept of radicals are found scattered in several papers on multiplicative lattices. However there were no concerted efforts for obtaining abstract formulation of radical theory of commutative rings with unity though there is a vast literature available. The study of radicals was carried out by Murata [17] and Zariski and Samuel [21] and the group of Thakare.

This chapter is based on the work of Thakare and Manjarekar [1982][20], where in they carry out the study of radicals. Here, we wish to report their finding in details. For this first, we recall the following definition.

Definition 2.1 :

A *multiplicative lattice* is a complete lattice L on which there is defined a multiplication that happens to be commutative, associative, and distributive over arbitrary joins and for which the largest element 1 is a multiplicative identity.

Definition 2.2 :

An element a of L is said to be <u>compact</u> if $a \leq \bigvee X$, $X \subseteq L$ implies the existence of a finite number of element x_1, x_2, ... , x_n of X such that $a \leq x_1 \vee x_2 \vee \ldots \vee x_n$.

According to Dilworth & Crawley [9], we report related result :

Lemma 2.3 :

Let L be a multiplicative lattice. Then the following statements are equivalent :

(1) L is complete lattice satisfying the ascending chain condition.

(2) Every element of L is compact.

Here after, we will assume that L is multiplicative lattice in which every element is compact.

Definition 2.4 :

The radical of an element a is $\bigvee \{ z \in L \,/\, z^s \leq a$, for some integer $s \}$ and is denoted by $\sqrt{a}$.

As stated in Dilworth [8], we have the following Lemmas without proof.

Lemma 2.5 :

If q is primary element then $\sqrt{q}$ is the minimal prime containing q. Note that $\sqrt{q}$ is called as the prime associated with q. Assume that an element a in L has an irredundant primary decomposition

$$a = q_1 \wedge q_2 \wedge \ldots \wedge q_n \qquad \ldots.(*)$$

§.2.RADICALS

The following most important lemmas that need to be stated without proof. First Lemma gives sufficient condition for a prime element p of L tc contain some associated prime element.

Lemma 2.6 :

If a prime element p contains a finite meet $\bigwedge_{i=1}^{m} q_i$ then it contains some q_i, if the q_i are primary then p contains the associated prime element p_i of one of them.

Lemma 2.7 :

If a prime element $p = \bigwedge_{i=1}^{n} p_i$ where the p_i are prime then p contains one of them by lemma 2.6 and thus is equal to it; the other p_j contain then this p_i. The next properties are immediate analogous of similar properties of radicals of an ideal in a commutative ring with unity.

Properties of Radicals 2.8 :

For a, b ∈ L

(p_1) $a \le \sqrt{a}$

(p_2) $a \le b \Rightarrow \sqrt{a} \le \sqrt{b}$

(p_3) $\sqrt{\sqrt{a}} = \sqrt{a}$

(p_4) $\sqrt{(a \wedge b)} = \sqrt{a} \wedge \sqrt{b} = \sqrt{(ab)}$

(p_5) $\sqrt{(a \vee b)} = \sqrt{(\sqrt{a} \vee \sqrt{b})}$.

Proof :

(p_1) $a^1 \le a \Rightarrow a \in \{ x \in L \mid x^s \le a$ for some integer $s\}$

and hence $a \le \sqrt{a}$. ∎

(p_2) Let $y \in \{ x / x^s \le a$ for some integer $s \}$. Then $y^s \le a \le b$ which gives $\{ x / x^s \le a$ for some integer $s \}$ $\subseteq \{ y / y^s \le b$ for some integer $s \}$. Therefore $\sqrt{a} \le \sqrt{b}$. ∎

(p_3) $a \le \sqrt{a} \Rightarrow \sqrt{a} \le \sqrt{\sqrt{a}}$. To prove $\sqrt{\sqrt{a}} \le \sqrt{a}$ take $y \in \{ x / x^s \le \sqrt{a}$ for some integer $s \}$. This implies $y^s \le \sqrt{a} = \bigvee\{ x / x^s \le a$ for some integer $s \}$. As each element of L is compact, $y^s \le \sqrt{a} = \bigvee_{i=1}^{n} \{ x_i / x_i^{si} \le a \}$.

Let $s1 + s2 + \ldots + sn = k$. Then

$(x_1 \vee x_2 \vee \ldots \vee x_n)^{s1+s2+\ldots+sn} \le x_1^{s1} \vee x_2^{s2} \vee \ldots \vee x_n^{sn} \le a$. Therefore $y^s \le (x_1 \vee x_2 \vee \ldots \vee x_n)$

$\Rightarrow y^{sk} \le (x_1 \vee x_2 \vee \ldots \vee x_n)^k \le a$.

i.e. $y \in \{ x / x^s \le a$ for some integer $s \}$ and $y \le \sqrt{a}$.

