CHAPTER - 4

APPLICATIONS OF A CHI-SQUARE DISTRIBUTION TO INDUSTRIAL STATISTICS :

4.1 Introduction :

In this chapter, we concentrate on the applications of chi-square distribution in industrial statistics. In section 4.2, we have used chi-square distribution to compute the probability of accepting a lot of quality θ. In the same section chi-square distribution is used to find the parameters n (sample size) and c (rejection number) of the single sampling plan by attributes, so that the resulting plan has oc (operating characteristic) function passing through the producer's risk point $(\theta_1, 1-\alpha)$ and consumer's risk point (θ_2, β).

Section 4.3 deals with applications of chi-square distribution in acceptance sampling by variables for exponential distribution to find n and k so that the resulting plan has oc function passing through the producer's risk point $(\theta_1, 1-\alpha)$ and consumer's risk point (θ_2, β) when lower and upper specification limit is given. In last section application of chi-square distribution is given in case of variable plan for normal distribution with mean μ (known) and variance 6^2 (unknown) to determine n and k, when lower and upper specification limit is given.

Single sampling plan : Here we shall explain single sampling plan by variables :

Suppose that a sample of n items are chosen at random without replacement from the lot. All the n items are measured.. Let $X_1, X_2, \ldots, X_n$ be the measurements. Using $X_1, \ldots, X_n$ an estimate $\hat{\theta}$ of $\theta = F(L; \eta)$ is obtained. If $\hat{\theta} \leq \theta_o$ (a given number) then the lot is accepted; otherwise the lot is rejected. Hence $\hat{\theta}$ is a function of $X_1, \ldots, X_n$. it is single sampling plan by variables and the probability of accepting the lot, which is a function of θ, based on the sampling plan SP, is called the operating characteristic (oc) function of the sampling plan sp. The oc function of the plan is denoted by $L_{sp}(\theta)$.

4.2 Applications of a chi-square distribution to determine the parameters n and c :

In this section, we apply a chi-square distribution to determine the parameters n and c, where 'n' is sample of size taken from a lot and 'c' is the rejection number. The probability of accepting a lot of quality θ is,

$$L(\theta) = \text{Prob.}\left[\text{accepting the lot when the number of defective in the lot are } N\theta\right]$$

$$= P\left[D_n \leq c \mid \text{when the lot quality is } \theta\right]$$

$$= \sum_{d=0}^{c} p\left[D_n = d \mid \text{when the lot quality is } \theta\right]$$

$$= \sum_{d=0}^{c} \binom{N\theta}{d}\binom{N-N\theta}{n-d} \Big/ \binom{N}{n} \qquad \ldots (4.2.1)$$

where N = total number of items in a lot

d = number of defectives in the lot

D_n = number of defectives in the sample of size n.

If N is large the hypergeometric distribution can be approximated by binomial distribution Duncan (1970). So (4.2.1) can be written as

$$L(\theta) \simeq \sum_{k=0}^{c} \binom{n}{k} \theta^k (1-\theta)^{n-k} \qquad \ldots (4.2.2)$$

Replacing the binomial probabilities by poisson probability having the same mean, then (4.2.2) can be written as

$$L(\theta) \simeq \sum_{k=0}^{c} \exp(-n\theta) \, (n\theta)^k / k! \qquad \ldots (4.2.3)$$

In sub-section 1.3.3, we have shown the relationship of chi-square distribution and poisson distribution. Therefore (4.2.3) can be written as

$$L(\theta) = p \left[\chi^2_{2(c+1)} \geq 2n\theta \right] \qquad \ldots (4.2.4)$$

where $\chi^2_{2(c+1)}$ is a chi-square variable with 2(c+1) d.f.

