CHAPTER 3 .

FACTORIAL EXPERIMENTS .

In this chapter we give the detail about 'factorial experiments'. In section 3.1 we give introduction of it along with the historical account. Section 3.2 is used for notaions and some definitions. Analysis of 'symmetrical factorial experiments' is given in the section 3.3 . The necessity of confounding and different methods of confounding are given in section 3.4 . In section 3.5 the need of 'fractional replication' is explained and different plans are given.

3.1. : INTRODUCTION :-

At the end of Chapter 1 , we have introduced the term , 'factorial experiments '. In this chapter and in the remaining chapters we will discuss about it , in detail.

In practical life factorial experiments are widely applied. As an illustration let us consider the following situtation :

We know that the yield of a particular type of crop mainly depends upon the irrigation levels and the different manures to be applied on it. Suppose there are different levels of irrigation and different types of marures. And suppose, the investigator has to find the effects of these factors on the yield of a crop.

One way to ascertain the effects is to carry out two separ-

ate simple experiments. One for testing the effects of different levels of irrigation and other to test the effects produced by different manures. In the first experiment one can find the optimum level of irrigation and thereafter in the second, one can determine the optimum level of manures.

In the above set up we have assumed that the two factors act independently, on the yield of crop. Obviously this cannot be true in practice. The yield of a crop does depend upon the combined effects of these two factors. For, many times high level of irrigation with strong dose of manures gives more yields and on the otherhand the lowest level of irrigation with strong dose of manure may give less yield. Hence it is clear that above two factors are correlated. In such situations it is essential to consider all possible combinations of different levels of different factors and to select which combinations affect the yield really. And, precisely this can be achieved by performing, 'factorial experiments '. In the next paragraph, we give in brief the historical development of 'factorial experiments '.

Prior to 1926 , factorial experiment was called the, 'complex experiment '. Fisher designated it as, 'factorial experiment '. Now it is known almost exclusively as a 'factorial experiment '. Yates (1935) states that the factorial experiments have been used on wheat trials at Broadbalk since 1843 and on barely trials at Hoosfield since 1852 .

Fisher and Yates are mainly responsible for the development and analysis of factorial experiments. The classic work on factorial experiment is, 'the pamphelt written by Yates (1937), enti-

tled, 'The Design And Analysis of Factorial Experiments'. Besides these two, Barnard (1936), Bose (1938, 1939, 1942, 1947, etc.) Cochran and Cox (1959), Kempthrone (1952), Fedrer (1955), among the others deserve no small amount of credit. And precise literature on this topic is available in the books by Kempthrone (1952), Fedrer (1955), Cochran and Cox (1959), John (1971), Ogawa (1974), Das and Giri (1979), Fedrer, Hedayat and Raktoe (1981) among others.

It must be mentioned here that in the literature, many a time, the phrase "Factorial Design" is being used. However, this is actually a misnomer for, there is no such thing as a factorial design. The adjective "factorial " refers to a special way of forming treatment combinations and not to any basic type of design. Yates (1937), and others have recognized this situation and they refer it to as 'factorial experiments' rather than 'factorial designs '. In our discussion, we will follow Yates .

In the next section we give some notations and difinitions which are helpful in further discussion.

3.2. : NOTATIONS AND DEFINITIONS :-

In the study of factorial experiments we will use the following notations .

The factors are denoted by capital letters such as by $A_1, A_2, A_3, - - - -$ or by $A, B, C, - - - - -$. We use 'X ' to denote a factor, in general. The i th factor is at s_i levels and these are denoted by $0, 1, 2, - - - s_{i-1}$ or $x_{1i}, x_{2i}, - - - -, x_{s_i,i}$; $1 \leq i \leq n$. A factorial experiment with n

factors : $A_1, A_2, - - -, A_n$ with $s_1, s_2, - - -, s_n$ levels

respectively is denoted by '$s_1 * s_2 * - - - * s_n$ - factorial experiment ', with $v = s_1 * s_2 * - - - * s_n$ treatment combinations.

Further, treatment combinations are denoted by corrosponding small letters, such as (1) , a_1 , a_2 , $a_1 a_2$, a_3 , $a_1 a_3$, $a_2 a_3$, - - -or

by (1), a, b, ab, c, ac, bc, - - - etc. or

by $a^x \; b^x \; c^x - - -$; where x_i ,

denotes the x_i th level of corrosponding factor ; $0 \leq x_i \leq s_{i-1}$.

Next, we use (x) and [x] to denote mean and total of all observations receiving treatment combination, 'x '. A main effect of factor X is denoted by X ; and interaction between the factors X, Y, Z is denoted by XYZ , and so on .

Further, let r_j denotes the number of replications of j th treatment combination ; $1 \leq j \leq s_1 * s_2 * - - - * s_n$.

SOME DEFINITIONS :-

Below we present few definitions [Raktoe, Hedayat and Federer (1981)] which are necessary in further discussion :

Definition : 3.2.1. : COMPLETE FACTORIAL EXPERIMENT (CFE):-

A factorial experiment is a complete factorial experiment or simply ' a complete replicate ', if $r_j \geq 0$ for all j ; $1 \leq j \leq v$ in each replication , i.e. every treatment combination should appear at least once in each replication. Shortly we can call it as CFE.

Definition : 3.2.2.: MINIMAL COMPLETE FACTORIAL EXPERIMENT (MCFE)

A complete factorial experiment (CFE) is said to be minimal complete if $r_j = 1$, for all j ; $1 \leq j \leq v$ i.e. each treatment combination occurs once and only once in each replication Shortly it can be called as 'M C F E '.

Remark :- The MCFE is often refered to as a single complete replicate of the $s_1 * s_2 * - - - * s_n$ --factorial experiment.

And if $r_j = r$, then a CFE is said to consist of r complete replicates of the $s_1 * s_2 * - - - * s_n$ --factorial experiment .

Definition : 3.2.3 : SYMMETRICAL FACTORIAL EXPERIMENT .

A factorial experiment is defined as ssymmetrical if $s = s_i$ for all i , $1 \leq i \leq n$; otherwise it is asymmetrical.

Further, a factorial experiment is said to be prime powered if $s_i = p_i^{u_i}$, for each i and p_i is a positive prime number and u_i is a natural number greater than or equal to 1 .

Remark :- An factorial experiment may be symmetrical prime powered or asymmetrical prime powered.

In symmetrical factorial experiment,

$s = s_1 = s_2 = - - - - = s_n = s$, so we get

$v = s_1 * s_2 * - - - * s_n = s^n$ --treatment combinations.

And factorial experiment is called s^n --factorial expriment.

Definition : 3.2.4. : INCOMPLETE FACTORIAL EXPERIMENT .

A factorial experiment is said to be an incomplete factorial experiment, or fractional factorial, or more simply fractional replicate, if $r_j = 0$ for some j ; $1 \leq j \leq v$ i.e. some of treatment combinations out of total, are not occuring in that replicate.

For the illustration, let us consider an exmple of a $2 * 3$ factorial experiment with two factors A and B at levels 2 and 3 respectively. The levels of factor A are { 0, 1 } and that of B are { 0, 1, 2 } . The possible treatment combinations with these levels are

(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2) .

A factorial experiment with treatment combinations , in each replication (0, 0), (0, 0), (0, 1), (0, 2), (1, 0), (1, 0), (1, 1), (1, 2) is a CFE and it is minimal complete with treatment combinations (0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2) . And a factorial experiment with treatment combinations , (0, 0), (0, 2), (0, 2), (1, 1), (1, 2) is fractional factorial . Further, since $s_1 =/= s_2$,it is asymmetrical factorial experiment and $s_1 = 2$, $s_2 = 3$ so it is asymmetrical prime powered factorial experiment.

Definition 3.2.5 :- THE ORTHOGONAL POLYNOMIAL MODEL

For simplicity, we develope the case for n=1 , i.e. for single factor and at later stage we generalise it for n factors.

Denote the levels of this single factor by $(x_1, x_2, - - -, x_s)$

and let Y_{xj} be the observation at the x_j th level.

Definition 3.2.5 :- The full polynomial model for the single factor factorial is given by the equation [Raktoe,Hydat etc.]

$$E(Y_x) = \beta_0 p_0(x_j) + \beta_1 p_1(x_j) + - - - + \beta_{s-1} p_{s-1}(x_j) \quad ----(3.2.1)$$

Where,

$$p_j(x_j) = x_j^w, \qquad j = 1, 2, - - -, s \; ; \; w = 0, 1, 2, - -, s-1.$$

and $\beta_0, \beta_1, - - -, \beta_{s-1}$ are the parameters. As in regression theory, here also β_0 is called intercept, β_1 is the linear regression coefficient, β_2 is the quadratic regression coefficient and so on. In matrix for the model (3.2.1) can be written as,

$$E(Y) = P\beta \qquad -------(3.2.2)$$

Where, P is a sxs matrix with (j,w) th entry being equal to x_j^w ; $w = 0, 1, 2, - - -, s-1$; $j = 1, 2, - - -, s$.

Let, H be the triangular Gram-Schmidt transformation matrix that orthonormalises the columns of P from left to right It follows that equation (3.2.2) can be written as

$$E(Y) = P H^{-1} H \beta = M\Phi \qquad -----(3.2.3)$$

Where, $M = P H^{-1}$ and $\Phi = H B$. This model is called the single factor orthogonal polynomial model.

The Gram-Schmidt process is explained as follows .

Indicate the columns of P and M by $P_0, P_1, - - -, P_{s-1}$ and $M_0, M_1, - - -, M_{s-1}$ respectively. Then, the Gram-Schmidt process is given by -

$$W_0 = P_0 \quad ,$$

$$W_1 = P_1 - \frac{W_{00}^{P_{01}}}{W_{00}} W_0 \quad ,$$

$$W_2 = P_2 - \frac{W_{11}^{P_{12}}}{W_{11}} W_1 - \frac{W_{00}^{P_{02}}}{W_{00}} W_0 \quad ,$$

$$W_{s-1} = P_{s-1} - \frac{W_{s-1\,s-1}^{W_{s-2}\,P_{s-1}}}{W_{s-1\,s-1}} W_{s-2} - \cdots - \frac{W_{00}^{P_{0\,s-1}}}{W_{00}} W_0 \quad ,$$

and

$$M_i = \frac{W_i}{||W_i||} \quad , \quad i = 0, 1, \ldots, s-1 \quad .$$

Here, U.V means the usual dot product $\sum_i u_i . v_i$ of the vectors U and V, and

$||U|| = \left(\sum_i u_i^2 \right)^{1/2}$, is the length of vector U .

For the illustration we consider a factor at three levels viz. 0, 1 and 2 . Then

$$P = \begin{bmatrix} 1 & x & x^2 \\ & 1 & 1 \\ & & 2 \\ 1 & x & x^2 \\ & 2 & 2 \\ & & 2 \\ 1 & x & x^2 \\ & 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ & & \\ 1 & 1 & 1 \\ & & \\ 1 & 2 & 4 \end{bmatrix} .$$

Then by using Grams-Schmidt procedure we get

$$M = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -\sqrt{2/3} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$$

and $\underset{\sim}{\Phi} = (\phi^0, \phi^1, \phi^2)$ is the parametric vector. If $\underset{\sim}{\Phi} = (\phi^0, \phi^1, \phi^2, - - - -, \phi^{s-1})$, then in regression theory ϕ^0 is called the intercept, ϕ^1 is the linear regression coefficient eliminating the intercept and ignoring all higher degree terms, ϕ^2 is the quadratic regression coefficient eliminating the intercept and the linear coefficient and ignoring all higher degree terms and so on. This is due to the orthogonality of matrix .