Thus $\sqrt{\sqrt{a}} \le \sqrt{a}$ and hence we have $\sqrt{a} = \sqrt{\sqrt{a}}$. ∎

(p_4) $\sqrt{(a \wedge b)} = \sqrt{a} \wedge \sqrt{b} = \sqrt{(ab)}$.

First we show that $\sqrt{(a \wedge b)} = \sqrt{a} \wedge \sqrt{b}$.

Let $c \in \{ x / x^s \le a \wedge b$ for some integer $s \}$ Then $c^s \le a \wedge b$ which implies $c^s \le a$ and $c^s \le b$. Hence $\{ x / x^s \le a \wedge b$ for some $s \in N \} \subseteq \{ x / x^s \le a$, s is an integer $\}$ and $\{ x / x^s \le a \wedge b \} \subseteq \{ x / x^s \le b$ for some integer $s \}$.

$\Rightarrow \sqrt{(a \wedge b)} \le \sqrt{a}$ and $\sqrt{(a \wedge b)} \le \sqrt{b}$. Therefore $\sqrt{(a \wedge b)} \le \sqrt{a} \wedge \sqrt{b}$.

Conversely let $y \le \sqrt{a} \wedge \sqrt{b} \Rightarrow y \le \sqrt{a}$ and $y \le \sqrt{b}$.

i.e $y \le \bigvee \{ x / x^s \le a \}$ and $y \le \bigvee \{ x / x^s \le b \}$. As each element of L is compact, suppose $y \le \bigvee_{i=1}^{r} \{ x_i / x_i^{si} \le a \}$ and $y \le \bigvee_{j=1}^{l} \{ z_j / z^{mj} \le b \}$.

Then $y \le x_1 \vee x_2 \vee \ldots \wedge x_r$ and $y \le z_1 \vee z_2 \vee \ldots \vee z_l$.

Let $s1 + s2 + \ldots + sr = n$ and $m1 + m2 + \ldots + ml = p$.

Then $y^n \le (x_1 \vee x_2 \vee \ldots \vee x_r)^{s1+s2+\ldots+sr}$ and hence we have

$x_1^{s1} \vee x_2^{s2} \vee \ldots \vee x_r^{sr} \leq a.$

$y^p \leq (z_1 \vee z_2 \vee \ldots \vee z_1)^p \leq z_1^{m1} \vee z_2^{m2} \vee \ldots \vee z_1^{ml} \leq b.$

Hence $y^n . y^p \leq ab \leq a \wedge b$ i.e. $y^{n+p} \leq a \wedge b.$

This implies $y \leq \sqrt{(a \wedge b)}$. Therefore $\sqrt{a} \wedge \sqrt{b} \leq \sqrt{(a \wedge b)}$.

Hence $\sqrt{(a \wedge b)} = \sqrt{a} \wedge \sqrt{b}$. Now we show that $\sqrt{(a \wedge b)} = \sqrt{(ab)}$.

Let $x \in \{ y / y^s \leq ab \} \Rightarrow x^s \leq ab \leq a \wedge b.$

$\Rightarrow x \in \{ y / y^s \leq a \wedge b \}$. Therefore $\sqrt{(ab)} \leq \sqrt{(a \wedge b)}$.

Let $x \in \{ y / y^s \leq a \wedge b \} \Rightarrow x^s \leq a \wedge b.$

$\Rightarrow x^s \leq a$ and $x^s \leq b \Rightarrow x^s . x^s \leq a.b$

Therefore $x \in \{ y / y^s \leq ab \}$.

Hence $\sqrt{(a \wedge b)} \leq \sqrt{(ab)}$ and therefore $\sqrt{(ab)} = \sqrt{(a \wedge b)}$.

i.e. $\sqrt{(a \wedge b)} = \sqrt{a} \wedge \sqrt{b} = \sqrt{(ab)}$. ∎

(p_5) Let $y \in \{ x / x^s \leq a \vee b \}$ then $y^s \leq a \vee b$ for some integer s. By (p_1) $y^s \leq \sqrt{a} \vee \sqrt{b}$. This implies $y \in \{ x / x^s \leq \sqrt{a} \vee \sqrt{b} \}$ and we have $\sqrt{(a \vee b)} \leq \sqrt{(\sqrt{a} \vee \sqrt{b})}$.

For the reverse inequality let $y \in \{ x / x^s \leq \sqrt{a} \vee \sqrt{b} \}$. Then by definition of radical, it follows that

$y^s \leq [\vee \{ x / x^s \leq a \}] \vee \{ z / z^l \leq b\}]$. Since every element of L is complete, we have

$y^s \leq [x_1 \vee x_2 \vee \ldots \vee x_m] \vee [z_1 \vee z_2 \vee \ldots \vee z_n]$ ....[2.8.$p_5$1]

where $x_i^{si} \leq a$, $i=1,2,\ldots,m$ and $z_j^{lj} \leq b$, $j=1,2,\ldots,n$ for some integers si and lj. Let $s1+s2+\ldots+sm+l1+l2+\ldots+ln = k$.