Now we use the equation (4.2.4) to find the parameters n and c of the single sampling plan by attributes, so that the resulting plan has oc function passing through the producer's risk point $(\theta_1, 1-\alpha)$ and consumer's risk point (θ_2, β). Since in (4.2.2) n and c are integers, we determine n and c such that

$$\sum_{k=0}^{c} \binom{n}{k} \theta_1^k (1-\theta_1)^{n-k} \geq 1-\alpha \quad \ldots (4.2.5)$$

and $$\sum_{k=0}^{c} \binom{n}{k} \theta_2^k (1-\theta_2)^{n-k} \leq \beta \quad \ldots (4.2.6)$$

Then n and c which satisfy the constraints (4.2.5) and (4.2.6), assume the producer's risk of atleast $(1-\alpha)$ and consumer's risk of atmost β. From (4.2.4) the inequalities (4.2.5) and (4.2.6) can be written as

$$p\left[\chi^2_{2(c+1)} \geq 2n\theta_1\right] \geq 1-\alpha \quad \ldots (4.2.7)$$

and $$p\left[\chi^2_{2(c+1)} \geq 2n\theta_2\right] \leq \beta \quad \ldots (4.2.8)$$

Let $\chi^2_{2(c+1),\, p}$ denote the lower p^{th} quantile of chi-square distribution with 2(c+1) d.f., that is

$$p(\chi^2_{2(c+1)} \leq \chi^2_{2(c+1),\, p}) = p \quad \ldots (4.2.9)$$

Using (4.2.9) in (4.2.7) and (4.2.8), we get

$$p(\chi^2_{2(c+1)} \geq 2n\theta_1) \geq p(\chi^2_{2(c+1)} > \chi^2_{2(c+1),\, \alpha}) \ldots (4.2.10)$$

and $$p(\chi^2_{2(c+1)} \geq 2n\theta_2) \leq p(\chi^2_{2(c+1)} > \chi^2_{2(c+1),\, 1-\beta}) \ldots (4.2.11)$$

so that

$$2n\theta_1 \leq \chi^2_{2(c+1),\, \alpha} \quad \ldots (4.2.12)$$

and $$2n\theta_2 \geq \chi^2_{2(c+1),\, 1-\beta}) \quad \ldots (4.2.13)$$

By taking ratio of (4.2.12) and (4.2.13), we get

$$\frac{2n\theta_2}{2n\theta_1} \geq \frac{\chi^2_{2(c+1),\,1-\beta}}{\chi^2_{2(c+1),\,\alpha}}$$

That is

$$\frac{\theta_2}{\theta_1} = \frac{\chi^2_{2(c+1),\,1-\beta}}{\chi^2_{2(c+1),\,\alpha}} = r(c) \qquad \ldots (4.2.14)$$

From (4.2.14) it is observed that for various values of α and β such that $1-\beta > \alpha$, $r(c)$ is decreasing function of c. This result is clear from tables I to IV given by S.N. Kulkarni (1987) in his dissertation. Since θ_2/θ_1 is given we choose c such that $r(c-1) > \frac{\theta_2}{\theta_1} \geq r(c)$ ... (4.2.15)

The ratio (4.2.15) is tabulated in Cameron (1952) for some chosen values of c, α and β. After determining c; n can be found out from (4.2.12) and (4.2.13) which give the condition that

$$\frac{\chi^2_{2(c+1),\,1-\beta}}{2\theta_2} \leq n \leq \frac{\chi^2_{2(c+1),\,\alpha}}{2\theta_1} \qquad \ldots (4.2.16)$$

If there is no n satisfying (4.2.16) then increase the value of c until such a n can be found.

4.3 Applications of chi-square distribution in case of acceptance sampling by variables for exponential distribution :

In this section, we apply chi-square distribution when the measurements on the items in the lot has an exponential distribution. Here we consider one parameter exponential distribution when lower specification limit L is given. Let X be the r.v. that follows an exponential distribution with p.d.f. is given by

$$f(x, \sigma) = \frac{1}{\sigma} \exp(-x/\sigma), \quad x > 0 \qquad \ldots (4.3.1)$$

where σ is the parameter of the distribution. Let θ be the probability of an item being defective, then

$$\begin{aligned} \theta &= P_{\sigma}(X_1 \leq L) \\ &= P_{\sigma}\left(\frac{2X_1}{\sigma} \leq \frac{2L}{\sigma} \right) \\ &= Q_2\left(\frac{2L}{\sigma} \right) \end{aligned}$$

That is

$$\frac{2L}{\sigma} = Q_2^{-1}(\theta) \qquad \ldots (4.3.2)$$

where $Q_2(.)$ is a chi-square distribution with 2 d.f.