Definition 3.2.6 :- A real n -tuple $(x_1, x_2, - - -, x_n)$ is said tobe less than a real r -tuple $(y_1, y_2, - - -, y_n)$ if and only if for the first u such that $x_u \neq y_u$, we have $x_u < y_u$; $1 \leq u \leq n$.

And a set of real n -tuples is said to be lexicographically ordered if it is ordered as above.

Definition 3.2.7 :- The left Kronecker product of two matrices $A_{mxn} = (a_{ij})$ and $B_{rxs} = (b_{ij})$ is equal to the m.r x n.s matrix

$$\begin{bmatrix} a_{11}B & a_{12}B & - - & a_{1n}B \\ a_{21}B & a_{22}B & - - & a_{2n}B \\ - & - & & - \\ a_{m1}B & a_{m2}B & - - & a_{mn}B \end{bmatrix}$$

Definition 3.2.8 :- The symbolic left Kronecker product of two vectors $X_{m\times 1}$ and $Y_{n\times 1}$ is equal to m.n x 1 vector

$$X \otimes Y = (x_1 y_1 , x_1 y_2 , - - -, x_1 y_n , - - -, x_m y_1 , x_m y_2 , - -, x_m y_n) .$$

With these definition now we generalise the orthogonal polynomial model for the case $n \geq 2$. Let the levels of each factor be ordered in increasing order. Let Y_p be the observation vector corrosponding to minimal complete factorial design p . The n -factor orthogonal polynomial model is given by

$$E (Y_p) = X_p \beta_p \qquad \text{-------(3.2.3)}$$

where, the subscripts of Y_p are lexicographically ordered .

$$X_p = M_1 \otimes M_2 \otimes - - - \otimes M_n \qquad \text{------(3.2.4)}$$

and

$$\beta_p = \Phi_1 \otimes \Phi_2 \otimes - - - \otimes \Phi_n \qquad \text{------(3.2.5)}$$

The matrix M_i and the vector Φ_i are the design matrix and parametric vector for the i th factor after left to right orthonormalisation. The Kronecker product $M_1 \otimes M_2 \otimes - - - \otimes M_n$ and the symbolic Kronecker product $\Phi_1 \otimes \Phi_2 \otimes \Phi_3 \otimes - - - \otimes \Phi_n$ are obtained by generalising the definitions 3.2.7 and 3.2.8 . It follows that the superscripts of the elements of B_p are also lexicographically ordered . The elements of B_p are called as factorial effects . For the illustration we refer the analysis of 3^2 -factorial experiments in the section 3.3.

In the remaning part of this chapter, we will discuss symmetrical factorial experiments in detail and in thapter 4, we will discuss about asymmetrical factorial experiments.

3.3 : SYMMETRICAL FACTORIAL EXPERIMENTS :-

Suppose there are n factors each at s levels. Then we will have $v = s^n$, total treatment combinations. And these v treatment combinations can be compared by using any one of standard designs such as CRD, RBD or LSD .

Now for the simplysity we study some particular cases, such as 2^2, 2^3, - - - etc. factorial experiments.

THE 2^2 --FACTORIAL EXPERIMENT :-

Suppose there are two factors, A and B each at two levels, so the total treatment combinations, $v = 4$. And these are (1) a, b and ab . These four treatment combinations can be tested by using RBD . For this we develop a model as

$$Y_{ij} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + e_{ij} \quad ; \quad i, j = 1, 2 \qquad \text{_____(3.3.1)}$$

Where, Y_{ij} denotes the random variable corrosponding to the observation y_{ij} is the observatior on i th level of factor A and j th level of factor B . μ is grand mean , α_i , is the effect due to ith level of factor A, β_j is the effect due to j^{th} level of factor B . $(\alpha\beta)_{ij}$ is the interraction effect of i-th level of factor A with j th level cf factor B and e_{ij} is the error term, which is assumed as distributed independently normally with mean and constant variance σ^2 .

To get least square estimates, we minimise

$$\sum_i \sum_j e_{ij}^2 = \sum_i \sum_j [\, y_{ij} - \mu - \alpha_i - \beta_j - (\alpha\beta)_{ij} \,]^2 \qquad \text{-----(3.3.2)}$$

with respect to , μ , α_i , β_j and $(\alpha\beta)_{ij}$. And to get unique solutions for normal equations we assume ,

$$\sum_{i=1}^{2} \alpha_i = 0 \ , \ \sum_{j}^{2} \beta_j = 0, \quad \sum_i (\alpha\beta)_{ij} = 0 \ , \ \sum_j (\alpha\beta)_{ij} = 0 \ ,$$

that is , $\alpha_1 + \alpha_2 = 0$.

which gives, $\alpha_2 = - \alpha_1 = \alpha$, say

Similarly ,

$$\beta_2 = - \beta_1 = \beta \ ,$$

and

$$(\alpha\beta)_{22} = - (\alpha\beta)_{12} = - (\alpha\beta)_{21} = (\alpha\beta)_{11} = (\alpha B) \text{. say.}$$

with this change, equation (3.3.2) can be rewritten as

$$\sum_i \sum_j e_{ij}^2 = \sum_i \sum_j [y_{ij} - u - (-1)^i \alpha - (-1)^j \beta - (-1)^{i+j} (\alpha B)\,]^2$$

$$\text{-----(3.3.2)}$$

$$i, j = 1, 2 \ .$$

Differentiating partially and equating with zero, we get the least square estimates $\hat{u}$, $\hat{\alpha}$ and $\hat{\beta}$ of u , α_i and β_j respectively.

Hence ,

$$4\,\hat{u} = y_{11} + y_{12} + y_{21} + y_{22}$$

i.e. $\hat{u} = (ab) + (b) + (a) + (v)$ _____(3.3.4).

Also,

$$4\,\hat{\alpha} = (ab) - (b) + (a) - (1) \qquad \text{_____(3.3.5).}$$

or, in the simplified form

$$4\,\hat{\beta} = (a-1)(b+1) \qquad \text{_____(3.3.6).}$$

Similarly, we get

$$4\,\hat{\beta} = (ab) - (a) + (b) - (1) \qquad \text{_____(3.3.7).}$$

or

$$4\hat{\beta} = (a+1)(b-1) \qquad \text{(3.3.8)}.$$

and

$$4(\hat{\alpha\beta}) = (ab) - (a) - (b) + (1) \qquad \text{(3.3.9)}.$$

or

$$= (a-1)(b-1) \qquad \text{(3.3.10)}.$$

From equation number (3.3,5),(ab) - (b) can be interpreted as the difference in average yield when factor A is changed from first level to second level, when factor B is at second level. And this difference is called 'simple effect ' of factor A at the second level of B . And (a) - (1) is simple effect of A when factor B is at first level. Further, the average of these two simple effects is called as, " main effect " of factor A . And it is denoted by A.

Hence ,

$$A = 1/2\,[(ab) - (b) + (a) - (1)] \qquad \text{(3.3.11)}.$$

By comparing equation rumbers (3.3.5) and (3.3.11), we get

$$A = 2\hat{\alpha} \quad .$$

Similarly, we get " main effect " due to factor B , as

$$B = 2\hat{\beta} \quad .$$

Further, when two simple effects of factor A at different levels of factor B are not equal to each other , it implies that factors A and B are depending upon each other . And the half of difference of (a) - (1) from (ab)-(b) is defined as interaction effect or measure of dependence between A and B.

$$\therefore \quad AB = 1/2\,[(ab) - (b) - (a) - (1)] \qquad \text{(3.3.12)}.$$

which is also

$$AB = 2(\hat{\alpha\beta}) \quad .$$

Here,the right hand sides of equations (3.3.6),(3.3.8) and (3.3.10) are expanded algebrically and treatment combinations are

replaced by corrosponding treatment means.

Remark :- Here from the equations (3.5.5), (3.5.7) and (3.5.9) it is seen that main effects and interaction are contrast and further we note that they are orthogonal contrasts to each other.

The s.s. due to factorial effect is obtained by dividing the square of factorial effect by $4r$., where r is the common number of replications. [we follow this from the definition] And this s.s. carries one d.f. Adding up these s.s. due to factorial effects we get s.s. due to treatments which carries 3 d.f.

However, it is more easy to obtain the factorial effects and their s.s. from the treatment totals rather than those obtained from treatment means .

Suppose factorial effect total due to factor A is defined as $[A] = [ab] - [b] + [a] - [1]$.

Similarly, we can define [B] and [AB]. Then s.s. due to main effect $A = s_A^2 = [A]^2/[4r]$ and it carries 1 d.f.

Similarly ,

$$s_B^2 = [B]^2/[4r] \quad \text{and} \quad s_{AB}^2 = [AB]^2/[4r]$$

are the s.s. due to main effect B and interaction AB respectively each carring 1 d.f.

Here our main hypothesis to be tested is to test the significance of interaction effect AB . We state the null hypothesis H_0 , as H_0 : Interaction effect is not significant

i.e. $(\alpha\beta)_{11} = (\alpha\beta)_{12} = (\alpha\beta)_{21} = (\alpha\beta)_{22} = 0$.

Let S.S.E. denotes the sum of squares due to error and it carries 3(r-1) d.f. Using standard results [Rao,(1985)], we can give ,

$$\frac{S\,S\,E}{\sigma^2} = \frac{M\,S\,E}{\sigma^2} \quad \text{is distributed as}$$

$\chi^2_{3(r-1)}$ with 3(r-1) d.f. and $\frac{s^2_{AB}}{\sigma^2}$ is distributed, under H_0, as χ^2 with 1 d.f.

And , since these two are distributed independently to each other [Rao,(1985)] , so to test H_0 , we use the following statistics

$$F = \frac{s^2_{AB}}{MSE} \quad ,$$ which follows Snedecor's F -distribution with [1, 3 (r-1)] d.f. under H_0 only. We reject H_0 at α % level of significance [Kempthrone (1966)p.12] (1.0.S), if

F calculated value $\geq F_{\alpha}[1,3\,(r-1)]$;

Otherwise, H_0 is accepted.

In the similar way the other hypotheses about the significance of factorial effects A and B are tested.

We below give the 'analysis of variance ' table (ANOVA).

TABLE NUMBER 3.3.1 .

ANOVA For 2^2 --Factorial Experiment In RBD With 'r' replications.

Sources of Variation	d.f.	Sum of Squares	M.S.	F
Replication	r-1	S.S.due to repl-cates .		
Treatment	3			
Main effect A	1	S^2_A	s^2_A	$F = \frac{s^2_A}{MSE}$
" " B	1	S^2_B	s^2_B	$F = \frac{s^2_B}{MSE}$
Interaction AB	1	S^2_{AB}	s^2_{AB}	$F = \frac{s^2_{AB}}{MSE}$
Error	3(r-1)	SSE	MSE	
Total	4r - 1	T S S		

For the construction purpose, we may have an alternative model. We associate co-oridanates x_1 and x_2 with factors A and B respectively.

Where,

$x_1 = 1$, when factor A is at second level.

$x_1 = -1$, when factor A is at first level.

In the same way, x_2 has significance about factor B. So the four treatment combinations are

(-1-1), (+1,-1), (-1+1), (+1+1).