Using well-known property see Dilworth[xx], we have

$[x_1 \vee x_2 \vee \ldots \vee x_m \vee z_1 \vee z_2 \vee \ldots \vee z_n]^k$

$\leq x_1^{s1} \vee x_2^{s2} \vee \ldots \vee x_m^{sm} \vee z_1^{l1} \vee z_2^{l2} \vee \ldots \vee z_n^{ln} \leq a \vee b$ [2.8$p_5$2]

from [2.8.$p_5$1] and [2.8.$p_5$2] $y^{sk} \leq a \vee b$ and

$y \in \{ x / x^s \leq a \vee b$ for some integer s $\}$

Therefore $\sqrt{(\sqrt{a} \vee \sqrt{b})} \leq \sqrt{(a \vee b)}$ and the proof is complete. ∎

Thakare and Manjarekar discussed the necessary and sufficient condition for an element to be equal to its own radical.

Theorem 2.9 :

Let $a \in L$ be such that it admits an irredundant primary decomposition $a = q_1 \wedge q_2 \wedge \ldots \wedge q_m$ ...(*)

Then $a = \sqrt{a}$ if and only if all the q_i are prime elements.

Proof :

Suppose each q_i in the representation (*) is prime. To prove $a = \sqrt{a}$ take $y \in \{ x / x^s \le a$ for some integer $s \}$. Then $y^s \le \wedge_{i=1}^{m} q_i \le q_i$ for all $i = 1, 2, \ldots, m$ and by primeness of q_i, we have $y \le q_i$ for each $i = 1, 2, \ldots, m$. Hence $y \le a$ i.e. $\sqrt{a} \le a$ and by properly (p_1) we have $a = \sqrt{a}$. Conversely, suppose $a = \sqrt{a}$ where a has the primary decomposition as in (*). Let p_i be an associated prime of q_i, $i = 1, 2, \ldots, m$. In view of representation (*) and by property (p_4) we have, $\sqrt{a} = \sqrt{q_i} \wedge \sqrt{q_2} \wedge \ldots \wedge \sqrt{q_m}$. But $a = \sqrt{a}$ gives $a = \wedge_{i=1}^{m} p_i$.

This representation of a as a meet of prime elements is irredundant because if for some j, $a = \wedge_{i \ne j} p_i \ge \wedge_{i \ne j} q_i \not> a$ then we get a contradiction as $a = \wedge_{i=1}^{m} q_i$, is an irredundant primary representation of a. To show that $q_i = p_i$ for each i, take any $y \le p_i$. As $\wedge_{i=1}^{m} p_i$ is irredundant decomposition for a, there is $z \le \wedge_{j \ne i} p_j$ such that $z \nleq p_i$. Now $yz \le \wedge_{i=1}^{m} p_i = \wedge_{i=1}^{m} \le q_i$ for each i. As each q_i is primary and $z \nleq p_i$ we have $y \le q_i$. In particular $p_i \le q_i$ and hence $p_i = q_i$ i.e. each q_i is a prime element. ∎

§. 3. PRIMARY DECOMPOSITIONS IN MULTIPLICATIVE LATTICES

According to Atiyah & MaCdonald [3] the decomposition of ideals is a traditional pillar to ideal theory. In modern treatment, with its emphasis on localization, primary decomposition is no longer, such a central tool in the lattice theory.

The notion of uniqueness of irredundant primary decomposition of the type $a = q_1 \wedge q_2 \wedge \ldots \wedge q_m$ ..... (*) is well discussed in [4]. Some results on primary decompositions concerning semi-modular multiplicative lattices are studied by McCarthy [16]. Richter,G. initiated the concept of irredundant decompositions in J-lattices.(see[18],[19]). Also they give a necessary condition for a complete lattice that each of its elements has an irredundunt decomposition .

In this section we report the uniqueness theorems for primary decompositions, which are represented by Thakare and Manjarekar [20].

We need to recall some basic definitions.

<u>Definition 3.1</u> :

The associated prime elements of the primary elements occurring in an irredundant primary representation $a = q_1 \wedge q_2 \wedge \ldots \wedge q_m$ of an element a are called <u>the associated prime elements of a</u> or simply the prime elements of a.

<u>Definition 3.2</u> :

A minimal element in the family of associated prime elements of a is called the isolated prime element of a. A prime element of a which is not isolated is said to be embedded.

Definition 3.3 :

If $a = \bigwedge_{i=1} q_i$ is an irredundant primary representation of a, the elements q_i are said to be primary components of a and q_i is called isolated or embedded according as its associated prime element p_i is isolated or embedded.

