CHAPTER II

ACCEPTANCE SAMPLING BY VARIABLES FOR NORMAL DISTRIBUTIONS

2.1. Introduction :

In this chapter we assume that the measurements on the items in the lot are normally distributed and we study the sampling plans by variables in the following cases :

Case (i): Mean is unknown ; variance is known and when.

(a) Lower specification limit L is given.

(b) Upper specification limit U is given.

Case (ii): Mean is known ; variance is unknown and when,

(a) Lower specification limit L is given.

(b) Upper specification limit U is given.

Case (iii): Both mean and variance are unknown when,

(a) Lower specification limit L is given.

(b) Upper specification limit U is given.

2.2. Variable plan when mean is unknown and variance is known:

Let X be the measurement on a randomly chosen item from the lot and suppose that lower specification limit L is given that is

Case(i)(a):Lower specification limit L is given :

Let θ be the probability of an item being defective then,

$$\begin{aligned}\theta &= P_{\mu}(X \leq L)\\ &= P_{\mu}\left(\frac{X-\mu}{\sigma} \leq \frac{L-\mu}{\sigma} \right)\\ &= \Phi\left(\frac{L-\mu}{\sigma} \right) \qquad (2.2.1)\end{aligned}$$

where $\Phi(.)$ is the cumulative distribution function of the standard Normal variate. We see that Θ is a decreasing function of μ. Let the criteria of selecting the lot be as follows.

Accept the lot if $\Theta \leq \Theta_o$ and reject otherwise. If Θ is known, the problem is quite trivial. If Θ is unknown the acceptance-rejection procedure can be developed by using an appropriate estimator of Θ and then comparing the estimator with the specified value Θ_o. Hence the first step is to finding the estimator of Θ based on the measurements of n items chosen at random from the lot.

Let $X_1, X_2, \ldots, X_n$ be the measurements on the n items chosen at random from the lot, so that $X_1, X_2, \ldots, X_n$ are i.i.d. normal with mean μ and variance σ^2. Now, one can take the suitable estimator of Θ as the maximum likelihood estimator (MLE) or the minimum variance unbiased estimator. We know that $\bar{X}$ is maximum likelihood estimator of μ and Φ is a one-to-one function of μ, so $\Phi(\frac{L-\bar{X}}{\sigma})$ is the maximum likelihood estimator of Θ.

In order to obtain MVUE of Θ define,

$$T_1 = \begin{cases} 1 & \text{if } X_1 \leq L \\ 0 & \text{otherwise.} \end{cases}$$

clearly T_1 is unbiased for Θ. Then by using Rao-Blackwell Lehmann Scheffe theorem the MVUE is given by,

$$E(T_1|\bar{X}) = P_\mu[X_1 \leq L|\bar{X}] \qquad (2.2.2)$$

In order to compute the R.H.S. of (2.2.2), we consider,

$$P_\mu[X_1 \leq L \mid \bar{X} = t]$$

$$= P_\mu[X_1-\bar{X} \leq L-t \mid \bar{X}=t]$$

Since the distribution of $X_1-\bar{X}$ is normal with mean 0 and variance $(n-1)\ \sigma^2/n$ which does not depend on μ. Hence by using Basu's Theorem we get

$$P\left[\frac{X_1-\bar{X}}{\sigma} \sqrt{\frac{n}{n-1}} \leq \left(\frac{L-t}{\sigma} \right) \sqrt{\frac{n}{n-1}} \right]$$

$$= \Phi\left(\frac{L-t}{\sigma} \sqrt{\frac{n}{n-1}} \right) \qquad (2.2.3)$$

Hence, the MVUE of Θ is given by

$$\hat{\Theta} = \Phi\left(\frac{L-\bar{x}}{\sigma} \sqrt{\frac{n}{n-1}} \right) \qquad (2.2.4)$$