Now we accept the lot if $\theta \leq \theta_o$ and reject it otherwise. If θ is known the problem is quite trivial, but if θ is unknown then we have to estimate $\hat{\theta}$ by choosing an appropriate method. By using Rao-Blackwell-lehmann-scheffe theorem and Basu's theorem the MVUE is given by

$$\hat{\theta} = 1 - \left(1 - \frac{L}{n\bar{X}}\right)^{n-1} \qquad \ldots (4.3.3)$$

Using (4.3.3) the oc function can be written as

$$L(\sigma) = P_\sigma \text{(accepting the lot)}$$

$$= P_\sigma\left[1 - (1 - \frac{L}{n\bar{X}})^{n-1} \leq \theta_o\right]$$

$$= P_\sigma\left[1 - \frac{L}{n\bar{\bar{X}}} \geq (1 - \theta_o)^{1/n-1}\right]$$

$$= P_\sigma\left[n\bar{X} \geq \frac{L}{1 - (1-\theta_o)^{1/n-1}}\right]$$

$$= P_\sigma\left[\frac{2n\bar{X}}{\sigma} \geq \frac{2L}{\sigma\left[1-(1 - \theta_o)^{1/n-1}\right]}\right]$$

$$= P_\sigma\left[\chi^2_{2n} \geq \frac{2KL}{\sigma}\right]$$

$$= 1 - Q_{2n}\left(\frac{2KL}{\sigma}\right) \qquad \ldots (4.3.4)$$

where $K = \dfrac{1}{1 - (1-\theta_o)^{1/n-1}}$ and Q_{2n} is chi-square distribution with 2n d.f. we find n and k so that the resulting plan* has oc function passing through the producer's risk point $(\theta_1, 1 - \alpha)$ and consumer's risk point (θ_2, β). Using (4.3.4) we get the following two equations

$$\frac{2KL}{\sigma_1} = Q^{-1}_{2n}(\alpha)$$

$$= \chi^2_{2n,\alpha} \qquad .. (4.3.5)$$

and $$\frac{2KL}{\sigma^2} = Q^{-1}_{2n}(1-\beta)$$

$$= \chi^2_{2n,\ 1-\beta} \qquad \ldots (4.3.6)$$

Using (4.3.2) in (4.3.5) and (4.3.6), we get

$$KQ_2^{-1}(\theta_1) = Q_{2n}^{-1}(\alpha)$$

That is $K\chi^2_{2,\theta_1} = \chi^2_{2n,\alpha}$ ... (4.3.7)

and $K\chi^2_{2,\theta_2} = \chi^2_{2n,1-\beta}$ ... (4.3.8)

dividing (4.3.8) by (4.3.7), we get

$$\frac{\chi^2_{2,\theta_2}}{\chi^2_{2,\theta_1}} = \frac{\chi^2_{2n,1-\beta}}{\chi^2_{2n,\alpha}} \quad \text{... (4.3.9)}$$

Using (4.3.9) by trial we can find the value of n and from (4.3.7) and (4.3.8) we get

$$k = \chi^2_{2n,\alpha} / \chi^2_{2,\theta_1} \quad \text{... (4.3.10)}$$

and $k = \chi^2_{2n,1-\beta} / \chi^2_{2,\theta_2}$ ... (4.3.11)

It is found that from (4.3.10) the resulting oc function of the plan passes through the producer's risk point $(\theta_1, 1-\alpha)$ and from (4.3.11) it passes through the consumer's risk point (θ_2, β).

Similarly chi-square distribution is used when upper specification limit U is given for exponential distribution.

4.4 : Application of chi-square distribution in case of variable plan for normal distribution :

In this section, we apply chi-square distribution when the measurements on the items in the lot has a normal distribution with mean μ(known) and variance σ^2 unknown when lower specification limit L is given. Let $X_1, \ldots, X_n$ be the measurements on the n items chosen at random from the lot and $X_1, \ldots, X_n$ are iid normal with mean μ and variance σ^2.

Let θ be the probability of an item being defective, then

$$\theta = P_\sigma (X \leq L)$$

$$= P_\sigma \left(\frac{X - \mu}{\sigma} \leq \frac{L - \mu}{\sigma} \right)$$

$$= \Phi \left(\frac{L - \mu}{\sigma} \right) \qquad \ldots (4.4.1)$$

where $\Phi(.)$ is standard normal distribution function.

