CHAPTER-I

DFR SYSTEM-PROBABILISTIC STUDY

CHAPTER - I

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1.1 Introduction :

When we buy any item we expect it to function properly for a reasonable period of time. Also a manufacturer would like to give some information about the average life of the item to his customers. The life testing experiments are designed to collect such information and using that the hypothesis under consideration is tested. In life testing experiments a number of items are subjected to test and the data consists of the recorded lives of all or some of the items. Since the item is likely to fail at any time it is customery to assume that the life of the item to be a random variable. So the probabilistic study of such random variable is needed. In this chapter a study of only continuous random variable is considered. Further it can be extended to discrete random variable. Also when life testing experiments conducted only two states of the item is considered, either a working state or failure state. No intermediate state is considered.

Every item exhibits its typical behaviour through its life length. Certain items become better and better as they work for longer period of time. This phenomenon can be explained through decreasing failure rate (DFR) distribution.

In section 1.2 we introduced the basic concepts of reliability such as life time distribution, survival function, failure rate function, various failure models and clouser property of DFR under mixture.

When items exhibit high failure rate at the initial period, it would be better to use the items for a reasonable amount of time before sending it to the market. This in a sense improves the quality of the items. This phenomenon we introduced in section 1.3 as a concepts of burn-in period, and obtain optimum burn-in period τ for items having generalized pareto distribution which we later discuss in the chapter II. Also we discuss the concept of censoring.

1.2 Basic concepts of Reliability :

1.2.1 Life Time Distribution and its Properties : Let T be a non-negative continuous random variable representing life time of items which is operated in an environment, for which it was designed and is defined as the time for which the item carries out its appointed functions satisfactorily, after which it passes into failed state. The life time distribution function of T is given by

$$F_T(t) = P\ [\text{that item fails before time } t]$$

$$= P\ [T \leqslant t] \qquad \ldots\ldots (1.1)$$

and corresponding probability density function of T is

$$f_T(t) = \frac{d}{dt} F_T(t) \qquad \ldots(1.2)$$

The distribution function defined in (1.1) must be satisfy following conditions,

a) F(t) is a right continuous function such that,

$$C \leqslant F(t) \leqslant 1 \qquad t > 0 \qquad \ldots\ldots (1.3)$$

$$F(t) = 0 \qquad t < 0 \qquad \ldots\ldots (1.4)$$

b) $\lim_{t \to \infty} F(t) = 1$ and $\lim_{t \to -\infty} F(t) = 0$ ..... (1.5)

For example, consider a series system consisting of n idependent components. That is the system functions if and only if all the n components functions and life time of the components are independent random variables. Let T_1, T_2, ... T_n denote the life times of n components respectively with respective distribution functions $F_1(t)$, $F_2(t)$ ... $F_n(t)$. If T denotes the life time of the system then,

T = min $(T_1, T_2, \ldots T_n)$ and its distribution function is given by,

$$F_T(t) = P[T \leqslant t]$$

$$= P[\min (T_1, T_2, \ldots T_n) \leqslant t]$$

$$= 1 - \prod_{i=1}^{n} [1-F_i(t)] \qquad \ldots\ldots (1.6)$$

which is a life time distribution of series system provided that all $F_i(t)$'s are distribution function satisfying conditions given in (1.3) - (1.5).

1.2.2 Survival Function : The probability that the component survives until some time t ($t > 0$) is called the survival function of the component.

Thus ,

$$R(t) = P[\,T > t\,]$$

$$= 1 - F_T(t)$$

$$= \int_t^{\infty} h(u)\ .\ du \qquad .....(1.7)$$

It is a monotone decreasing, differentiable function with $R(o) = 1$ and $R(\infty) = \lim_{t \to \infty} R(t) = 0$