The following theorem gives a characterization for a prime element of L to be equal to some associated prime p_i.

Theorem 3.4 :

Let $a \in L$ have primary decomposition

$$a = q_1 \wedge q_2 \wedge \ldots \wedge q_m. \qquad (*)$$

Let p_i (i= 1, 2, ..., m) be associated primes of q_i. Then the following statements are equivalent :

(1) A prime element p of L is equal to some p_i.

(2) There exists an element $b \in L$ not contained in a and such that (a : b) is primary for p.

Proof :

(1) => (2): Suppose a prime element $p = p_i$ for some i and p_i are as given in the hypothesis. For this i there exist

$$b \le \bigwedge_{j \neq i} q_j. \quad \text{Such that } b \nleq q_i \qquad (3.4.1)$$

Since the primary decomposition (*) is irredundant . First we show that for such an element b the element (a : b) evidently contains q_i and is contained in p_i.

Let $x \in \{ y \,/\, yb \le a \}$ then $xb \le q_j$ for every j.

But for our fixed i, $b \nleq q_i$ and hence $x^s \le q_i$ for some integer s. Thus $x \in \{ y \,/\, y^s \le q_i$ for some integer s $\}$ which shows that

$\bigvee \{ y \,/\, yb \le a \} \le \bigvee \{ y \,/\, y^s \le q_i \}$ i.e. $(a:b) \le p_i$ (3.4.2)

On the other hand select $y \le q_i$. From (3.4.1) we have

$yb \leq \wedge_{j=1} q_j = a$ which implies that $y \leq (a : b)$ and hence

$q_i \leq (a : b)$ ...(3.4.3)

From (3.4.2) and (3.4.3), we have $\sqrt{q_i} = p_i \leq \sqrt{(a : b)} \leq \sqrt{p_i} = p_i$.
i.e. $\sqrt{(a : b)} = p_i$. To show that $(a:b)$ is primary take $yz \leq (a : b)$ and suppose that $z \nleq p_i$. Clearly $yzb \leq a \leq q_i$ and thus $yb \leq q_i$. Use of (3.4.1) gives $yb \leq \wedge_{i=1}^{m} q_i = a$ yielding $y \leq (a : b)$. Thus $(a : b)$ is primary and we conclude that $(a : b)$ is p_i primary and hence p primary. Thus (1) => (2).

<u>(2) => (1)</u>: Suppose that for some element b such that $b \nleq a$, the element $(a : b)$ is primary for a given prime element p. Since $a = q_1 \wedge q_2 \wedge \ldots \wedge q_m$, it follows that $(a : b) = \wedge_{i=1}^{m} (q_i : b)$. (See Dilworth [7]). By property (p_4) of radicals and our assumption, we have

$\sqrt{(a : b)} = \wedge_{i=1}^{m} \sqrt{(q_i : b)} = p$ ....(3.4.4)

From assumption $b \nleq a$ and the irredundant decomposition (*), we have $b \nleq q_i$ for some i and $b \leq q_i$ for remaining ones.

<u>Case (1)</u> When $b \leq q_i$, we have by Dilworth [7]

$\sqrt{(q_i : b)} = 1$.

<u>Case (2)</u> When $b \nleq q_i$. Let $x \leq q_i$. Then

$xb \leq q_i$ and $x \in \{ y / yb \leq q_i \}$ i.e. $x \leq (q_i : b)$

Hence $q_i \leq (q_i : b)$. This implies $p_i \leq \sqrt{(q_i : b)}$.

For the reverse irrequality suppose $x \leq (q_i : b)$.

Then $xb \leq q_i$. But $b \nleq q_i$ yields $x^s \leq q_i$ for some integer s

i.e. $x \leq \sqrt{q_i}$ and $\sqrt{(q_i : b)} \leq p_i$. Therefore $\sqrt{(q_i : b)} = p_i$.

From (3.4.4) and the above two cases it follows that

$p = \wedge_T p_j$ for some subset T of $\{ 1, 2, \ldots, m \}$. Therefore by Lemma 2.7 we conclude that $p = p_j$ for some j. ∎

The following characterization is obvious.

Theorem 3.5 :

Let L be a lattice, in which a has an irredundant decomposition $a = q_1 \wedge q_2 \wedge \ldots \wedge q_m$. (*). A prime element $p \in L$ contains a, if and only if p contains some p_i, Where the p_i are the associated primes of the q_i's respectively.