With these values, we can have the model

$$y_{ij} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + e_{ij} \qquad \text{-----(3.3.13)}$$

and, we shall have, $\beta_0 = u$, $\beta_1 = \alpha$, $\beta_2 = \beta$, and $\beta_{12} = (\alpha\beta)$.

THE 2^3 FACTORIAL EXPERIMENTS :-

Suppose there are three factors A,B and C each at two levels, so the total number of treatment combinations are eight, and they are -- (1), a, b, ab, c, ac, bc, abc. These eight treatment combinations can be tested with the help of RBD, with r' replications. For this, we develop a model as

$$y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \nu_k + (\alpha\nu)_{ik} + (\beta\nu)_{jk} + (\alpha\beta\nu)_{ijk} + e_{ijk} \qquad \text{---(3.3.14)}$$

$$; \quad i,j,k = 1,2 \ .$$

All the terms in the model have same corrosponding meaning as earlier.

To get unique solutions for normal equations, we impose the conditions ;

$$\sum_i \alpha_i = 0 \ , \sum_j \beta_j = 0, \ \sum_k \nu_k = 0, \sum_i (\alpha\beta)_{ij} = 0, \sum_j (\alpha\beta)_{ij} = 0 \ .$$

and so on , which implies

$$\alpha_1 + \alpha_2 = 0, \quad \text{i.e.} \quad \alpha_2 = -\alpha_1 = \alpha_1 \text{ say.}$$

Similarly, $\beta_2 = \beta_1$ $\nu_2 = \nu_1$,

$$(\alpha\beta)_{11} = (\alpha\beta)_{22} = -(\alpha\beta)_{12} = -(\alpha\beta)_{21} = (\alpha\beta), \text{ say, and so on.}$$

Further, the eight second order interaction terms $(\alpha\beta\nu)_{ijk}$ have the same absolute value. If they are not all zero, four of them are positive and the other four are negative [John(1971)]

Hence, we have,

$$\bar{\sum}_i \bar{\sum}_j (\alpha\beta\gamma)_{ijk} = 0, \quad \bar{\sum}_i \bar{\sum}_k (\alpha\beta\gamma)_{ijk} = 0, \quad \bar{\sum}_j \bar{\sum}_k (\alpha\beta\gamma)_{ijk} = 0, \text{ lead to}$$

$$(\alpha\beta\gamma)_{222} = -(\alpha\beta\gamma)_{221} = -(\alpha\beta\gamma)_{212} = -(\alpha\beta\gamma)_{122} = (\alpha\beta\gamma)_{211} = (\alpha\beta\gamma)_{121}$$

$$= (\alpha\beta\gamma)_{112} = -(\alpha\beta\gamma)_{111} = (\alpha\beta\gamma) \quad \text{------ say.}$$

Hence, the least square estimates are given as,

$$8\hat{\mu} = (abc)+(bc)+(ab)+(ac)+(a)+(b)+(c)+(1) \qquad \text{----(3.3.15)}$$

$$8\hat{\alpha} = (abc)-(bc)+(ac)-(c)+(ab)-(b)+(a)-(1) \qquad \text{----(3.3.16)}$$

$$8\hat{\beta} = (abc)+(bc)-(ac)-(c)+(ab)+(b)-(a)-(1) \qquad \text{----(3.3.17)}$$

$$8\hat{\gamma} = (abc)+(b)+(ac)+(c)-(ab)-(b)-(a)-(1) \qquad \text{----(3.3.18)}$$

$$8(\hat{\alpha\beta}) = (abc)+(bc)-(ac)+(ab)-(a)-(b)+(c)+(1) \qquad \text{----(3.3.19)}$$

$$8(\hat{\alpha\beta\gamma}) = (abc)-(bc)-(ac)-(ab)+(a)+(b)+(c)-(1) \qquad \text{----(3.3.20)}$$

In equation (3.3.16), (abc)-(bc) gives difference in average response by changing the levels of factor A from first to second, when factors B and C are at second level. And it is called as the simple effect of factor A, when factors B and C are at second levels. Similarly we have the other three simple effects of factor A, at the various levels of factors B and C. And an average of these four simple effects is defined as, 'main effect' of factor A, which is denoted by A.

Hence,

$$A = \frac{(abc)-(bc)+(ab)-(b)+(ac)-(c)+(a)-(1)}{4}$$

which can be written in more simplified form as,

$$A = \frac{(a-1)\,(b+1)\,(c+1)}{4} \qquad \text{----(3.3.21)}$$

or $\quad A = 2\hat{\alpha}$

In the similar way, we will have four simple effects of factor B, at various levels of factors A and C. And these simple effects are

$$(abc)-(ac),\quad (ab)-(a),\quad (bc)-(c,\ (b)-(1).$$

And an average of these four simple effects is defined as the "main effect of factor B".

Hence,

$$B = \frac{(abc)-(ac)+(ab)-(a)+(bc)-(c)+(b)-(1)}{4}$$

$$= \frac{(a+1)\ (b-1)\ (c+1)}{4} \qquad \text{--------(3.3.22)}$$

or,

$$B = 2\hat{\beta}\ .$$

Also, in equation (3.3.19)

$$\frac{(abc)-(ac)-(bc)+(c)}{2},$$

gives an interaction A B when C is at first level. Then average of these above two interactions is called, interaction AB.

Hence,

$$AB = \frac{1}{4}\ [\ (abc)-(bc)-(ac)+(c)+(ab)-(b)-(a)+(1)\ ] \qquad \text{----(3.3.23)}$$

In the similar way, we define BC and AC. And, one half difference of second from first is defined as the interaction ABC.

Hence,

$$ABC = \frac{1}{4}\ [\ (abc)-(bc)-(ac)+(c)-(ab)+(b)+(a)-(1)\ ]$$

which can be written in the more simplified form, as

$$ABC = \frac{1}{4}\ (a-1)\ (b-1)\ (c-1) \qquad \text{-------(3.3.24)}$$

The right hand side of equations

(3.3.21), (3.3.22), (3.3.23) and (3.3.24)

have same meaning as earlier.

Further, we note that all main effects and interactions are contrasts and they are orthogonal to each other. Hence S.S.due to any Factorial effect, X is given by

$$S_X^2 = \frac{[X]^2}{8.r} \qquad \text{------(3.3.25)}$$

which carries 1 d.f.

Here we are mainly interested to test the significance of interaction effects we put H_0 : Interaction effect is absent

i.e. $(\alpha\beta\gamma)_{111} = (\alpha\beta\gamma)_{121} = (\alpha\beta\gamma)_{122} = - - - = 0$.

Then from the well known results from Rao (1985) , we have test statistic as, under H_0 .

$$F = \frac{S^2_{ABC}}{SSE/7(r-1)} ,$$

Which follows Snedecor's F -distribution with [1,7(r-1)] d.f. Where SSE is a sum of squares due to error. We reject H_0 , if,

$$F \text{ calculated} > F_{\alpha} : [1,7(r-1)] ,$$

Otherwise we accept it.

In the similar way we can have tests for the significance of other effects.

Below we give structure of 'ANOVA' for 2^3 -factorial experiment in RBD with r replications.

Table No.

Structure of 'ANOVA '

Sources of Variation	d.f.
Replicates	r − 1
Treatments	7
Main effects	
A	1
B	1
C	1
I st order interactions	
AB	1
AC	1
BC	1
II nd order interactions	
ABC	1
Error	7(r − 1)
Total	8r − 1

2^n -FACTORIAL EXPERIMENT :-

Suppose there are n factors A, B, C - - - each at levelS two, so we will have 2^n -treatment combinations. Let X is some effect (main effect or interaction). Then X is given by

$$X = \frac{1}{2^{n-1}.r} (a\pm1)(b\pm1)(c\pm1)\cdots\cdots \quad ----(3.3.26)$$

Where the sign of ± is negative for all those letters appe-aring in X and positive elsewhere. And the right hand side has

same interaction as earlier, and r is the number of replications.

We have following model,

$$Y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\beta\gamma)_{jk} + (\alpha\gamma)_{ik} + (\alpha\beta\gamma)_{ijk} + e_{ijk} \qquad \text{-----(3.3.27)}$$

Where the terms in the model have same meaning as earlier.

The total 2^n treatment combinations are compared by using any of the standard designs with 'r' replications. The treatment S.S. carrying $2^n - 1$ d.f. is split up into $2^n - 1$ orthogonal components each carrying 1 d.f.

The effects in 2^n -factorial experiments may be enumerated as :

Factorial Effects	Numbers
Main effects	n
2 -factor interactions	$\frac{n(n-1)}{2}$
3 -factor interactions	$\frac{n(n-1)(n-2)}{3!}$
-	
-	
Total	$2^n - 1$.

Further, the S.S. due to any factorial effect X is given by

$$S_X^2 = \frac{[X]^2}{2^n . r} ,$$

and it carries 1 d.f. Here we wish to test the significance of

various factorial effects. In general,

H_0 : Effect due to factor X is insignificant,

against,

H_1 : Factor X shows significant effect.

This is H_0 is tested, with the test statistic

$$F = \frac{S_X^2}{MSE} ;$$

which follows $F(1,v)$, under H_0. Where v represents d.f. corrosponding to error S.S. We follow the usual criteria to make the decision.

The procedure of getting the factorial effect totals and sum of squares due to main effects and interaction is laborious. For this, Yates (1937) developed a technique of getting factorial effect totals and S.S. due to factorial effects. It is known as, 'Yates Algorithm' or 'Yates procedure' .

We discuss below Yates algorithm for 2^n-factorial experiment (John - 1971).

Yates Algorithm

Following are the different steps involved in it.

Step - 1 : Write all the treatment combinations in the standard order viz. (1), a, b, ab, c, ac, bc, abc - - - etc., in the first column.

Step -2 : Every column out of the remaining is partitioned into two parts. Entries in the first part of the subsequent columns are obtained by 'addition'. The entry in

the i th row of the (k+1) th column is the sum of the (2i-1) th and (2i) the entries in the K th column, for $1 \leq i \leq 2^{n-1}$.

Step -3 : The entries in the second half are obtained by subtraction. The entry in $(2^{n-1} + i)$th row in the (k+1)th column is obtained by substracting (2i-1)th entry from (2i)th entry of K th column.

Step -4 : The first entry in (n+1)th column is the grand total G and remaining entries are corrosponding contrast total with treatment combinations in first column arranged in standard order.

We consider 2^3 -factorial experiment,

Column 1	2	3	4
(1)	a+(1)	(a+(1))(b+(1))	(a+(1))(b+1)(c+1)
a	b(a+(1))	c(a+(1))(b+(1))	(a-1) (b+1) (c+1)
b	c(a+(1))	(a-(1))(b+(1))	(a+b) (b-1) (c+1)
ab	bc(a+(1))	c(a-(1))(b+(1))	(a-1) (b-1) (c+1)
c	(a-(1))	(a+(1))(b-(1))	(a+1) (b+1) (c-1)
ac	b(a-(1))	c(a+(1))(b-(1))	(a-1) (b+1) (c-1)
bc	c(a-(1))	(a-(1))(b-(1))	(a+1) (b-1) (c-1)
abc	bc(a-(1))	c(a-(1))(b-(1))	(a-1) (b-1) (c-1)

Here symbols are used to represent the corrosponding totals. We note that the entries in column '4th', irrespective of divisors are effects due to different factors. First entry is total effect, denoted by G and next subsequent are A, B, AB, C, AC, BC and ABC effects respectively.

In the similar way the Yates procedure can be justified for

2^n -factorial' experiment.