1.2.3 Failure Rate Function : The probability that the item fails in the time interval $(t\ , t + \Delta t)$ given that it was survived upto time t is defined as failure rate function h(t) and is given by,

$$h(t) = \lim_{\Delta t \to \infty} 1/\Delta t \quad P[\ t < T < t + \Delta t / \Delta T > t\ ]$$

$$= f_t\ (t) / \ 1 - F_T(t) \qquad .....(1.8)$$

Thus, failure rate function is the instantaneous rate of death or failure. h(t) must satisfy $h(t) \geqslant 0$ and $\int_0^{\infty} h(u).du = \infty$

1.2.4 Cummulative Failure Function : The cummulative failure rate function is given by ,

$$H(t) = \int_0^t h(u)\ du\ , \qquad t > 0 \qquad .....(1.9)$$

1.2.5 Relationship Between The Functions : $F_T(t)$, $f_T(t)$, R(t) h(t) : The relationship between various functions is stated and proved in the following theorems.

Theorem 1.1 : The failure rate function h(t) is related with survival function R(t) as

$$h(t) = -\frac{d}{dt} \log R(t) \qquad .....(1.10)$$

Proof : From 1.2 we know that,

$$f(t) = \frac{d}{dt} F_T(t)$$

$$= - R'(t) \qquad .....(1.11)$$

From equation 1.8 ,

$$h(t) = \frac{f_T(t)}{1-F_T(t)}$$

Let $V = 1 - F_T(t)$ and $dv/dt =- f_T(t)$

Hence,

$$h(t) = -\frac{dv/dt}{V}$$

$$= - d/dt (\log V)$$

$$= - d/dt \log (1- F_T(t))$$

$$= d/dt \log (R(t))$$

Hence the proof,

Theorem 1.2 : The relation between failure rate function h(t) with corresponding distribution fjnction F(t) is determined by,

$$F(t) = h(t) \exp (- \int_0^t h(u) du) \qquad .....(1.12)$$

Proof : From theorem (1.1)

$$h(t) = - d/dt \log (R(t))$$

Thus $\int_0^t h(u)\, du = - \log R(t) \Big|_0^t$

$$= - \log R(t)$$

Therefore, $R((t) = \exp (- \int_0^t h(u).du$

But $R(t) = 1 - F(t)$

Thus, $1-F(t) = \exp (- \int_0^t h(u\, du)$

$$F(t) = 1 - \exp (- \int_0^t h(u).du)$$

From equation (1.2)

$$f(t) = d/dt\, F(t)$$

$$= h(t) \exp (- \int_0^t h(u)\, du) \qquad ....(1.13)$$

From (1.12) we get,

$$R\, (t) = \exp (- H(t) \qquad .....(1.14)$$

With the help of failure function we state one important property of series system as,

Remark 1.1: If h(t) denotes the failure rate function of the distribution function F(t) given by,

$$F(t) = 1 \prod_{i=1}^{n} (1 - F_i(t)) \quad , \quad t > 0$$

and is differentiable. Where $F_i(t)$ is satisfy all the conditions given in (1.3) - (1.5) , and h(t)'s denotes the failure rate of $F_i(t)$'s respectively. Then,

$$h(t) = \prod_{i=1}^{n} h_i(t) \qquad \ldots(1.15)$$

Proof : Proof can be obtained from Barlow and Proschan (1975).

1.2.6 Increasing Failure Rate Function (IFR) :

The distribution function F is an increasing failure rate (IFR) distribution if,

$$\bar{F}(x/t) = P\,[\text{that a unit of age } t \text{ will survive for an additional period } x]$$

$$= \frac{\bar{F}(t+x)}{\bar{F}(t)} \qquad \ldots\ldots(1.16)$$

is decreasing in t for each $x \geqslant 0$

That is,

$$\bar{F}\,(x/t) \downarrow \text{ in } t \qquad x \geqslant 0 \qquad \ldots\ldots(1.17)$$

Theorem 1.3 : If F is an increasing failure rate distribution then its failure rate increases with t .