Proof :

Suppose $p_i \leq p$ for some i. Then $a \leq q_i \leq p_i \leq p$. Conversely suppose $a \leq p$, then we have $q_1 q_2 \ldots q_m \leq q_1 \wedge q_2 \wedge \ldots \wedge q_m \leq p$. This gives $q_i \leq p$ for some i, Since p is prime. Let $x \in \{ y \,/\, y^s \leq q_i$ for some integer s $\}$, this implies that $x^s \leq q_i \leq p$ for some integer s. But p is prime implies $x \leq p$ Hence $p_i \leq p$. ∎

By using Theorem 3.4, we conclude that the associated primes p_i arising from the irredundant primary decomposition of an element $a = \wedge_{i=1}^{m} q_i$ are uniquely determined.

In the next result, Thakare and Manjarekar showed that even those q_i's can be uniquely determined which are isolated primary components of $a \in L$.

<u>Theorem 3.6</u> :

Let $a \in L$ have an irredundant primary decomposition

$$a = q_1 \wedge q_2 \wedge \ldots \wedge q_m \qquad \ldots.(*)$$

and p_i's be associated primes of q_i's. The element $q_i' = \vee \{ z \in L \,/\, (a:z) \leq p_i \}$ is an element of L which is contained in q_i. If q_i is an isolated primary component of a then $q_i = q_i'$.

Proof :

Take any element $x \in \{ z \in L / (a:z) \nleq p_i \}$

$= \{ z \in L / \vee \{ x / xz \leq a \} \nleq p_i \}$.

Then there is an element $\beta \in L$ such that $\beta z \leq a$ and $\beta \nleq p_i$.

Then $\beta^n \nleq q_i$ for all integers n.

Since $\beta z \leq q_i$ and $\beta^n \nleq q_i$ for every integer n and q_i is primary it follows that $z \leq q_i$. Therefore $q_i' \leq q_i$. This completes the proof of first part.

Next if q_i is isolated primary component of a then p_i is a minimal associated prime of a and hence $p_j \nleq p_i$ whenever $i \neq j$. Then there exists $b_j \leq p_j$ such that $b_j \nleq p_i$. Since every element of L is compact, we have

$b_j \leq p_j = \vee_{r=1}^{n} \{ z_r / z_r^{s_r} \leq q_j$ for some integer $s_r \}$

Puting $s_1+s_2+\ldots+s_n = k(j)$.

Then $b_j^{k(j)} \leq (z_1 \vee z_2 \vee \ldots \vee z_n)^{k(j)} \leq q_j$.

Obviously we have $b = \prod_{j \neq i} b_j^{k(j)} \nleq p_i$ (as p_i is prime).

However $b \leq \wedge_{j \neq i} q_j$ Next we take any $x \leq q_i$.

Then $bx \leq \wedge_{i=1}^{m} q_i = a \Rightarrow b \leq (a:x) = \vee \{ c \in L / cx \leq a \}$.

Since $b \nleq p_i$ it follows that $(a:x) \nleq p_i$. This shows that $x \in \{ z \in L / (a:z) \nleq p_i \}$. Hence we have the in equality $x \leq \vee \{ z \in L / (a:z) \leq p_i \} = q_i'$.

Thus $q_i \leq q_i'$ and by using the first part we have $q_i = q_i'$. ∎

§. 4. FURTHER RESULTS ON RADICALS AND PRIMES.

Thakare and Manjarekar [20] settled the relation between the radical of a and isolated primes of $a \in L$. We shall report the same here.

Theorem 4.1 :

The radical of a having irredundant primary decomposition $a = q_1 \wedge q_2 \wedge \ldots \wedge q_m$ (*)

is the meet of isolated primes of a.

Proof :

By using property (p_4) the radical of a is the meet of all associated primes p_i $(i = 1, 2, \ldots, m)$ of a

i.e. $\sqrt{a} = p_1 \wedge p_2 \wedge \ldots \wedge p_m$.

If some p_k is not isolated then we have $p_k \geq p_i$ for some p_i. Hence we delete such embedded elements from the above representation and hence the proof. ∎

Recall the notion of nilpotent element in a multiplicative lattice which plays an important role in a Noether lattice.

Definition 4.2 :

An element *a* of a multiplicative lattice L is nilpotent if $a^n = 0$ for some integer n.

The following consequence is worth noting.

Corollary 4.3 :

In a Noether lattice L, the join of the set of all nilpotent elements is the meet of the isolated primes of 0 i.e. the minimal prime elements of 0.

Proof :

Let $0 = \bigwedge_{i=1}^{m} q_i$ be a finite irrdeundant primary decomposition of 0. Let p_i (i = 1, 2, ... , m) be associated primes of q_i (i = 1, 2, ... , m) respectively. By Theorem 4.1 we have $\sqrt{0} = p_1 \wedge p_2 \wedge \ldots \wedge p_r$ where $p_1, p_2, \ldots, p_r$ are isolated primes of 0. By definition of the radical, we then have

$\vee \{ a / a \text{ is nilpotent} \} = p_1 \wedge p_2 \wedge \ldots \wedge p_r$. ∎

Thakare and Manjarekar have given the characterization of the primeness of radical of a in the following;

Corollary 4.4 :

Let L be a multiplicative lattice.