Step 5 : The entries in (n+2)th column are obtained by squaring the corrosponding entries in the (n+1)th column and dividing by 2^n and these entries are the sum of square of corrosponding contrast. The sum of the entries in this column is equal to the sum of squares of the original observations.

The calculations may be error prone at step 3. To verify whether the calculations are correct or incorrect we have following check procedure. Using the fact that' the sum of the squares of the entries in the (n+1)th column is 2^n times the sum of squares of the data ', Quenouille (1953) suggested a check.

Consider a 2^2 -factorial experiment. According to Quenouille (1953) replace the first and third entries in th second column by any numbers x and y . It is seen that entries in the third column are ab+b+x, ab-b+y, ab+b-x and ab-b-y . And sum of the entries in third column is 4ab, irrespective of the values of the values of x and y. It is obvious that this check is villnerable to errors of sign in the (n+1)th column. However, Rayner (1967) has shown that this 'check' is true inspite of mistakes in sign at any stage of the algorithm.

Below we give justification of Yates algorithm.

THE 3^n -FACTORIAL EXPERIMENTS

In 3^n -experiment there are n factors each at 3 levels. Instead of taking two levels if we take 3 levels the scope of the experiment increases and it becomes more informative. Further,

for quantitative levels we can study the pattern of change of responce in better way with the increase of the levels of factors When factors are at three levels, then it is possible to investigate whether the change is linear or quadratic. From this point of view it is better to have more number of levels. However, the number of levels of factors cannot be increased too much as the size of the experiment increases too rapidly with them.

3^2 -FACTCRIAL EXPERIMENT

Suppose there are two factors A and B each at three levels We may use 0, 1 and 2 as the level codes and treat them as the three elements of mod 3. For 3^2 -experiment we have the following treatment combirations.

00, 01, 02, 10, 11, 12, 20, 21, 22.

or these may be denoted by

$$(1),\ b_1,\ b_2,\ a,\ ab_1,\ ab_2,\ a_2,\ a_2b_1,\ a_2b_2 .$$

or also by

$$a^0b^0,\ a^0b^1,\ a^0b^2,\ a^1b^0,\ a^1b^1,\ a^1b^2,\ a^2b^0,\ a^2b^1,\ a^2b^2 .$$

The mathematical model for two factors each at three levels may be given as

$$E(Y) = \beta_{00} + \sum_{i=0}^{2}\beta_{i0}X_1^i + \sum_{j=0}^{2}b_{oj}x_2^j + \sum_{i=0}^{2}\sum_{j=0}^{2}\beta_{ij}x_1^i x_2^j \qquad \text{----(3.3.28)}$$

Where x_1 and x_2 are co-ordinates for the levels of factors A and B respectively. Replacing x_1^i by the ith degree orthogonal polynomial for 3 levels, u_i and x_2^j by the corrosponding orthogonal polynomial v_j, we obtain

$$E(y)=\phi_1^o\phi_2^o + \sum_i \phi_1^o\phi_2^i u_i + \sum_j \phi_1^o\phi_2^j v_j + \sum_i\sum_j \phi_1^o\phi_2^j u_i v_j \quad \text{---(3.3.29)}$$

$i = 0,1,2.$
$j = 0,1,2.$

Consider a two factor factorial with

$G_1 = \{0,1,2\}$ and $G_2 = \{0,1,2\}$. The two Vander Monde matrics are

then --

$$P_1 = \begin{bmatrix} 1 & x_{11} & x_{11}^2 \\ 1 & x_{12} & x_{12}^2 \\ 1 & x_{13} & x_{13}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

and

$$P_2 = \begin{bmatrix} 1 & x_{21} & x_{21}^2 \\ 1 & x_{22} & x_{22}^2 \\ 1 & x & x^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

on orthonormalising the columns of P_1 and P_2 as described in definition 3.2.5 we obtain

$$M_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} ; M_2 = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

The two parametric vectors for this case are --

$$\Phi_1^1 = (\phi_1^0, \phi_1^1, \phi_1^2) ; \Phi_2^1 = (\phi_2^0, \phi_2^1, \phi_2^2) .$$

Hence the design matrix X_P and parametric vector β_P are obtained as --

$$X_P = M_1 \circledast M_2 = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} \circledast \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$
= \begin{bmatrix}
\frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{18}} & \frac{-1}{\sqrt{6}} & \frac{1}{2} & \frac{-1}{\sqrt{12}} & \frac{1}{\sqrt{18}} & \frac{-1}{\sqrt{12}} & \frac{1}{6} \\
\frac{1}{\sqrt{3}} & 0 & \frac{\sqrt{2}}{3} & \frac{-1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{18}} & 0 & \frac{-1}{3} \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{8}} & \frac{-1}{\sqrt{6}} & \frac{-1}{2} & \frac{-1}{\sqrt{12}} & \frac{1}{\sqrt{18}} & \frac{1}{\sqrt{12}} & \frac{1}{6} \\
\frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{18}} & 0 & 0 & 0 & \frac{-\sqrt{2}}{3} & \frac{1}{\sqrt{3}} & \frac{-1}{3} \\
\frac{1}{\sqrt{3}} & 0 & \frac{-\sqrt{2}}{3} & 0 & 0 & 0 & \frac{-\sqrt{2}}{3} & 0 & \frac{2}{3} \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{18}} & 0 & 0 & 0 & \frac{-\sqrt{2}}{3} & \frac{-1}{\sqrt{3}} & \frac{-1}{3} \\
\frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{18}} & \frac{1}{\sqrt{6}} & \frac{-1}{2} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{18}} & \frac{-1}{\sqrt{12}} & \frac{1}{6} \\
\frac{1}{\sqrt{3}} & 0 & \frac{-\sqrt{2}}{3} & \frac{1}{\sqrt{6}} & 0 & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{18}} & 0 & \frac{-1}{3} \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{18}} & \frac{1}{\sqrt{6}} & \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{18}} & \frac{1}{\sqrt{12}} & \frac{1}{6}
\end{bmatrix}
$$

and $\beta_p^1 = (\phi_1 \odot \phi_2)^1$

$$
= (\phi_1^{0}\phi_2^{0}, \phi_1^{0}\phi_2^{1}, \phi_1^{0}\phi_2^{2}, \phi_1^{1}\phi_2^{0}, \phi_1^{1}\phi_2^{1}, \phi_1^{1}\phi_2^{2}, \phi_1^{2}\phi_2^{0}, \phi_1^{2}\phi_2^{1}, \phi_1^{2}\phi_2^{2}).
$$

Appling the least squares procedure to the model

$Y_P = X_P \beta_P + \epsilon_P$, Where X_P is in orthogonal form. We obtain the following best linear unbaised estimator (BLUE) for β_P from equation (3.3.29).

$$\hat{\beta}_P = [X_P' X_P]^{-1} X_P' Y_P = X_P^{-1} Y_P \quad . \qquad \text{-----(3.3.30)}$$

Then we get ,

$$\phi^{o\,o}_{1\,2} = \frac{y_{00} + y_{01} + y_{02} + y_{10} + y_{11} + y_{12} + y_{20} + y_{21} + y_{22}}{3}$$

$$= \frac{A_0 + A_1 + A_2}{3}$$

$$= \frac{G}{3} \quad .$$

$$\phi^{o\,1}_{1\,2} = \frac{1}{6}\left[-y_{00} + y_{02} - y_{10} + y_{12} - y_{20} + y_{22} \right]$$

$$= \frac{1}{6}\left[y_{02} + y_{12} + y_{22} - y_{00} - y_{10} - y_{20} \right]$$

$$= \frac{1}{6}\left[B_2 - B_1 \right]$$

$$\phi^{o\,1}_{1\,2} = \frac{1}{18}\left[y_{00} - 2y_{01} + y_{02} + y_{10} - 2y_{11} + y_{12} + y_{20} - 2y_{21} + y_{22} \right]$$

$$= \frac{1}{18}\left[y_{00} + y_{10} + y_{20} - 2(y_{01} + y_{11} + y_{21}) + y_{02} + y_{12} + y_{22} \right]$$

$$= \frac{1}{18}\left[B_0 - 2B_1 + B_2 \right] .$$

$$\phi^{1\,o}_{1\,2} = \frac{-1}{6}\left[y_{00} + y_{01} + y_{02} - y_{20} - y_{21} - y_{22} \right]$$

$$= \frac{1}{6}\left[A_2 - A_0 \right]$$

$$\phi^{1\,1}_{1\,2} = \frac{1}{2}\left[y_{00} + y_{22} - y_{02} - y_{20} \right]$$

$$\phi^{1\ 2}_{1\ 2} = \frac{1}{12}\,[\ -y_{00} + 2y_{01} - y_{02} + y_{20} - 2y_{21} + y_{22}\ ]$$

$$= \frac{1}{12}\,[(y_{20} - 2y_{21} + y_{22}) - (y_{00} - 2y_{01} + y_{02})]$$

$$\phi^{2\ o}_{1\ 2} = \frac{1}{18}\,[\ y_{00} + y_{01} + y_{02} - 2y_{10} - 2y_{11} - 2y_{12} + y_{20} + y_{21} + y_{22}\ ]$$

$$= \frac{1}{18}\,[\ A_0 - 2A_1 + A_2\ ]$$

$$\phi^{2\ 1}_{1\ 2} = \frac{1}{12}\,[\ -y_{00} + y_{02} + 2y_{10} - 2y_{12} - y_{20} + y_{22}\ ]$$

$$= \frac{1}{12}\,[\ y_{02} - y_{00} - 2(y_{12} - y_{10}) + y_{22} - y_{20}\ ]$$

$$\phi^{2\ 2}_{1\ 2} = \frac{1}{6}\,[\ y_{00} - 2y_{01} + y_{02} - 2y_{10} + 4y_{11} - 2y_{12} + y_{20} - 2y_{21} + y_{22}\ ]$$

$$= \frac{1}{6}\,[\ y_{00} - 2y_{01} + y_{02} - 2(y_{10} - 2y_{11} + y_{12}) + y_{20} - 2y_{21} + y_{22}\ ]$$

The estimates of $\phi^{i\ j}_{1\ 2}$ are orthogonal contrasts. The contrasts corrosponding to $\phi^{i\ o}_{1\ 2}$ and $\phi^{o\ j}_{1\ 2}$ are the linear and quadratric contrast for A and B and give the subdivision of S^2_A and S^2_B into single degrees of freedom as before -

And $\phi^{i\ j}_{1\ 2}$ (i > o , j > o) give us subdivision of the interaction sum of squares into single degrees of freedom.

The contrast corrosponding to $\phi^{1\ 1}_{1\ 2}$, $\phi^{1\ 2}_{1\ 2}$, $\phi^{2\ 1}_{1\ 2}$ and $\phi^{2\ 2}_{1\ 2}$ are usually called the lin A x lin B, lin A x quad B, quad A x lin B

and quad A x quad B contrasts respectively. All 4 d.f. for interaction can be accounted for in this way by associating single d.f. for each interaction effect.

A randomised block design can be adopted for the experiment. If there are r replications the analysis of variance for testing the hypotheses that factorial effects are not significant is given as follows --

Table No. 3.3.3

ANOVA For 3^2 -Factorial experiment in RBD with r -replications .