Proof : let F is IFR then from (1.17),

$$\bar{F}\,(x/t) \downarrow \quad t$$

$$\frac{\bar{F}(t)+x)}{\bar{F}\,(t)} - 1 \downarrow \quad t$$

$$\frac{\bar{F}\,(t+x) - \bar{F}(t)}{\bar{F}(t)} \downarrow \quad t$$

That is,

$$\frac{F(t) - F(t+x)}{\bar{F}(t)} \downarrow \quad t \qquad x > 0$$

Hence
$$\lim_{x \to 0} \frac{F(t+x) - F(t)}{x\,\bar{F}(t)} = \frac{f(t)}{\bar{F}(t)} \uparrow \quad t$$

That is, $\quad r(t) \uparrow \quad t$

This indicates that fcr a population of some components of same size surviving some time t, the expected number of failure in the interval (t, t + Δ t) increased as t increases.

1.2.7 Decreasing Failure Rate Function (DFR) : In contrast the distribution functon F is said to have decreasing failure rate function (DFR) if,

$$\bar{F}(x/t) = \frac{\bar{F}(t + x)}{\bar{F}(t)}$$

is increasing in t for each $x > 0$, $-\infty < t < \infty$ or by theorem (2.3) conversely, If $\bar{F}(x/t) \uparrow t$ then its $r(t) \downarrow t$.

The DFR distribution may arise , when there is a mixture of distribution or when a failure rate decreasing in the initial period,

Remark (1.2) If each F is DFR distribution then its mixture is also DFR distribution.

Proof : The general proof is fallowed by the theorem (4.5) and Lemma (4.6) given in Barlow and Proschan (1975; P 102, P.103). We prove this result for the exponential mixture as fallows

Example 1.1 : Let F be a mixture of exponential distribution given by,

$$F(x) = \int_0^\infty (1 - e^{-\lambda x}) \, dG(\lambda) \qquad .....(1.18)$$

Where λ is a random variable having any continuous distribution. From (1.18) the probability density function is given by,

$$f(x) = \int_0^\infty \lambda e^{-\lambda x} \, dG(\lambda) \qquad .....(1.19)$$

Then the failure rate function is,

$$r(x) = \frac{f(x)}{\bar{F}(x)}$$

$$= \frac{\int_0^\infty \lambda\, e^{-\lambda x}\; dG(\lambda)}{\int_0^\infty e^{-\lambda x}\; dG(\lambda)} \qquad \ldots.(1.20)$$

and,

$$r'(x) = \frac{\int_0^\infty e^{-\lambda x} dG(\lambda) \int_0^\infty \lambda^2\, e^{-\lambda x} dG(\lambda) + [\int_0^\infty \lambda e^{-\lambda x} dG(\lambda)]^2}{[\int_0^\infty e^{-\lambda x} dG(\lambda)]^2} \qquad \ldots..(1.21)$$

Now we have by cauchy-Schwarz inequality,

Let X and Y be two random variables with finite variance. Then Cov(X,Y) exists, morever,

$$E^2(XY) \leqslant EX^2.EY^2 \qquad \ldots..(1.22)$$

With equality if and only if there exists real numbers a and b not both zero such that, P [a X+bY=0]=1

Here, with $X^2 = e^{-\lambda x}$, $Y^2 = \lambda^2 e^{-\lambda x}$, from equations (1.22) and (1.21)we get,

$$\int_0^\infty e^{-\lambda x}\; dG(\lambda) \int_0^\infty \lambda^2\; e^{-\lambda x} dG(\lambda) > [\int_0^\infty \lambda e^{-\lambda x} dG(\lambda)]^2$$

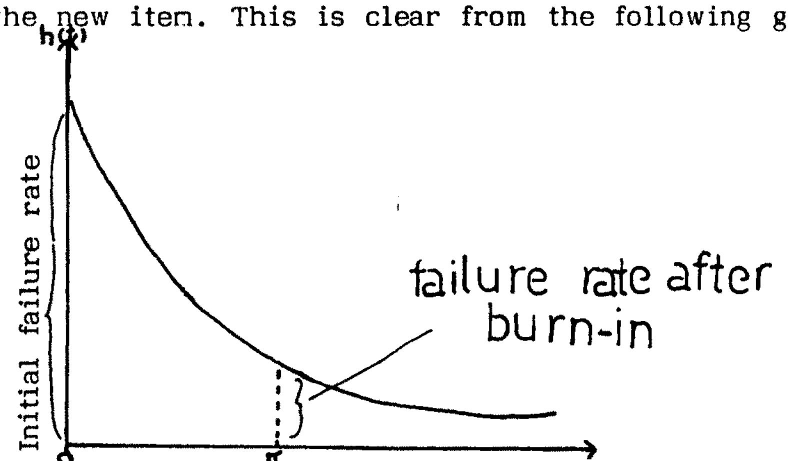
Thus, r'(x) < 0 that is r(x) is non-increasing function of x. This follows that the mixture of exponential given by (1.18) is DFR.