For $a \in L$, following statements are equivalent,

(1) $\sqrt{a}$ is prime element.

(2) a has single isolated prime element.

Proof :

(1) => (2): Suppose that $\sqrt{a}$ is prime element and $\sqrt{a} = p_1 \wedge p_2$ where p_1 , p_2 are isolated primes. Hence there exists $x, y \in L$ such that $x \leq p_1$, $x \leq p_2$ and $y \leq p_2$, $y \leq p_1$. Hence $xy \leq \sqrt{a} = p_1 \wedge p_2$. But $p_1 \wedge p_2$ is prime implies $x \leq p_1 \wedge p_2$ or $y \leq p_1 \wedge p_2$. In any case we get the contradiction $x \leq p_2$ or $y \leq p_1$. Hence $\sqrt{a} \neq p_1 \wedge p_2$. This can be proved for any finite number of isolated primes.

(2) => (1): Suppose that an element a has single isolated prime p then by Theorem 4.1 it is clear that $\sqrt{a} = p$. ∎

Hereafter L will denote a Noether lattice. Thakare and Manjarekar investigated the notable characteristics between associated prime elements and residuation. We call, q is p-primary if $p = \sqrt{q}$.

Theorem 4.5 :If q is p-primary and if $a \in L$ such that $a \nleq q$ then $(q : a)$ is again p-primary . Also, if $a \leq q$ then $q : a = 1$.

Proof :

If $a \leq q$ then we have $(q : a) = \bigvee \{ z \in L / za \leq q \}$. Since $za \leq q$ for all $z \in L$ it follows that $q{:}a = 1$.

Next suppose that $a \nleq q$ where q is p-primary we show that $(q : a) = q'$ is p-primary. If $y \leq q' = q : a$ then $ay \leq q$. But $a \nleq q$ and q is p-primary imply that $y \leq p$. This shows that $q' \leq p$.

Now let $x \leq p = \sqrt{q}$ then for some positive integer m, $x^m \leq q \leq (q : a) = q'$ (by proposition 2.6). Hence $p \leq \sqrt{q'}$.

Finally, assume that $cd \leq q'$ and $c \nleq p$. Then for $y \leq a$ we have $ycd \leq cda \leq q$ where $c \nleq p$. As q is primary we have $ad \leq q$. Consequently we have
$d \leq (q : a) = q'$. Thus, $(q : a) = q' \leq p$
Hence $q' = (q : a)$ is p-primary. ∎

Theorem 4.6 :

Let L be a Noether lattice and $a \neq 1$ belongs to L and b be any element of L. Then $a : b = a$ iff b is not contained by any prime ideal belonging to a. (i.e. $a : b = a$ iff no prime element of a contains b).

Proof :

Let $a = q_1 \wedge q_2 \wedge \ldots \wedge q_m$ be a normal decomposition of a. Suppose b is not contained by any associated prime of a. i.e. $b \nleq \sqrt{q_i} = p_i$ for any $i = 1, 2, \ldots , m$.
We claim that $a : b = a$. Obviously $a : b \geq a$. Let $x \leq a : b$.
Then $xb \leq a \leq q_i$ for each $i = 1, 2, \ldots , m$.
As q_i is primary, $xb \leq q_i$ and $b \nleq \sqrt{q_i}$ it follows that $x \leq q_i$ for each $i = 1, 2, \ldots , m$. This shows that $x \leq \bigwedge_{i=1}^{m} q_i = a$

and hence $a : b \le a$. Therefore $a : b = a$.

Now assume that $a : b = a$.

To prove that b is contained in no associated prime element of a.

Suppose $b \le p_i$ for some i, say $b \le p_1$ we have $a = a : b$ which implies $a = a : b = (a: b) : b = a : b^2$. And hence,

In general $a = a : b^n$ for all n. As $b \le p_1 = \sqrt{q_1}$ we have $b^r \le {p_1}^r$. We choose n such that ${p_1}^r \le q_1$ so that $b^r \le q_1$.

Then we have $a = (\bigwedge_{i=1} q_i) : b^r$

$$\Rightarrow a = (q_1: b^r) \wedge (q_2 : b^r) \wedge \dots \wedge (q_m : b^r)$$

Now $(q_1 : b^r) = \bigvee \{ z \in L / b^r z \le q_1 \} = 1$.

$$\Rightarrow a = 1 \wedge (q_2 : b^r) \wedge \dots \wedge (q_m : b^r).$$

$$\Rightarrow a = (q_2 : b^r) \wedge \dots \wedge (q_m : b^r).$$

By using Theorem 4.5 it follows that each $(q_i : b^r)$ is either 1 if $b^r \le q_i$ or $(q_i : b^r)$ is p_i-primary.