Sources of Variation	d. f.	S. S.
Replication	r − 1	S^2_R
Treatments	8	
S^2_A	2	S^2_A
lin. A	1	S^2_{AL}
quad. A	1	S^2_{AQ}
S^2_B	2	S^2_B
lin.B	1	S^2_{BL}
quad. B	1	S^2_{BQ}
S^2_{AB}	4	S^2_{AB}
lin A.lin B	1	S^2_{ALBL}
lin A.quad B	1	S^2_{ALBQ}
quad.A.lin B	1	S^2_{AQBL}
quad. A. quad B	1	S^2_{AQBQ}
Error	8(r−1)	S S E
Total	9r − 1	T S S

Where S^2_R, is S.S. due to replication, S^2_{AL} is a S.S. due to the linear contrast due to factor A and the other terms have the similar meanirgs.

3^n FACTORIAL EXPERIMENT

Let the levels of factor be ordered in increasing order and let Y_P be the observation vector corrosponding to minimal complete factorial design. The n -factor arthogonal polynomial model is given by

$$E(Y_P) = X_P \beta_P \qquad \text{------(3.3.31)}$$

Where the subscripts of Y_P are lexico graphically ordered.

$$X_P = M_1 \otimes M_2 \otimes - - - \otimes M_n \; . \qquad \text{------(3.3.32)}$$

and

$$B_2 = O_1 \otimes O_2 \otimes - - - \otimes O_n \; . \qquad \text{------(3.3.33)}$$

The matrix M_i and the vector O_i are the design matrix and parametric vector for the i th factor after left to right orthonormalisation (or orthogonalisation). It follows that the superscripts of the elements of B_P are also lexicographically ordered. The elements of β_P also called factorial effects, have traditionally named in the following manner $\phi^{o}_{1}\phi^{o}_{2} - - \phi^{o}_{n}$ is called the general mean or intercept. $\phi^{o}_{1}\phi^{o}_{2} - - - \phi^{P}_{q} - - \phi^{o}_{n}$ is called the p th main effect of the q th factor, $\phi^{s}_{1}\phi^{s}_{2}\phi^{o}_{3} - - \phi^{o}_{n}$ is called the s_1 th degree of factor F_1 by s_2 th degree of factor F_2 interaction effect, and so on. Also, an effect $\phi^{i}_{1}\phi^{i}_{2} -$

$- - \phi_n^i$ is said to be an k th order interaction if k exponents out of $i_1, i_2, - - i_n$ are non zero.

Yate's Algorithm To Find Main Effects And Interactions For 3^n -Factorial .

Like 2^n -factorial, Yates developed the same technique for 3^n -factorial experiment. For it following are the different steps --

Step I :- Arrange the data in standard order ; such as

$$(1),\ a_1,\ a_2,\ b_1,\ a_1b_1,\ a_2b_1,\ b_2,\ a_1b_2,\ a_2b_2,\ c_1,$$

$$a_1c_1,\ a_2c_1,\ \text{and so on}$$

Step II :- Denote the entries in the k th column by k_i . The entries in the (k+1) th column fall into three groups. In the first group the i th entry is $k_{3i} + k_{3i-1} + k_{3i-2}$; $i > 1$. In the second group the i th entry is $k_{3i} - k_{3i-2}$; in the last group the i th entry is $k_{3i} - 2k_{3i-1} + k_{3i-2}$.

Checking procedure :-

Let $S_1 = k_1 + k_1 + k_4 + k_7 + - - -$

$S_2 = k_2 + k_2 + k_5 + k_8 + - - -$

$S_3 = k_3 + k_3 + k_6 + k_9 + - - -$

further let $X = S_1 + S_2 + S_3$ and $Y = S_1 - S_2 + 3s_3$.

The check is that X in any column should equal Y th prec-

eding column.

For n factors each at 3 levels we get 3^n number of treatment combinations . As the number of factors increase there will be more number of treatments. To handle the situations having large number of treatments we will discuss in next section.

THE s^n -FACTORIAL EXPERIMENT

Suppose there are n factors $A_1, A_2, - - - A_n$, each at s levels. Clearly there are s^n treatment combinations carrying $s^n - 1$ d.f. between them. In factorial experiment we partition the treatment sum of squares into $s^n - 1$ -orthogonal contrasts each possessing one d.f. corrosponding to the main effects and interactions of n -factors. For s^n -factorial experiment the main effects and interactions are defined as follows .

Definition (3.3.1) MAINEFFECT :-

A contrast may be said to represent the main effect of the ' i th' factor if the coefficients in the linear function constituting the contrast are independent of the levels of factors other than the factor A_i ; $i = 1, 2, - - , n$. It is clear that there are $s - 1$ contrasts for each main effect of s^n -factorial experiment.

Definition (3.3.2) INTERACTION :-

A contrast may be said to represent a two-factor (or first order) interaction [Bose(1937)] of 'i th' and 'j th' factors if (a) The coefficients in the linear function constituting the co-

ntrast are independent of the levels of factors other than the levels of factors A_i and A_j and

(b) The contrast is orthogonal to any contrast representing the main effect of factors A_i and A_j.

And there are $(s-1)^2$ contrasts representing the two factor interaction of A_i and A_j.

By induction, after defining the (k-1) th order interaction of k factors, for k= 2, 3, - - - r-1, we define the 'r' factors interaction.

Definition :(3.3.3) : A conntrast is said to belongs to (r-1)th order inteaction of r factors, $i_1, i_2 - - - i_r$ if,

(a) The coefficients in the linear function constituting the contrast are independent of the levels of factors other than $A_{i1}, A_{i2}, - - - A_{ir}$; and

(b) The contrast is orthogonal to all contrasts belonging to all possible main effects and interaction of the $A_{i1}, A_{i2} - - A_{ir}$ factors. And there are $(s-1)^r$ contrasts representing the r -factor interaction of factors $A_{i1}, A_{i2}, - - - A_{ir}$.

Here we have in all s^n -treatment combinations. For small values of n and s we can test these s^n treatment combinations either in CRD, or in RBD or in L.S.D.

But if either n or s or both large, we will discuss analysis for such experiments in next section.

//

3.4 CONFOUNDING

When the number of factors n and/or the number of levels of the factor increase, the total number of treatment combinations becomes large. Obiviously, such an experiment requires large number of experimental units in each replication. Due to large size, it may not be possible to obtain sufficient homogeneous field for the complete replication, which results in increase in unmanagable cost and error. To overcome this difficulty, Yates (1933) suggested a technique of 'confounding '.

By 'confounding ' we mean the splitting of a replicate into desired number of incomplete blocks in such a way that certain effect contrasts (main effect or interaction) are identical with block comparisons. A factorial effect contrast which is mixed up with block effects is called as 'confounded effects '. And a factorial experiment containing confounded factorial effect is termed as confounded experiment.

By splitting up replication into different incomplete blocks and allocating treatment combinations arbitrarly, at least one interaction or main effect is confounded. But, if we follow some systematic procedure of allocation of treatment combinations to the different incomplete blocks we can minimise the number of confounded effects. With above description, below we present the precise definition of confounding (Raktoe, Fedrer etc.(1981)).

Consider a minimal complete factorial experiment. Let

$$E(Y_P) = X_P B_P \qquad \text{-----(3.4.1)}$$

With $cor(Y_P) = \sigma^2 IN$, and rank of X_P is less than or

equal to the number of columns of X and β is a parametric vector arising in a model.

Definition (3.4.1) : CONFOUNDING :-

Let ψ_1 and ψ_2 be two algebrically independent linear parametric functions of β. For a given desing D, let $\hat{\psi}_1$ be linear unbaised estimator of ψ_1 under the assumption that $\psi_2 = 0$. If $\psi_2 =/= 0$ and $E(\hat{\psi}_1) = \psi_1 + c\,\psi_2$; $c =/= 0$, then ψ_1 is said to be 'confounded 'with ψ_2 under design D, and it is referred as a "confounded design".

Note that the confounded design is a biased design for ψ_1 if $\psi_2 =/= 0$. Further, the confounding is symmetric; that is, if ψ_1 is confounded with ψ_2 under design D, then by applying above definition of confounding, we can conclude that ψ_2 is confounded with ψ_1 under D.

As an illustration, consider the following situation in 2^3 factorial experiment.

Suppose eight treatment combinations are split up in to two blocks, as

Block I	Block II
abc, a, b, c	ab, ac, bc, (1) .

The model as discussed earlier can be written, with block effects as

$$Y_{ijkl} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \gamma_k + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk} + (\alpha\beta\gamma)_{ijk} + \delta_l + e_{ijkl} \quad ---.(3.4.2)$$

Where δ_l is the effect due to l th block and all other terms have same significances as discussed earlier.

Now the interaction ABC is estimated by

$$4\,ABC = 1\ (\ abc + a + b + c\ ) - 1\ (\ ab + ac + bc + (1)\ )$$

Then, it follows that

$$E\,(4ABC) = (\alpha\beta\gamma)_{100} + (\alpha\beta\gamma)_{010} + (\alpha\beta\gamma)_{001} + (\alpha\beta\gamma)_{111} - (\alpha\beta\gamma)_{110} - (\alpha\beta\gamma)_{101} - (\alpha\beta\gamma)_{011} - (\alpha\beta\gamma)_{000} + 4(\delta_1 - \delta_2)\ . \qquad \text{------(3.4.3)}$$

Which is equivalent to

$$E\ (\ \Delta\ ABC\ ) = \psi_1 + C\ \psi_2\ .$$

Where,

ψ_1 = treatment effect,

$\psi_2 = \delta_1 - \delta_2$

= Block effect

and

$C = 4$.

Which implies, interaction ABC is confounded with block effects.

TYPES OF CONFOUNDING :-

When there are two or more replications, then question arises whether the same factorial effects are confounded in each replication or different sets of factorial effects are confounded in different replications. Depending upon this fact there are following two types of confounding.

1) Complete or total confounding :-

If the same interaction is confounded in all replications, the confounding is called as the 'complete Confounding ' or 'total confounding '.

2) Partial confounding :-

If different interactions are confounded in different replications then confounding is called a 'partial confounding '.

No information will be available on totally confounded interactions and partial information will be available on partially confounded interactions. If an interaction is confounded in r_1 replications and is unconfounded in r_2 replications, the loss of information on that interaction is $\frac{r_1}{r_1 + r_2}$.

Many problems in factorial experiment theory turn out to have a geometric, an algebric, or a combinatorial property. As a consequence finite mathematical structures such as 'groups', 'rings', 'fields', 'Projective and Euclidean Geometrics ' have been used successfully in clarifying, extending and resolving many issues related to factorial experiments.

In the second chapter we have already discussed 'Finite Fields (Galois Fields)', 'Projective Geometric' and 'Euclidean Geometry ' and their properities. Their properities are very useful in the construction of factorial designs.

As we have told earlier that due to the systematic arrangement of treatment combinations to the different blocks we can minimise the confounded effects, so we allocate treatments scientifically to blocks that is, we construct factorial designs.

In the construction of factorial experiments we take the helps of 'Galois Field ','Projective Geometry 'and 'Euclidean Geometry '. We already have discussed about these in chapter two

It is clear that in symmetric factorial experiment s is a prime or power of prime, so there exists $GF(s)$, and for any integer n, we have $EG(n,s)$. A point in $EG(n,s)$ denoted by $(x_1, x_2, x_3, - - x_n)$ is considered as identical to the treatment combination $T(x_1, x_2, - - x_n)$ where factor A_1 is at x_1 level, A_2 at x_2 level, A_3 at x_3 level and so on. And the s^n treatment combinations in s^n-factorial experiment can be represented by s^n-points of $EG(n,s)$.