Using the estimator $\hat{\Theta}$, the criteria for accepting or rejecting the lot is as follows. Accept the lot if $\hat{\Theta} \leq \Theta_o$ otherwise reject the lot. But,

$$\hat{\Theta} \leq \Theta_o \text{ iff } \frac{L-\bar{x}}{\sigma} \sqrt{\frac{n}{n-1}} \leq \Phi^{-1}(\Theta_o)$$

which implies that,

$$\bar{x} \geq L - k\sigma\sqrt{\frac{n-1}{n}} \qquad (2.2.5)$$

where $k = \Phi^{-1}(\Theta_o)$. We note that $\Theta = \Theta_1$ corresponds to $\mu = \mu_1$ and $\Theta = \Theta_2$ corresponds to $\mu = \mu_2$. Using (2.2.5) OC function can be written as,

$$L(\mu) = P_\mu[\text{Accepting the lot})$$

$$= P_\mu\left[\bar{x} \geq L - k\sigma\sqrt{\frac{n-1}{n}} \right]$$

$$= P_\mu\left[\frac{\bar{x}-\mu}{\sigma/\sqrt{n}} \geq \left(\frac{L-\mu}{\sigma} \right) \sqrt{n} - k\sqrt{n-1} \right]$$

$$= 1 - \Phi \left(\frac{(L-\mu)\sqrt{n}}{\sigma} - k\sqrt{n-1}\right)$$

$$= 1 - \Phi (Z_\theta \sqrt{n} - k \sqrt{n-1}) \qquad (2.2.6)$$

where $Z_\theta = \frac{L-\mu}{\sigma}$. We find n and k so that the resulting plan has OC function passing through the producer's risk point $(\theta_1, 1-\alpha)$ and consumer's risk point (θ_2, β). Using (2.2.6) we get the following two equations.

$$\Phi (Z_{\theta_1} \sqrt{n} - k \sqrt{n-1}) = \alpha \qquad (2.2.7)$$

$$\Phi (Z_{\theta_2} \sqrt{n} - k \sqrt{n-1}) = 1 - \beta \qquad (2.2.8)$$

The equations (2.2.7) and (2.2.8) can be wiritten as

$$Z_{\theta_1} \sqrt{n} - k \sqrt{n-1} = Z_\alpha \qquad (2.2.9)$$

and

$$Z_{\theta_2} \sqrt{n} - k \sqrt{n-1} = - Z_\beta \qquad (2.2.10)$$

Solving (2.2.9) and(2.2.10) simultaneously we get the value of n, that is

$$n = \left[\frac{Z_\alpha + Z_\beta}{Z_{\theta_1} - Z_{\theta_2}}\right]^2 \qquad (2.2.11)$$

from equation (2.2.9) we get

$$k = \frac{Z_{\theta_1} \sqrt{n} - Z_\alpha}{\sqrt{n-1}} \qquad (2.2.12)$$

and from equation (2.2.10) we get

$$k = \frac{Z_{\theta_2} \sqrt{n} + Z_\beta}{\sqrt{n-1}} \qquad (2.2.13)$$

Having determined n by (2.2.11) k can be found by substituting the value of n either in equation (2.2.12) or (2.2.13). It is found from (2.2.12) that the resulting OC function of the plan passes through the producer's risk point(θ_1, $1-\alpha$) and if it is found from (2.2.13) it passes through the consumer's risk point (θ_2, β).

Case (b) : Upper specification limit U is given :

In this case θ is given by

$$\theta = P_{\mu}[\, X > U \,]$$

$$= P_{\mu}[\, \frac{X-\mu}{\sigma} > \frac{U-\mu}{\sigma} \,]$$

$$\theta = 1 - \Phi\left(\frac{U-\mu}{\sigma} \right) \tag{2.2.14}$$

where $\Phi(.)$ is the cumulative distribution function of the standard Normal variate and θ is a decreasing function of μ. Proceeding similarly as per case (a), we can find the minimum variance unbiased estimate of θ. In order to obtain the MVUE of θ, define,

$$T_2 = \begin{cases} 1 & \text{if } X_1 \geq U \\ 0 & \text{otherwise.} \end{cases}$$

Clearly T_2 is unbiased for θ, then MVUE is given by,

$$E(T_2 \mid \bar{X}) = P_{\mu}[X_1 \geq U \mid \bar{X}] \tag{2.2.15}$$

In order to compute the R.H.S. of (2.2.15) consider,

$$P_{\mu}[X_1 \geq U \mid \bar{X} = t]$$

$$= P_{\mu}[X_1 - \bar{X} \geq U-t \mid \bar{X}=t]$$

Since the distribution of $X_1-\bar{X}$ is normal with mean zero and variance $(n-1)\sigma^2/n$, which does not depend on μ. Hence by using Basu's Theorem we get

$$= P_\mu \left[\frac{X_1-\bar{X}}{\sigma} \sqrt{\frac{n}{n-1}} \geq \left(\frac{U-t}{\sigma}\right) \sqrt{\frac{n}{n-1}} \right]$$