1.3 CONCEPT OF BURN-IN PERIOD :

Manufacturers want to sell the product or items produced in their factories after improving the quality, if possible. One way of improving the quality is to reduce the failure rate before the item is sold. This can be achieved by using every item produced in the factory itself for some reasonable period. This type is called "Burn-in " and the duration for which "Burn-in is continued is called the Burn-in period or debugging period. Naturally the life of the item after burn-in is not initial life but it is residual or remaining life. The life time after burn-in is called the residual life. Eventhough the concept of burn-in is valid for any item it is usually item with DFR distribution.

Definition 1.1: Let T be a continuous non-negative random variable having distribution function F then the burn-in perod is defined as the period for which the item is put to use before it sold or used for the purpose it is mean for.

If F is DFR, the failure rate of the item after burn-in is less than the new item. This is clear from the following graph.



Here 0 to τ is the burn-in period.

Definition 1.2 : Let T be a non-negative continuous random variable with distribution function F. Then after burn-in of length τ ($\tau > 0$) the life of T is denoted by T_τ which is the residual life of T at age τ and is given by,

$$P[\, T_\tau > t\,] = P[\, T > t + \tau / T > \tau\,]$$

$$= \frac{P[T > t + \tau\,]}{p[\, T > \tau]} \qquad .....(1.23)$$

Remark 1.3 : If a component has a GPD as its life time distribution, then the residual life after burn-in has a GPD.

The applicability of the burn-in property is shown in the follwing example with the help of graphs.

Example 1.2 : Assume the test data which we gave in Chapter II is distributed as a GPD with shape and scale parameters α and β [See Definition 2.1.2]. Then the estimaters of α and β are $\hat{\alpha}$ =0.518 and $\hat{\beta}$ = 1.08 respectively, which we have been obtained in section (3.2).Suppose the burn-in period of item is 48 hours, then after 48 hours burn-in, the residual life of that item $T_{48hours}$ is GPD with parametrs α and $\tilde{\beta}_f$ = β/ 1+48β[See Fig.1.1].

Suppose we plot graph of burn-in period against the $\tilde{\beta}_f$. Then it shows at the initial period that is when $\tau = 0$ the $\tilde{\beta}_f = \hat{\beta}$ and as burn-in time increases $\tilde{\beta}_f$ decreases [See Fig.1.2).Suppose the reliability is estimated at 20 minutes that is $\hat{R}(20) = (1+ 20\, \tilde{\beta}_f)^{-\hat{\alpha}}$ and plot the graph of $\hat{R}(20)$ against the burn-in time, it shows increase in reliability and approach to 1 as burn-in time increases [See Fig. 1.3].

Fig. 1.1 :