By using Rao-Blackwell-Lehmann-Scheffe theorem and Basu's theorem, the MVUE of θ is given by

$$\hat{\theta} = F_{t_{n-1}} \left[\frac{(n-1)^{1/2} \left(\frac{(L-\mu)}{S} \right)}{\left(n - \left(\frac{L - \mu}{S} \right)^2 \right)^{1/2}} \right] \qquad \ldots (4.4.2)$$

where $F_{t_{n-1}}$ (X) is the df of t variate with (n-1) d.f.

using the estimator $\hat{\theta}$, the criteria for accepting or rejecting the lot is as

Accept the lot if $\hat{\theta} \leq \theta_o$ and reject it otherwise.

where $S^2 = \sum_{i=1}^{n} (X_i - \mu)^2/n-1$

But $\hat{\theta} \leq \theta_o$ iff $F_{t_{n-1}} \left[\dfrac{(n-1)^{1/2} \left(\frac{L-\mu}{S} \right)}{\left(n - \left(\frac{L-\mu}{S} \right)^2\right)^{1/2}} \right] \leq \theta_o$

That is

$$\frac{(n-1)^{1/2} \left(\frac{L-\mu}{S} \right)}{\left(n-\left(\frac{L-\mu}{S} \right)^2 \right)^{1/2}} \leq - F^{-1}_{t_{n-1}} (\theta_o) = - k' \quad \ldots (4.4.3)$$

where $k' = - F^{-1}_{t_{n-1}} (\theta_o)$. Solving (4.4.3) we get

$$S^2 = \left(\frac{n-1}{k'^2} + 1\right) \left(\frac{(L-\mu)^2}{n} \right) \quad \ldots (4.4.4)$$

Using (4.4.4) the oc function can be obtained as

$$\begin{aligned} L(\sigma) &= P_{\sigma} \text{(Accepting the lot)} \\ &= P_{\sigma} \left[S^2 \leq \left(\frac{n-1}{k'^2} + 1\right) \frac{(L-\mu)^2}{n} \right] \\ &= P_{\sigma} \left[(n-1)S^2/ \sigma^2 \leq \frac{(n-1)}{n} \left(\frac{n-1}{k'^2} +1\right) \left(\frac{L-\mu}{\sigma}\right)^2 \right] \\ &= P_{\sigma} \left[\chi^2_n \leq k Z^2_\theta \right] \\ &= Q_n(K Z^2_\theta) \end{aligned} \quad \ldots (4.4.5)$$

where $k = \frac{(n-1)}{n} \left(\frac{n-1}{k'^2} + 1\right)$ and $Q_n(\cdot)$ chi-square distribution with n d.f., $Z_\theta = \frac{L-\mu}{\sigma}$.

Now we find n and k so that the resulting plan has oc function passing through the producer's risk point (θ_1, $1-\alpha$) and consumer's risk point (θ_2, β)

Using (4.4.5), we get following two equations

$$KZ^2_{\theta_1} = Q^{-1}_n (1-\alpha)$$

$$= \chi^2_{n',} (1-\alpha) \qquad \ldots (4.4.6)$$

and $$KZ^2_{\theta_2} = \chi^2_{n',} \beta \qquad \ldots (4.4.7)$$

From (4.4.6) and (4.4.7), we have

$$Z^2_{\theta_2} / Z^2_{\theta_1} = \chi^2_{n',} \beta / \chi^2_{n',} 1-\alpha \qquad \ldots (4.4.8)$$

and from (4.4.6), we get

$$K = \chi^2_{n',} 1-\alpha / Z^2_{\theta_1} \qquad \ldots (4.4.9)$$

and from (4.4.7), we have

$$K = \chi^2_{n} , \beta / Z^2_{\theta_2} \qquad \ldots (4.4.10)$$

It is found that from (4.4.9) the oc function of the plan passes through the producer's risk point (θ_1, $1-\alpha$) and if it is found from (4.4.10) it passes through the consumer's risk point (θ_2, β).

Similarly chi-square distribution is used when upper specification limit is given for normal distribution with mean μ(known) and variance σ^2 (unknwon).

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