$$\hat{\theta} = 1 - \Phi \left(\frac{U-\bar{x}}{\sigma} \sqrt{\frac{n}{n-1}}\right)$$

$$= \Phi\left(\frac{\bar{x}-U}{\sigma}\sqrt{n/n-1}\right) \qquad (2.2.16)$$

so, accept the lot if $\hat{\theta} \gtrless \theta_o$ otherwise reject the lot, but

$$\hat{\theta} \geq \theta_o \text{ iff } \Phi \left(\frac{\bar{x}-U}{\sigma}\right) \sqrt{\frac{n}{n-1}}) \geq \theta_o$$

i.e. $\frac{\bar{x}-U}{\sigma} \sqrt{\frac{n}{n-1}}) \gtrless \Phi^{-1}(\theta_o)$

which implies that

$$\bar{x} \gtrless U + k\sigma \sqrt{\frac{n-1}{n}} \qquad (2.2.17)$$

where $k = \Phi^{-1}(\theta_o)$. Using (2.2.17) the OC function can be written as

$$L(\mu) = P_\mu[\text{ Accepting the lot}]$$

$$= P_\mu[\bar{x} \gtrless U + k\sigma\sqrt{\frac{n-1}{n}}]$$

$$= P_\mu[\frac{\bar{x}-\mu}{\sigma/\sqrt{n}} \gtrless \frac{(U-\mu)}{\sigma} \sqrt{n} + k \sqrt{n-1}]$$

$$= 1 - \Phi [(\frac{U-\mu}{\sigma}) \sqrt{n} + k \sqrt{n-1}]$$

$$= 1 - \Phi (Z_\theta\sqrt{n} + k \sqrt{n-1}) \qquad (2.2.18)$$

where $Z_\theta = \frac{U-\mu}{\sigma}$. We find n and k so that the resulting plan has OC function passing through the producer's risk point $(\theta_1, 1-\alpha)$ and consumer's risk point (θ_2,β).

Using (2.2.18) we get the following two equations.

$$\Phi\,(Z_{\theta_1}\sqrt{n} + k\sqrt{n-1}) = 1-\alpha \qquad (2.2.19)$$

$$\Phi\,(Z_{\theta_2}\sqrt{n} + k\sqrt{n-1}) = 1-\beta \qquad (2.2.20)$$

The equations (2.2.19) and (2.2.20) can be written as

$$Z_{\theta_1}\sqrt{n} + k\sqrt{n-1} = -Z_\alpha \qquad (2.2.21)$$

and

$$Z_{\theta_2}\sqrt{n} + k\sqrt{n-1} = +Z_\beta \qquad (2.2.22)$$

Solving (2.2.21) and (2.2.22) simultaneously we get the value of n, that is,

$$n = \left[\frac{Z_\alpha + Z_\beta}{Z_{\theta_1} - Z_{\theta_2}} \right] \qquad (2.2.23)$$

from equation (2.2.21) we get

$$k = \frac{-Z_\alpha - Z_{\theta_1}\sqrt{n}}{\sqrt{n-1}} \qquad (2.2.24)$$

and from equation (2.2.22) we get

$$k = \frac{(Z_\beta + Z_{\theta_2}\sqrt{n})}{\sqrt{n-1}} \qquad (2.2.25)$$

It is found that (2.2.24) the resulting OC function of the plan passes through the producer's risk point $(\theta_1, 1-\alpha)$ and if it is found from (2.2.25) it passes through the consumer's risk point (θ_2, β).

Example 2.1 :

Suppose that we are given $\alpha = .1$, $\beta = .1$, $\theta_1 = .01$ and $\theta_2 = .0383$ using (2.2.11) we will get the value of n. That is n = 22. We can find the value of k from equation (2.2.12) and from equation (2.2.13), that is

$$k = -1.5319$$

and

$$k = -2.6605$$

Taking the average of k, we have k = 2.0962. Similar tables can be prepared for upper specification limit.

In the following tables I to IV for different values of α, β and θ_2 the n and k is computed and is compared with the attribute plan parameters n and c.