Any $(n-1)$-flat of $EG(n,s)$ has an equation of the form

$$a_0 + a_1 x_1 + a_2 x_2 + - - - + a_n x_n = 0 \quad ; \quad \text{-----(3.4.4)}$$

$$a_i \in GF(s).$$

and contains s^{n-1} points. By keeping $a_1, a_2, - - a_n$ constant and varying a_0 over the elements of $GF(s)$, we generate s parallel $(n-1)$-flats that have no common point and constituting a pencil, denoted by $P(a_1, a_2, - - - a_n)$ of $(n-1)$ flats. This pencil, divides the s^n treatment combinations into s sets which give rise to $s-1$ independent contrasts. Hence the pencil $P(a_1, a_2, - - - a_n)$ is said tc carry $s-1$ d.f.

We observe that the pencils $P(a_1, a_2, - - -, a_n)$ and $P(b_1, b_2, - - -, b_n)$ are identical if and only if $b_i = \rho a_i$, where $\rho \in GF(s)$ and $\rho \neq 0$.

By the definitions of main effects and interactions [Bose (1937)] as discussed in previous section of this chapter, we can

say that the pencil $P(a_1, a_2, - - -, a_n)$ represents the interaction of i_1 th, i_2 th, $- - -$, i_r th factors if and only if $a_{i_1}, a_{i_2}, - - -, a_{i_r}$ are non-zero and the other co-ordinates in the pencil $P(a_1, a_2, - - -, a_n)$ are zero.

The pencil (3.4.4) can be written as

$$a_1 x_1 + a_2 x_2 + - - - + a_n x_n = \alpha_j .$$

Where $a_1, a_2, - - -, a_n$ and α_j are the elements of GF(s).

When $\alpha_j = 0$, we have

$$a_1 x_1 + a_2 x_2 + - - - + a_n x_n = 0 ,$$

(n-1) -flat with s^{n-1} points.

Let

$\bar{\backslash}$
/_ $(a_1, a_2, - - -, a_n)^{(0)}$ be the sum of observations on s^{n-1} treatment combinations in this flat. Similarly, for $\alpha_j = 1$ we get,

$$a_1 x_1 + a_2 x_2 + - - - + a_n x_n = 1$$

have (n-1) -flat with s^{n-1} points and denote sum of observations on these treatment combinations by

$\bar{\backslash}$
/_ $(a_1, a_2, - - -, a_n)^{(1)}$.

Lastly

$\bar{\backslash}$
/_ $(a_1, a_2, - - -, a_n)^{(s-1)}$ denote the sum of observations on the treatment combination in the (n-1) -flat

formed by the equation

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = s - 1 \ .$$

These sets of s^{n-1} equations are disjoint and each will have s treatment combinations. Consider the contrast

$$L = l_0 \bar{\sum}_{/} (a_1, a_2, \cdots, a_n)^{(0)} + l_1 \bar{\sum}_{/} (a_1, a_2, \cdots, a_n)^{(1)} + \cdots + l_{s-1} \bar{\sum}_{/} (a_1, a_2, \cdots, a_n)^{(s-1)} .$$

This contrast belongs to the pencil $P(a_1, a_2, \cdots, a_n)$ and there are (s-1) independent contrasts representing $P(a_1, a_2, \cdots, a_n)$

Hence each pencil has (s-1) d.f.

We shall prove the following theorem for different two pencils.

Theorem : (3.4.1) :-

If $P(a_1, a_2, \cdots, a_n)$ and $P(b_1, b_2, \cdots, b_n)$ are two different pencils then the linear contrast corrosponding to them are orthogonal.

Proof :- Consider a pencil $P(a_1, a_2, \cdots, a_n)$ that is

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = \alpha_j \ ,$$

$$\alpha_j \in GF(s) \ .$$

and the corrosponding contrast is

$$L_1 = l_0 \bar{\sum}_{/} (a_1, a_2, \cdots, a_n)^{(0)} + l_1 \bar{\sum}_{/} (a_1, a_2, \cdots, a_n)^{(1)} + \cdots$$

$$- - + l_{s-1} \sum^{(s-1)} (a_1, a_2, - - -, a_n) , \quad \text{with} \quad \sum_{i=0}^{s-1} l_i = 0$$

And consider another pencil $P(b_1, b_2, - - -, b_n)$ which is different from $P(a_1, a_2, - - -, a_n)$ with corrosponding contrast

$$L_2 = l'_0 \sum^{(0)} (b_1, b_2, - - -, b_n) + l'_1 \sum^{(1)} (b_1, b_2, - - -, b_n) + -$$

$$- - + l'_{s-1} \sum^{(s-1)} (b_1, b_2, - - -, b_n)$$

with

$$\sum_{i=0}^{s-1} l'_i = 0 \quad .$$

These two contrasts are orthogonal if, $\sum_{i=0}^{n} l_i l'_i = 0$.

There are s^{n-1} terms in $\sum^{(0)} (a_1, a_2, - - -, a_n)$ with coefficient l_0 and out of these only s^{n-2} terms are in $\sum^{(0)} (b_1, b_2, - - -, b_n)$, that is we have $(n-2)$ -flat with s^{n-2} points in

$$\sum^{(0)} (b_1, b_2, - - -, b_n) \quad .$$

Similarly s^{n-2} points are common in

$$\sum^{(0)} (a_1, a_2, - - -, a_n) \quad \text{and} \quad \sum^{(1)} (b_1, b_2, - - -, b_n)$$

and so on. Thus the s^{n-2} points

$\overline{\backslash\!\_}^{(0)}$ $(a_1, a_2, - - -, a_n)$. are distributed in

$$b_1 x_1 + b_2 x_2 + - - - + b_n x_n = \alpha_i \ ;$$

$$\alpha_i = 0, 1, 2, - - -, s-1 \ .$$

Hence the sum of products of coefficients corrosponding to $\overline{\backslash\!\_}^{(0)}$ $(a_1, a_2, - - -, a_n)$ with the coefficients corrosponding

$\overline{\backslash\!\_}^{(0)}$ $(b_1, b_2, - - -, b_n)$, $\overline{\backslash\!\_}^{(1)}$ $(b_1, b_2, - - -, b_n)$, - -, $\overline{\backslash\!\_}^{(s-1)}$ $(b_1, b_2, - -, b_n)$

is equal to

$$s^{n-2} \left(l_0 l_0' + l_0 l_1' + l_0 l_2' + - - - + l_0 l_{s-1}' \right)$$

which is equal to

$$s^{n-2} \, l_0 \left(l_0 + l_1 + - - - + l_{s-1} \right) = 0 \ .$$

Similarly the sum of products of coefficients corrosponding to $\overline{\backslash\!\_}^{(1)}$ $(a_1, a_2, - - -, a_n)$ is

$$s^{n-1} \left(l_1 l_0' + l_1 l_1' + - - - + l_1 l_{s-1}' \right)$$

which is equal to zerc .

And same is true for other contrasts. Which implies the sum of products of coefficients in two contrasts is zero. Hence the two different pencils are orthogonal to each other.

Using these properities we give construction of symmetric factorials in the next section.

3.4.1 CONFOUNDING IN s^n -FACTORIALS THROUGH PENCILS .

The ideas of confounding through pencils were first introduced by Bose and Kishen (1940) and later improved by Bose (1947) Constructed designs are given by Cochran and Cox (1957) and Kitagawa and Nitone (1953). Method due to Bose (1947) has been discussed here. In the confounding of s^n -factorial experiment constant block size must be necessarily a power of 's ' and we say such an experiment as (s^n , s^k). It means , the total number of treatment combinations, s^n are arranged in ' s^k ' blocks each of size 's^{n-k}'. To construct (s^n ,s^k) -factorial , we use following steps. [Raghavrao (1971)].

Step 1 :- We first choose k independent pencils, such as

P_1 , P_2 , - - - , P_k ; where ,

$$P_i = P(a_{i1}, a_{i2}, - - - a_{in}) \; ; \quad i = 1, 2, - - -, k \; .$$

Let

$\bar{\backslash}_{/\_}\, i1$ $\bar{\backslash}_{/\_}\, i2$ - - - $\bar{\backslash}_{/\_}\, ik$ be flats belonging to the independent pencils P_1 ,P_2 , - - -, P_k , respectively.

These flats pass through a common (n-k) -flat $\bar{\backslash}_{/\_}\, i1$, i2, - -,ik

Step 2 :- We denote the treatment combinations on the (n-k) -flat by $\bar{\backslash}_{/\_}\, i1$, i2 , - - - , ik .

In this way the totality of s^n treatment combinations will be divided into s^k sets of the type $\bar{\backslash}_{/\_}\, i1$, i2 ,- - - , ik .

Step 3 :- We form the (i1,i2, - - -, ik) th block with the treatment combinations

$$\left(\bar{\bigvee}_{} \right) i1,i2, - - -, ik \quad \text{for}$$

i1, i2 , - - - , ik = 0, 1, 2, - - -, s-1 .Then it can be verified that the d.f. carried by generalised pencils also are confounded with s^k blocks. The pencils of the type

$$P\left[\bigvee_{i=1}^{k} a_{i\,i1}, \bigvee_{i=1}^{k} a_{i\,i2}, - - -, \bigvee_{i=1}^{k} a_{i\,in} \right]$$

Where $a_i \in GF(s)$, are called as 'Generalised Pencils '.

Further it can be shown that a block containing (0,0,- -,0) treatment combination forms a group called the 'inlablock sub-group' or ' key block '.

Step 4 :- Let $(x_1, x_2, - - -, x_s)$ be a key block. From this block, remaining blocks are generated. We can take

$(x_1+y, x_2+y, - - -, x_s+y)$ as a second block

where y is a treatment combination not belonging to the key block. Here addition of treatment combinations will be vector addition where each component is in GF(s). If z is a treatment combination not in the first and second blocks, the third block can be taken as $(x_1+z, x_2+z, - - -, x_s+z)$ continuing in this way we can construct the whole replication plan of (s^n, s^k) experiments. As an illustration of this method we consider the following example.

Example (3.4.1.1):- onsider the construction of $(2^5, 2^2)$ by confounding the pencils P(1,1,1,0,0) and ρ(1,0,0,1,1).

Step 1 :- Here we have $2^5 = 32$ total treatment combinations. And pencils $P(1,1,1,0,0)$ and $\rho(1,0,0,1,1)$ are independent pencils.

Step 2 :- We divide the 32 treatment combinations into 4 blocks each of size 8 . A generalised pencil $P(0,1,1,1,1)$ is also confounded. The key block is constituted by the solutions of following two simultaneous equations .

$$x_1 + x_2 + x_3 = 0$$

$$x_1 + x_4 + x_5 = 0 .$$

Step 3:- Hence the cortents of key block is 00000, 01100, 00011, 10110, 10101, 11010, 11001, 01111.

The remaining three blocks are obtained from the key block as we have discussed earlier. Hence the complete plan of entire replication is given as

Table No. 3.4.1.1

key Block	Block 2	Block 3	Block 4
0 0 0 0 0	0 0 1 0 0	0 0 0 0 1	1 0 0 0 0
0 1 1 0 0	0 1 0 0 0	0 1 1 0 1	1 1 1 0 0
0 0 0 1 1	0 0 1 1 1	0 0 0 1 0	1 0 0 1 1
1 0 1 1 0	1 0 0 1 0	1 0 1 1 1	0 0 1 1 0
1 0 1 0 1	1 0 0 0 1	1 0 1 0 0	0 0 1 0 1
1 1 0 1 0	1 1 1 1 0	1 1 0 1 1	0 1 0 1 0
1 1 0 0 1	1 1 1 0 1	1 1 0 0 0	0 1 0 0 1
0 1 1 1 1	0 1 0 1 1	0 1 1 1 0	1 1 1 1 1 .