As $a \neq 1$ it follows that each term cannot be 1. Thus we obtain for a primary decomposition with which the prime element p_1 is not associated. This primary decomposition can be refined into a normal decomposition of a with which p_1 is not associated and has less than m components.

Thus a has two normal decomposition with different number of components. But this contradicts <u>The Fundamental Theorem of normal decomposition</u> which states that ;

" Any two normal decompositions of an element have the same number of components and the same set of associated primes. "

Hence the proof. ∎

The other elegent form of the above theorem 4.6 is as follows :

Corollary 4.7 :

Let L be Noether lattice. Then an element b of L is contained in some associated prime elements of a in L if and only if $a : b \neq a$.

The above corollary gives the uniqueness of the maximal associated primes of a in the following form.

Corollary 4.8 :

Let L be a Noether lattice. An element a is contained in some associated prime element of c if and only if there exist some $b \nleq c$ for which $ab \leq c$.

Proof :

Let $c = q_1 \wedge q_2 \wedge \dots \wedge q_n$ be an irredundant primary decomposition of c and p_i be an associated prime of q_i i.e. $p_i = \sqrt{q_i}$. Suppose there exist some $b \nleq c$ for which $ab \leq c$. To prove that $a \leq p_i$ for some associated prime .

Let if possible $a \leq p_i = \sqrt{q_i}$ for each $i = 1, 2, \dots, m$.

Now $ab \leq c \Rightarrow ab \leq q_i$ for each $i = 1, 2, \dots, m$.

$\Rightarrow b \leq q_i$ for each i.

$\Rightarrow b \leq q_1 \wedge q_2 \wedge \dots \wedge q_m = c$ i.e. $b \leq c$

which is contradiction as $b \nleq c$. Hence $a \leq p_i$ for some i. Conversely, suppose a is contained in some associated prime p_i of $c = q_1 \wedge q_2 \wedge \dots \wedge q_m$. To prove that there exist some $b \nleq c$ for which $ab \leq c$. We known that if $a \neq 1$ then b is contained in no associated prime element of a iff $a : b = a$. Now a is contained in some associated prime element of c implies $c:a \neq c$.

By proposition 2.6 we have $c : a \geq c$.

Hence $c : a \neq c \Rightarrow (c : a) > c$. Put $b = (c : a)$. Then $b = (c : a) \nleq c$ and $ab = a(c : a) = (c : a)a \leq c$ (by proposition 2.1). Therefore there exist some b such that $b = c : a \nleq c$ for which $ab \leq c$. ∎

The concept of zero divisor in the context of r-lattices was introduced by Anderson [1]. After building so many things, Thakare and Manjarekar [20] also discussed the concept of zero divisor in multiplicative lattices which is restated as follows :

Definition 4.9 :

Let L be a Noether lattice. An element a of L is called <u>zero divisor</u> if $(0 : a) \neq 0$.

(i.e. $\bigvee \{ z \in L \mid za = 0 \} \neq 0$).

Thus a is zero divisor in L if there exist at least one non zero z in L such that $za = 0$.

Thakare and Manjarekar relate zero-divisor with associated primes in the next result.

Proposition 4.10 :

Let L be a Noether lattice. Then the join of zero divisor is contained in the join of all associated prime element of 0.

Proof :

Suppose $0 = q_1 \wedge q_2 \wedge \dots \wedge q_m$ is the irredundant primary decomposition of 0. Suppose x is zero divisor in L. Therefore $(0 : x) \neq 0$. By applying corollary 4.7 to $a = 0$ we get $x \leq p_i$ for some associated prime p_i of q_i.

Thus the join of zero divisor is contained in the join of all associated primes p_i of zero. ∎

Theorem 4.11 :

Let L be a multiplicative lattice. For an element $a \in L$. There exist an element $r \leq b$ such that $ar \leq b$ iff $b : a \nleq b$.

Proof :

Suppose for an element $a \in L$, there exist an element $r \leq b$ such that $ar \leq b$. To prove that $b : a \nleq b$. We known that $b : a = \bigvee \{ z \in L / za \leq b \}$. Obviously $b : a \geq b$. It is enough to show that $(b : a) \leq b$. By hypothesis there exist $r \leq b$ such that $ar \leq b$.Hence $(b : a) = \bigvee \{ x \in L / ax \in b \} \leq b$. Hence $b : a \nleq b$.

Conversely, suppose $b : a \nleq b$. As $b : a \geq b$ (by proposition 2.6), $b : a \nleq b \Rightarrow b : a \leq b$. Therefore there exist $r \in \{ z \in L / az \leq b \}$ such that $r \leq b$. i.e. $ar \leq b$ and $r \leq b$. ∎

Thakare and Manjarekar proved an abstract formulation of well known lemma for Noetherian rings which is useful to prove Krull's theorem (see Zariski and Samuel [21]). Which is reported as follows.