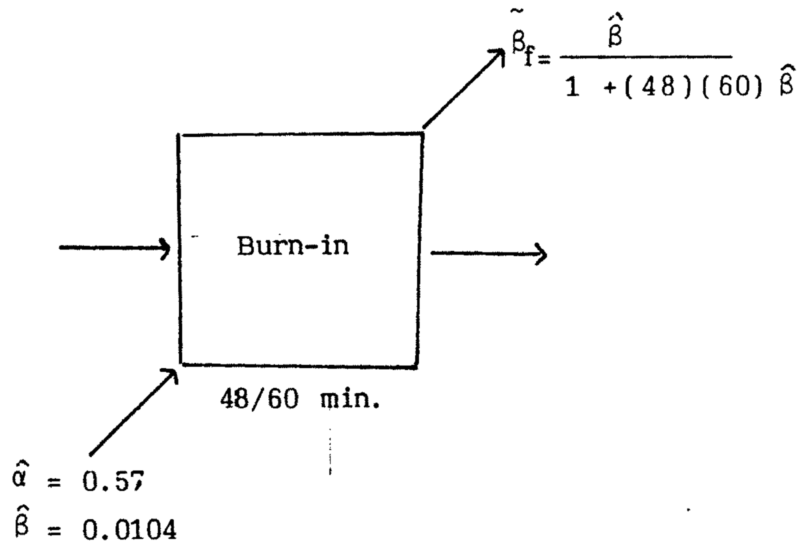


Fig. 1.2 :

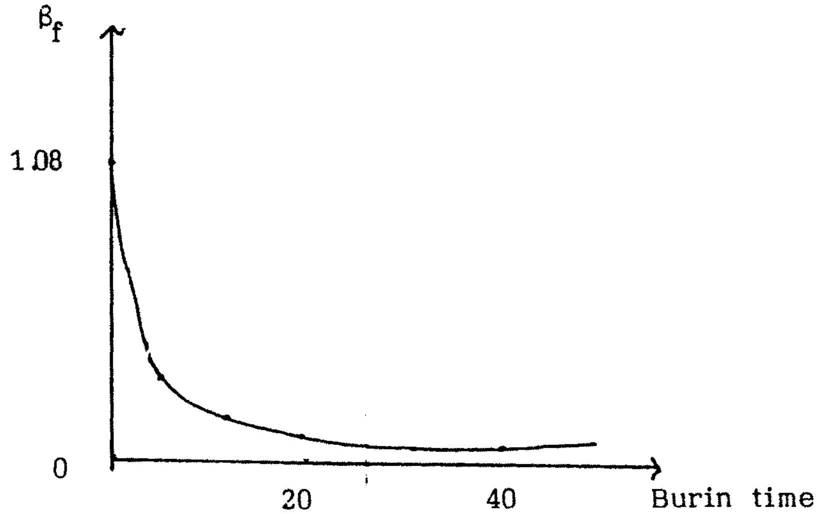
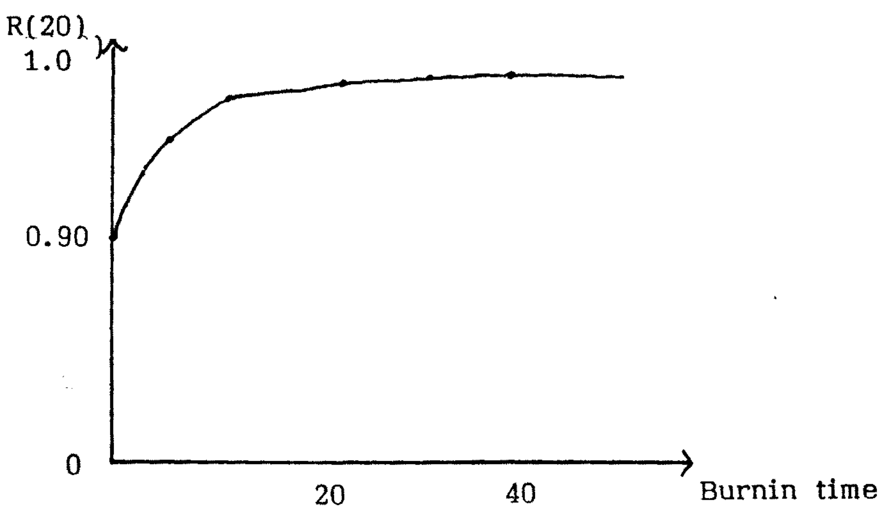


Fig. 1.3 :



But for conducting such burn-in procedure manufacturer has incur some costs. Saunder and Myhre (1983) gives some criterions by which the optimum burn-in period for the item having GPD can be obtained. It is discussed in the following section.