Remark : For GF(2), it is clear that addition and multiplication operations are identical. So $(2^n, 2^k)$ type of experiment can be constructed in some diffrent style. Here 'key block ' is constituted by the treatment combinations having an 'even ' number of symbols in common with each interaction $P_1, P_2, - - -, P_k$.

The block form a group under the binary operation, that square of subject to the restriction any symbol will be replaced by 1 .

Let (x_1 , x_2 , - - -, x_2) be a key block, which is taken as block 1. If y is a treatment combination not belonging to block 1,then the block 2 will be obtained as (x_1 y,x_2 y,- -,x_2 y). Where the square of any symbole is replaced by 1. IF z is a treatment combination not occuring in first two blocks, then the third block will be (x_1 z, x_2 z, - - -, x_2 z), where the square of any symbol is replaced by 1. By the similar way we can generate 2^k blocks of replication. Here not only P_1 , P_2 , - -, P_k pencils are confounded but also the interactions obtained by multiplying P_1 , P_2 , - - -, P_k in all possible ways and replacing the square of any symbol by one, are also confounded.

Example 3.4.1.2:- Let us construct (2^5 , 2^2) experiments with factors A, B, C, D and E by confounding the pencils ABC and ADE . When we confound above two interactions automatically

ABC . ADE = A^2 B C D E = B C D E is also confounded.

Here key block is

((1), acd, ace, de, abd, abe, bcde, bc)

From this key block , the remaining 3 blocks are obtained by multiplying a, b and ab .

Table No. 3.4.1.2.

Block	Constituents of the Block
Key block	(1, acd, ace, de, abd, abe, bcde, bc)
2	(a, cd, ce, ade, bd, be, abcde, abc)
3	(b, abcd, abce, bde, ad, ae, cde, c)
4	(ab, bcd, bce, abde, d, e, acde, ac)

or, the key blok is obtained by the solutions of following two simulations equation

$$x_1 + x_2 + x_3 = 0$$

and

$$x_1 + x_4 + x_5 = 0 \ .$$

Hence the constitutents of the key block are given as below,

Key Block :-

(00000, 01100, 00011, 10110, 10101, 11010, 11001, 01111)

The remaining three blocks are obtained from the key block as discussed earlier.

Table No 3.4.1.3.

Key Block	Block 2	Block 3	Block 4
0 0 0 0 0	0 0 1 0 0	0 0 0 0 1	1 0 0 0 0
0 1 1 0 0	0 1 0 0 0	0 1 1 0 1	1 1 1 0 0
0 0 0 1 1	0 0 1 1 1	0 0 0 1 0	1 0 0 1 1
1 0 1 1 0	1 0 0 0 1	1 0 1 0 1	0 0 1 1 0
1 0 1 0 1	1 0 0 0 1	1 0 1 0 0	0 0 1 0 1
1 1 0 1 0	1 1 1 1 0	1 1 0 1 1	0 1 0 1 0
1 1 0 0 1	1 1 1 0 1	1 1 0 0 0	0 1 0 0 1
0 1 1 1 1	0 1 0 1 1	0 1 1 1 0	1 1 1 1 1

This above discussed method is due to Bose (1947). Das (1964) developed a some what alternative method to the above method of confounding. Das's method incorporates all the prin-

ciples of Bose's method but differes in procedure. In the method of Bose the the interactions for confounding are chosen first and the block contents follow. In the method of Das the block conteents are choosen first and the interactions confounded follow therefrom. The method of Das has been described in detail in the book due to Das and Giri (1979).

3.4.2 CONFOUNDING WITH THE HELP OF PSEUDOFACTORS .

For s^n factorial experiment , where $s = p^m$, p is a prime and $m > 1$ can be considered as p^{mn} -factorial experiment in m.n psedofactors. Hence, confounding in such experiment can be made to accommodate the experiment in p^k blocks of p^{mn-k} plots. After construction, the levels of pseudofactors are replaced by appropriate levels of original factors.

Example 3.4.2.1:- We consider a 4^2 experiment with two factors P and K each at four levels, 0, 1, 2 and 3. We can construct such types of designs with the help of psedofactors. Identify the four levels 0, 1, 2, and 3 of factor with treatment combinations 00, 01, 10 and 11 of 2^2 -factorial experiment in two pseudofactors A and B. And levels 0, 1, 2, and 3 with factor K are identified with the treatment combinations 00, 01, 10 and 11 of 2^2 -experiment in two pseudofactors C and D. Hence 4^2 experiment can be taken as identical with 2^4 -experiment with four psedofactors each at levels two viz. 0 & 1 .

Suppose interaction ABCD is confounded, then the contents of a key block is obtained by solving equation

$$x_1 + x_2 + x_3 + x_4 = 0$$

Hence we get key block as -

(0000, 0011, 0101, 0110, 1010, 1100, 1001, 1111)

The second block is obtained in the usual way.

The required plan is given as,

Table No.3.4.2.1 .

Block	Constituents of the Block
1	(0000, 0011, 0101, 0110, 1010, 1100, 1001, 1111)
2	(0001, 0010, 0100, 0111, 1011, 1101, 1000, 1110)

Replacing these treatment combinations by the treatment combinations in the original factors, P and k, we get the following :

Table No. 3.4.2.2

Block	Constituents of the Block
1	(00, 03, 11, 12, 22, 30, 21, 33)
2	(01, 02, 10, 13, 23, 31, 20, 32).

We have discussed the method of confounding to reduce the size of block. There is also another method in which only a fraction of the treatment combinations are experimented. Such a method is called as 'Fractional Replication '. In the next section we will discuss in detail about ' fractional replication'.

3.5 :- FRACTIONAL FACTORIALS

When there are large number of factors, even each at two levels, there will be large number of treatment combinations. For example, with 8 factors each at 2 levels, we will have in all $2^8 = 256$ treatment combinations. In such cases it is impo-

ssible to carry out experiment with entire replication because the expenditure may go beyound the budget or in some cases non-experimental error, e.g. the labelling of treatments may be changed, the plots may be wrongly numbered etc. may enter. For such cases Finney (1945) proposed a method in which only a fraction of total number of treatment combinations are experimented. Such a type of factorial experiment with a fraction of total number of treatments is called as, 'Fractional Factorials ' or 'Fractional Replication'. Plackett and Burman (1946) studied the problem in more detail and gave different fractional factorials. Then pointed out their utility in physical and industrial research, since then, fractional factorials have found many applications, particularly in industrial and research development. In fractional factorial the size of the experiment is reduced, however information on certain higher order interaction is lost. The main problem of fractional factorial is the suitable choice of non-estimable interaction used to select a fraction. Such non-estimable interactions used for the selection of treatment combinations are called as , 'defining contrasts ' or 'identity relation '. They are equated with I .

After selecting a fraction of treatment combinations one can easily show that any contrast of selected treatment combinations, in the fraction represents more than one effects or interactions. Such effects or interactions which are represented by same contrast of treatment combinations are called as, 'alias '. By assuming, other interactions in the alias are negligible when compared with one of them in same alias, the interaction under interest

is estimated by the contrast with the selected treatment combinations. The salient features of fractional factorial are given as below.

1. The fraction of s^n -experiment is of the order $\frac{1}{s^k}$. It is written as $\frac{1}{s^k}(s^n) = s^{n-k}$.

2. The fraction consists of the key block of the confounded factorial s^n in blocks of size s^{n-k} , obtained by confounding $\frac{s^k - 1}{s - 1}$ interactions .

3. These $\frac{s^k - 1}{s - 1}$ interactions are called as 'defining contrasts'. They are written in a row begining with I -and separeted by equality sign.

4. From the defining contrast only one block is obtained so they are not estimable.

We illustrate these ideas with following example:

Consider a 2^3 Factorial experiment in which only the 4 treatment combination viz. a, b, c and abc are tested. This is half of the complete replicate of 2^3 factorial experiment with these treatment combiration the main effect of A , is given by

$$A = \frac{1}{2}[\ (abc) + (a) - (b) - (c)\ ] \quad ---(3.5.1)$$

Similarly the main effects of B and C are given. Note that A, B and C contrasts are mutually orthogonal. Thus we can estimate there three main effects independently.

Also for the above set of treatment combinations, the interaction BC, is given by

$$BC = \frac{1}{2}\left[(abc) + (a) - (b) - (c) \right] \qquad \text{-----(3.5.2)}$$

It is seen that in the equation (3.5.1) and (3.5.2) the right hand sides are same. Hence the main effect A and interaction BC are confounded with the same quantity. So A and BC are aliases.. Also for the same set of treatments we have B and C are alias with AC and AB respectively, and further B is confounded with AC and C is confounded with AB .

If 8 treatment combinations are available, then interaction ABC would be computed from

$$ABC = \frac{1}{8}\left[(abc)+(a)+(b)+(c)-(ab)-(ac)-(bc)-(1)\right] \qquad \text{----(3.5.3)}$$

The treatment combinations which we have used earlier are from ABC contrast which have positive signs. That is, ABC is used to split up the factorial into two half replications. So ABC is a defining contrast or identity relation. Hence we can write --

$$I = A\ B\ C$$

So by using a half replicate we loose one factorial effect, ABC entirely and each main effects are get mixed up with one of the two -factor interactions. With the ABC as defining contrast we have following three sets of alias

$$A = BC$$

and

$$B = AC$$

$$C = AB \ .$$

These are obtained from the defining contrast by multiplying both sides with main effect and replacing the square of a symbol by 1 .

Remark :- The experiment could have been conducted with treatment combination (1), ab, ac, and bc from the remaining half. In that case also ABC is confounded with replicates and main effects are get mixed up with any of the 2 factor interactions. And alias sets are given on next page --

$$A = -BC \quad , \qquad B = -AC \quad , \qquad C = -AB$$

$$= BC \qquad\qquad = AC \qquad\qquad = AB \quad .$$

Example : 3.5.2 :- Consider a half replicate of a 2^5 -factorial Suppose a five factor interaction 'ABCDE' is chosen as defining contrast. Hence ,

$$I = A\,B\,C\,D\,E$$

Now we can write down the alias of any other factorial effect. For instance, the alias of A is A^2BCDE i.e. BCDE .

The complete set of alias is given as below --

Defining contrast,

$$I = A\,B\,C\,D\,E$$

Main effect	Alias	Twofactor Interaction	Alias
A	B C D E	A B	C D E
B	A C D E	A C	B D E
C	A B D E	A D	B C E
D	A B C E	-	-
E	A B C D	D E	A B C

From this it is seen that, every main effect has four-factor interaction as alias, and every two-factor interaction has three factor interaction as alias. No main effect is mixed up with 2 -factor interaction. So , to get information on main effects and two-factor interactions, their alias should contain interact-

ions of at least three factors. In such a case the variation due to alias group can be considered to be due to the main effect or two factor interaction. And variation due to higher order interactions is pulled with error. To get such a type of alias group our defining contrasts do not include any interaction with less than five factors. This serves as criterion for selecting an appropriate defining contrasts. Below we discuss in general

$\frac{1}{2^k}$ replicate of 2^n -factorial experiments.