Theorem 4.12 :

Let L be a Noether lattice, a be any two elements of L. Then there exist an integer k and an element a' of L such that $ab = a \wedge a'$ and $a' \geq b^k$.

Proof :

Let $ab = q_1 \wedge q_2 \wedge \ldots \wedge q_m$ be the primary decomposition of ab and $p_1, p_2, \ldots, p_m$ be associated primes of $q_1, q_2, \ldots, q_m$. Obviously, $q_i \leq p_i$ ($i = 1, 2, \ldots, m$) and $ab \leq q_i$.

Let $\{ q_i' \}$ be the set of primary components of ab whose associated primes contain b. i.e. $a \leq p_i'$ and let $\{ q_j'' \}$ be the set of primary component of ab whose associated primes do not contain b i.e. $b \nleq p_j''$.

Take $a' = \wedge_i q_i'$ and $a'' = \wedge_j q_j''$.

Then $ab = (\wedge_i q_i') \wedge (\wedge_j q_j'')$.

Therefore $ab = a' \wedge a''$.

$\Rightarrow ab = q_1 \wedge q_2 \wedge \ldots \wedge q_m$ and $ab \leq a' = \wedge_i q_i'$

and $ab \leq a''$. Now $ab \leq a'' = \wedge_j q_j''$.

$\Rightarrow ab \leq q_i'$ for each i and $b \leq p_i' = \sqrt{q_i'}$.

$\Rightarrow b^{r_i} \leq q_i'$ for some i.

Let $k = \max \{ r_i \,/\, (i = 1,2,\ldots,n)\}$ $(n \leq m)$

$\Rightarrow b^k \leq \wedge q_i' = a'$. Take any element $y_j \leq b$ such that $y_j \nleq \sqrt{q_j''} = p_j''$. For any element $x \leq a$, $y_j x \leq ab \leq q_j''$ $(j = 1,2,\ldots,t)$. Now $ab \leq q_j''$ and $b \nleq p_j''$ $\Rightarrow$ $a \leq q_j''$ (by definition of primary element)

$$\Rightarrow a \leq \wedge_j q_j'' = a''.$$

Since $ab = ab \wedge a = (a' \wedge a'') \wedge a \Rightarrow ab = a' \wedge (a'' \wedge a)$

Therefore $ab = a \wedge a'$ where $b^k \leq a'$. ∎

Theorem 4.13 :

If q is p-primary and if $a \nleq p$ then $(q : a) = q$.

Proof :

By proposition 2.1 we have $a (q : a) \leq q$. Let $x \in \{ z \in L \,/\, za \leq q \}$. Since q is p-primary we have $xa \leq q$ and $a \nleq p$. We have $x \leq q$.

Hence $(q : a) = \vee \{ z \in L \,/\, za \leq q \} \leq q$. Thus $q : a \leq q$.

Also by proposition 2.6, we have $q \leq q : a$.

Therefore $(q : a) = q$. ∎

Finally, we give our original result.

Theorem 4.14 :

Let p, q be elements of a multiplicative lattice L such that

(1) $q \le p \le \sqrt{q}$.

(2) if $ab \le q$ with $a \nleq p$ then $b \le q$.

Under these conditions q is a primary element of L with $p = \sqrt{q}$.

Proof :

To show that q is primary element. Suppose that $ab \le q$ but $b \nleq q$. Using (2) we conclude that

$a \le p \le \sqrt{q} = \bigvee \{ z \in L \ / \ z^n \le q,$ for some integer $n \}$.

This shows that $a^n \le q$ and q is primary. Next, to show that $p = \sqrt{q}$, we have to prove that $\sqrt{q} \le p$.

Let $x \le \sqrt{q} = \bigvee \{ z \in L \ / \ z^n \le q$ for some integer $n \}$. This implies that $x^n \le q$ for some integer n. Suppose m is the least positive integer such that $x^m \le q$. If $m = 1$ then $x \le q \le P$. If $m > 1$ then $x^m = x^{m-1} \cdot x \le q$ with $x^{m-1} \nleq q$ and hence $x \le p$ [by using (2)]. In any case we have $x \le \sqrt{q} \Rightarrow x \le p$. Hence $\sqrt{q} \le p$. Therefore we conclude that $p = \sqrt{q}$. ∎

We wish to mention that the efforts of Professor Thakare and his group with the active involvement of Professor Shichiro Maeda of Japan have extended not only the theory of symmetricity but also the theory of multiplicative lattices. We wish to abstract several analogues of decomposition theorems for commutative rings and various ideal theoretical results in future.

*** THE END ***