1.3.2 Optimum Burn-in period for the GPD : for a system with failure rate gave in equation(3.3.1), a burn-in of length $\tau > 0$ will yield a residual life X_{τ} with a failure rate in the same family but with different parameters (See Remark 2.3.3). It is given by,

$$h(t+\tau) = \frac{\alpha\beta}{1+(t+\tau)\beta} + \alpha\beta\gamma$$

$$= \alpha\beta' \, q(t \;\; \beta' : \gamma') \qquad .....(1.24)$$

wher, $\beta' = \beta/1+\tau\beta$ and $\gamma' = \gamma\,(1+\tau\beta)$ .....(1.25)

After burn-in the initial failure rare lowered at $\alpha\beta'(1+\gamma')$. From the value of $\alpha\beta\;(1+\gamma)$. While the terminal failure rate remains $\alpha\beta'\gamma' = \alpha\beta\gamma$

Given the three parameters which have been estimated by the methods of section (3.3). For such system, to answer the question that what is the benefit due to .burn-in period ? that depends on following criterion cf benefit. From it we obtain optimum burn-in period.

Chiterion I : Let the cost per unit time of burn-in is Rs.c and the benefit is Rs.B for each failure per hour of the maximum failure rate reduced by the burn-in, then increased gain per system for a burn-in duration τ is ,

$$g(\tau) = B\;[\alpha\beta\;(1+\gamma) - \alpha\beta'(1+\gamma')]-C\tau$$

From equation (1.25)

$$g(\tau)= B\ [\alpha\beta\ (1+\gamma)-(\alpha\beta/1+\tau\beta(1+\gamma(1+\tau\beta))]-C\tau$$

$$= \alpha\beta B[1+\gamma - \frac{1}{1+\tau\beta} - \gamma]- C\tau$$

$$= \alpha\beta B\ [\ 1- 1/1+\tau\beta]-C\tau \qquad ....(1.26)$$

Then, $g'(\tau) = 0$ gives,

$$\alpha\beta B \frac{\beta}{(1+\tau\beta)^2} = C$$

$$C\ (1+\tau\beta)^2 = \alpha\beta^2 B$$

that is,

$$\tau = \sqrt{\frac{\alpha B}{C}} - 1/\beta$$

Since burn-in can not be negative, the optimum burn-in for the criterion I is

$$\tau = [\sqrt{\frac{\alpha B}{C}} - 1/\beta\ ]^+ \qquad .....(1.27)$$

Where $x^+ = \max(x,0)$. Now for various value of α, β, B and C we compute optimum burn-in period τ. A computer programme in BASIC is developed as :

```
10  REM " PROGRAMME TO COMPUTE OPTIMUM "
20  REM " BURN-IN PERIOD FOR THE ITEMS "
30  REM " HAVING  GPD MODEL "
40  REM " STATEMENT (80) INPUTS PARAMETERS "
50  REM " OF GPD ,COST OF BURN-IN AND "
60  REM  " BENEFIT DUE TO BURN -IN "
```

```
70   FOR   I = 1  TO N
80   INPUT ALPHA, BETA, COST,BENEFIT
90   E = SQR ( ALPHA * BENEFIT /COST)
100  T = ABS ( E - 1/BETA)
110  PRINT "OPTIMUM BURN-IN PERIOD = ";T
120  NEXT I
130  END
```

Remark 1.4 : From the above programme we can get,

ALPHA	BETA	COST IN RS.	BENEFIT IN IN Rs.	OPTIMUM BURN-IN PERIOD IN MINUTES
0.518	1.08	1	5	0.68
2.493	0.00402	1	5	245.22
0.57	0.0104	1	5	94.46
0.5739	0.1106	1	5	7.34

Criterion II : Suppose the initial failure rate is bring to within 100 P % of the ultimate failure rate that is,

$$\alpha\beta' \ (1 + \gamma') = \alpha\beta\gamma(\ 1+ \ p)$$

$$\alpha\beta \ / \ 1+ \tau\beta \ (\ 1+ \ \gamma(1+\tau\beta)\ )= \alpha\beta\gamma \ (1+ \ p)$$