$\frac{1}{2^k}$ REPLICATE OF 2^n -FACTORIAL EXPERIMENTS

Consider a problem of construction of 2^{-k} fraction of a 2^n factorial experiment. Out of $2^n - 1$ total main effects and interactions, $2^k - 1$ will be confounded with means and remaining $2^n - 2^k$ will be mutually mixed up in sets of 2^k.

To construct such a design, first we select k independent interactions $X_1, X_2, - - -, X_k$ such that none of them is obtainable from the others by multiplication and none of them should belong in the same alias set. Then we select treatment combinations of the same sign as the control, in the above interactions. Then the generalised interactions of $X_1, X_2, - - -, X_k$ will also be confounded with mean. We have identity relation as,

$$I = X_1 = X_2 = X_1X_2 = X_3 = X_1X_3 = X_2X_3 = X_1X_2X_3 \quad \text{etc.}$$

From these defining contrasts and alias sets we select a 2^{-k} fraction of 2^n. Such an experiment is denoted by 2^{n-k}.

To get the clear idea about above discussion we consider an example.

Example 2: 3.5.3 :- Consider an experiment with 8 factors A, B, C,D, E, F, G, H each at two levels. And suppose we wish to have $\frac{1}{4}$ th replication of 2^8. It is denoted by 2^{6}. Here we use only 64 treatment combinations out of 256. To save main-effects and 2-factor interactions we choose defining contrast containing at least five letters. Let ABCDE and ABFGH are two independent defining contrasts and generalised one is CDEFGH. Hence,

$$I = A\,B\,C\,D\,E = A\,B\,F\,G\,H = C\,D\,E\,F\,G\,H .$$

Suppose we wish to arrange these 64 treatment combinations in 4 blocks each of size 16. For this we have to confound other three interactions of order at least two, with blocks. Hence,

$$I = A\,B\,C\,D\,E = A\,B\,F\,G\,H = C\,D\,E\,F\,G\,H$$

$$A\,C\,F = B\,D\,E\,F = B\,C\,G\,H = A\,D\,E\,G\,H$$

$$B\,D\,G = A\,C\,E\,G = A\,D\,E\,H = B\,C\,E\,F\,H$$

$$A\,B\,C\,D\,F\,G = E\,F\,G = C\,D\,H = A\,B\,E\,H .$$

The intrablock sub-group is given as --

(1)	ach	aef	cefh
bdh	abcd	abdefh	bcdefh
beg	abcegh	abfg	bcfgh
degh	acdeg	adfgh	cdfg .

From this,intrablock subgroup taking as block I the remainingblocks can be obtained in the usual way on the next page -.

Table No. 3.5.1

$\frac{1}{4}$ Replicate Of 2^8 Experiment In 4 Blocks

Block I	Block II	Block III	Block IV
(1)	ab	ce	de
ach	bch	aeh	acdeh
aef	bef	acf	adf
cefh	abcefh	fh	acdfh
bdh	adh	bcdhe	bch
abcd	cd	abde	abce
abdefh	defh	abcdfh	abfh
bcdefh	acdef	bdf	bcf
bcgh	aeg	bcg	bdg
abcegh	cegh	abgh	abcdgh
abfg	fg	abcefg	abdefg
bcfgh	acfgh	befgh	bcdefgh
degh	abdegh	cdh	gh
acdeg	bcdeg	adg	acg
adfgh	bdfgh	acdefgh	aefgh
cdfg	abcdfg	defg	cefg

The 63 contrasts are obtained by Yates procedure and these are tested using the analysis of variance. The analysis may be performed by dropping letters 'a' and 'f' from the treatment combinations. Then we have to rename some of of the treatment combinations involving B, C, D, E, G, H factors by using identity relation $I = -ABCDE = -ABFGH = CDEFGH$.

The structure of analysis of variance is given as below -

Table No. 3.5.2 .

Sources of Variation	d.f.
Blocks	3
Main effects	8
2 -Factor interactions	28
Error	24
Total	63

To perform an experiment with such large sized blocks leads to increase in experimental error. So we try to reduce the size of blocks.

Suppose wish to have 8 blocks, each of size 8 . Construction of such an experiment is not simple and some of the 2 -factor interactions are confounded, with blocks. Let, we confound the following interactions with blocks.

$$ACF = BDEF = BCGH = ADEGH$$

$$BDG = ACEG = ADFG = BCEFH$$

$$ABCDFG = EFG = CDH = ABEH$$

$$CDF = ABEF = ABCDH = EGH$$

$$AD = BCE = BDFGH = ACEFGH$$

$$BCFG = ADEFG = ACH = BDEH$$

$$ABG = CDEG = FH = ABCDEFH \quad .$$

The plan is given as below -

Table No. 3.5.3 .

$\frac{1}{4}$ Replicate of 2^{8} Design In 8 Blocks Of Size 8

I	II	III	IV	V	VI	VII	VIII
(1)	ab	ce	de	fg	gh	acf	bdh
abcd	cd	abde	abce	abcdfg	abcdgh	bdf	ach
cefh	abcefh	fh	cdfh	cegh	cefg	aeh	bcdef
abdefh	defh	abcdfh	abfh	abdegh	abdefg	acdeh	aef
beg	aeg	bcg	bdg	bef	beh	abcefg	degh
acdeg	bcdeg	adg	acg	acdef	acdeh	defg	abcegh
bcfgh	acfgh	befgh	bcdefgh	bch	bcf	abgh	cdfg
adfgh	bdfgh	acdefgh	aefgh	adh	adf	cdgh	abfg .

The analysis of variance may be performed with 8 blocks each of size 16, by dropping letters, namely, a and f . Using the Yates procedure we find effects. In the analysis some of the interactions are renamed according to the identity

$$I = - ABCDE = - ABFGH = CDEFGH \quad .$$

The structure of analysis of variance will be

Table No. 3.5.4 .

Sources of Variation	d.f.
Interaction AD, FH	2
Block error	5
Main effects	8
2 -Factor interactions	26
Error (with in blocks)	22
Total	63

The two confounded interactions will be estimated with a variance equal to (block error mean square/16) and all the main effects and the other interactions with variance equal to (within block error mean square / 16) .

s^{-k} REPLICATION OF s^{n} FACTORIAL EXPERIMENTS:

A s^{-k} replicate of an s^{n} experiment is denoted by s^{n-k} .

To select s^{-k} replicate, we first select k independent pencils $P(a_{i1}, a_{i2}, - - -, a_{in})$ $(i = 1, 2, - - -, k)$ such that none is obtained from the others and will not result in two pencils in the same alias set. Then we select the treatment combinations occuring on the same flat as the clear that they occur on the same flat in each of the generalised interactions of pencil $P(a_{i1}, a_{i2}, - - -, a_{in})$ $(i = 1, 2, - - -, k)$ the defining contrast is , then

$$I = P(\geq_{i}^{-} l_{i} a_{i1} a_{i2} \ldots a_{in} , - - -, \geq_{i}^{-} l_{i} a_{i1} a_{i2} \ldots a_{in}) ;$$

Where, $l_i \in GF(s)$.

and there will be $\left(\frac{s^k - 1}{s - 1}\right)$ pencils in the defining contrast.

The remaining $\left(\frac{s^k - 1}{s - 1}\right)$ pencils will be divided into $\left(\frac{s^{n-k} - 1}{s - 1}\right)$ alias sets of s^k pencils. The alias set of the pencil $P(b_1, b_2, \ldots, b_n)$ contrasts of pencils representing generalised interactions of it with each of the $\left(\frac{s^k - 1}{s - 1}\right)$ pencils of the defining contrast.

Example 3.5.5 . :- We construct a $\frac{1}{3}$ replicate of 3^5 experiment based on the defining contrast

$$I = A\ B\ C\ D\ E$$

Here out of 243 possible treatment combinations we use only 81.

We have the following alias sets, with respect to defining contrast

$$I = A\ B\ C\ D\ E$$

$$A = A^2\ B\ C\ D\ E = B\ C\ D\ E$$

$$AB = A^2\ B^2\ C\ D\ E = C\ D\ E$$

$$AB^2 = A\ B\ C\ D\ E = B^2\ C\ D\ E \ .$$

If we confound ABC with blocks, we will also confound

$ABC\ (ABCDE) = ABCD^2\ E$ and $ABC\ (ABCDE) = DE^2$, Here two factor interaction is confounded. If we confound ABC^2 with blocks we

will also confound ABC^2 (ABCDE) = ABD^2E^2 and CD^2E^2 and these then constitute a suitable system of confounding for blocks of 27 plots. And to use blocks of 9 plots it is necessary to confound at least one 2 -factor interaction. The following may be confounded with blocks to give this result :

$$ABC^2 = ABD^2E^2 = CD^2E^2$$

$$AB^2D = AC^2DE^2 = BC^2E^2$$

$$ACD^2 = AB^2CE^2 = BD^2E$$

$$BCD^2 = AB^2C^2D^2E = AE \quad .$$

The intrablock subgroup consists of the treatment combinations which satisfy -

$$\left.\begin{array}{l} x_1 + x_2 + x_3 + x_4 + x_5 = 0 \\ x_1 + x_2 + 2x_3 = 0 \\ x_1 + 2x_2 + x_4 = 0 \end{array}\right| \quad \text{mod } 3 \ .$$

These treatment combinations form a group.

The intra block subgroup is as follows :

$$[(1), acd^2e^2, ac^2de^2, bcd, abce^2, abd^2e, bc^2d^2, abde^2, ab^2ce^2]$$

The other blocks are obtained bymultiplying this block by

$$c^2e, ce^2, de^2, de, cde^2, cd^2, cd^2e^2, cde .$$

The multiplication is ordinary multiplication with conditions that $a^3 = b^3 = c^3 = d^3 = e^3 = 1$.

The plan is given as the ~~next page~~ -

Table No. 3.5.5 .

$\frac{1}{3}$ Replicate of 3^{5} Design In 9 blocks of size 9 .

Block 1 -(1),acd e ,a c de,bcd,abc e ,a bd e,b c d ,ab de ,a b ce

Block 2 - c e,ad ,a cde ,bde,abc,a bc d e ,b cd e,ab c d,a b c

Block 3 - ce ,ac d e, a d ,bc de ,abe,a bcd ,b d e ,ab de,a b c

Block 4 - d e,acd,a c e ,bce,abc d ,a bde ,b c de,ab ,a b cd e

Block 5 - de ,ace,a c d ,bcd e ,abc e,a b,b c e ,ab d e,a b cd

Block 6 - c d e ,ade, a c,be ,abce,a bc d,b cde ,ab c e,a b d

Block 7 - c d,ae ,a cd e,bd ,abcde ,a bc e,b c,ab c d e ,a b de

Block 8 - cd ,ac de ,a e,bc ,abd e ,a bcde,b d,ab e ,a b c d e

Block 9 - cde,ac ,a d e ,bc d e,abd,a bce ,b e,ab d ,a b c de

The technique of fractional factorials has been made by Addelman (1963), Chakarbarti (1956), Mukerjee (1980), Nishii (1981), Bose (1982). Adhikari and Das (1986), among the others. For plans of standard fractional replicate designs we refer Brownlee, Kelly and Lcvarine (1948); Connor and Zelen (1959) ; Cochran and Cox (1957) ; Kitaggawa and Mitome (1953) and publication by the National Bureu of standars.

Upto here we have considered the symmetric factorial experiments. In the next chapter we will discuss about the , 'asymmetric experiments ' .

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