Solving for τ we get,

$$\tau \ = \ 1/\beta \ [\ 1/\gamma p-1]^{+} \qquad .....(1.28)$$

Now for different value of β , γ , p we compute τ by a programme in BASIC developed as below :

```
10   REM " PROGRAMME TO COMPUTE OPTIMUM "
20   REM " BURN-IN PERIOD "
30   INPUT " BETA " ; B
40   INPUT " GAMMA " ; G
50   INPUT " PERCENTAGE " ; P
60   E = ABS (( 1/G*P)-1)
70   T = E/B
80   PRINT " OPTIMUM BURN-IN PERIOD = ";T
90   END
```

Remark 1.5 : The results of the above programme is as follows,

BETA	GAMMA	PERCENTAGE	OPTIMUM BURN- IN PERIOD IN MINUTES
1.08	1	0.1	0.83
1.08	1	0.5	0.46
1.08	1	0.8	0.80

Let the total money available for a burn-in period is Rs.D per system. Then the percentage of initial to terminal failure rate for a burn-in of length τ is,

$$\frac{\alpha\beta\ (1+\gamma)}{\alpha\beta\gamma} = 1 + \frac{1}{\gamma(\ 1+\tau\beta)}$$

Since the number of units of time that the burn-in can utilize is $D/C = \tau$

Hence,

$$P = \frac{1}{\gamma(\ 1+D/C\ \beta)} \qquad .....(1.29)$$

We compute P for various values of D,C, γ and β by using programme in BASIC is developed below.

```
10   REM "PROGRAMME TO COMPUTE VALUE "
20   REM " OF    PERCENTAGE "
30   INPUTE"BETA   "; BE, "GAMMA ";G
40   INPUT "BENEFIT " ; B, " TOTAL COST" ; D
50   F = G * ( 1 + D* B/C)
60   P = 1/F
70   PRINT " PERECENTAGE = " ; P
80   END
```

Remark 1.6 : From the above programme we get, when Beta = 1.08, Cost of Burn-in = Rs. 1, Gamma = 0.5 and total cost available for burn-in = Rs.3, Then Percentage P = 0.47.

Now the concept of censoring is discussed in the following section.

1.3.2 Concept of Censoring : Censoring is a concept which is introduced for the purpose of reducing the cost of conducting the experiments censored samples are quite often used in the study of life time behaviour. In life testing experiment when certain experiment is carried out on the sample. A complete sample is available where the failure times of all n times are recorded. But it is not possible in all situation. Some times sample contains only partial information. The term censoring is used in that sense. It contains partial information. Following are types of censoring.

Type I Censoring : The factor that affects the life testing experiment is the amount of time required to complete sample. In such case we may put n items to test and terminate the experiment at a pre-assigned time t_0. The samples obtained from such an experiment are called "time-censoring" samples. Here the data consists of the life times of items that failed before to say n and the fact that (n-m) items have survived beyond t_0.

Type II Censoring : In life testing experiment once the item failed can not use again. This limits the number of item under test. That is we may put n times on test and terminate the experiment when k (a pre-assigned item) $<$ n have failed. The sample obtained from such an experiment are called "failure censored samples." In this case data consists of the life times of the k items that failed and the fact that (n-k) items have survived beyond k Censoring may also conducted by progressive that is described as below

Progressive Censoring : In this type of censoring items may be removed from life test throughout the duration of the test. Suppose that n items are put on to test, then at random times some units even if they have not failed, are removed from the test and their alive times are recorded. Also if any unit fails before it removed the failure times are recorded. Suppose there are k failures. Thus the data consists of observed failure times t_1 $t_2 \ldots k_k$ and $t_{k+1} \ldots t_n$ are observed alive times. This type of censoring is a particular case of progressive censoring. In such censoring case the likelihood function is written as,

$$L = \prod_{i=1}^{n} f(t_i) \prod_{i=k+1}^{n} \bar{F}(t_i)$$

Remark 1.7 : This progressive censoring may be considered as the generalization of type II censoring.