TABLE -I

$\alpha = .1$, $\beta = .1$, $\theta_1 = .01$

θ_2	Attribute plan parameters		Variable plan parameters	
	n	c	n	k
.0383	174	3	27	-2.0962
.0329	243	4	28	-2.1212
.0253	465	7	47	-2.1611
.0206	863	12	81	-2.1952
.0176	1536	20	141	-2.2259

TABLE-II

$\alpha = .01 \quad \beta = .05 \quad \theta_1 = .01$

θ_2	Attribute plan parameters		Variable plan parameters	
	n	c	n	k
.0942	82	3	15	-1.7901
.0716	127	4	22	-1.8678
.0412	350	8	45	-1.9992
.0356	476	10	57	-2.0355
.0319	609	12	72	-2.0670
.0305	677	13	75	-2.0723
.0293	746	14	83	-2.0829
.0258	1034	18	112	-2.1151
.0252	1106	19	118	-2.1204
.0246	1181	20	125	-2.1258

TABLE-III

$\alpha = .05, \quad \beta = .05, \quad \theta_1 = .01$

θ_2	Attribute plan parameters		Variable plan parameters	
	n	c	n	k
.1335	35	1	8	-1.8366
.077	47	2	14	-1.9437
.0465	197	4	26	-2.0427
.0275	616	10	66	-2.1393
.0263	692	11	73	-2.1478
.0253	768	12	77	-2.1520
.0237	923	14	91	-2.1649
.0215	1241	18	116	-2.1824
.0207	1403	20	133	-2.1913

TABLE-IV

$\alpha = .1, \quad \beta = .05, \quad \theta_1 = .01$

θ_2	Attribute plan parameters		Variable plan parameters	
	n	c	n	k
.0376	243	4	29	-2.1234
.0334	314	5	35	-2.1392
.0242	700	10	68	-2.1860
.0225	864	12	81	-2.1966
.0189	1537	20	142	-2.2260

2.3 Variable plan when mean is known and variance is unknown :

In this section we shall consider the variable plan when mean is known and variance is unknown in the case of lower specification limit and upper specification limit.

Case(a): Lower specification limit L is given :

Let θ be the probability of an item being defective then,

$$\begin{aligned} \theta &= P_\sigma (X \leq L) \\ &= P_\sigma \left(\frac{X-\mu}{\sigma} \leq \frac{L-\mu}{\sigma} \right) \\ &= \Phi \left(\frac{L-\mu}{\sigma} \right) \end{aligned} \tag{2.3.1}$$

Let $X_1, X_2, \ldots, X_n$ be the measurements on the n items chosen at random from the lot so that $X_1, X_2, \ldots, X_n$ are i.i.d. normal with mean μ and variance σ^2. In order to obtain minimum variance unbiased estimate of θ, define,

$$T_1 = \begin{cases} 1 & \text{if } X_1 \leq L \\ 0 & \text{otherwise,} \end{cases}$$

and $S^2 = \frac{1}{n}\sum_{i=1}^{n} (X_i-\mu)^2$. Clearly T_1 is unbiased for θ. When μ is known S^2 is sufficient and complete for σ^2. Then by using Rao-Blackwell-Lehmann-Schefte Theorem we get the MVUE of θ as

$$E\ (T_1|S) = P_\sigma\ [X_1 \leq L \mid S] \tag{2.3.2}$$

In order to evaluate the right hand side of (2.3.2) we need to know the conditional distribution of X_1 given S=s. Consider,

$$P_\sigma\ [\ X_1 \leq L \mid S=s\ ]$$

$$= P_\sigma\ [\ \frac{X_1-\mu}{S} \leq \frac{L-\mu}{s} \mid S=s\ ]$$

$$= P_\sigma\ [\ T \leq t_0 \mid S=s\ ] \tag{2.3.3}$$

where

$$\left. \begin{array}{l} T = (X_1-\mu)/S \\ t_o = (L-\mu)/s \end{array} \right| \tag{2.3.4}$$

Let,

$$T' = \frac{(X_1 - \mu)}{[\ \frac{1}{n-1} \sum_{2}^{n} (X_i-\mu)^2\ ]^{1/2}} \tag{2.3.5}$$

we note that the numerator and denominator of (2.3.5) are independent. Now the right hand side of (2.3.5) can be written as,

$$T' = \frac{\sqrt{n-1}\,(X_1-\mu)}{\left[\sum_{i=1}^{n}(X_i-\mu)^2-(X_1-\mu)^2\right]^{1/2}}$$

that is

$$T' = \frac{\sqrt{n-1}\;T}{(n-T^2)^{1/2}} \qquad (2.3.6)$$

where T is an defined in (2.3.4). Also $T \leq t_o$ is equivalent to

$$T' \leq \frac{(n-1)^{1/2}\,t_o}{(n-t_o^2)^{1/2}} \quad , \quad t_o < \sqrt{n}$$

and T' follows student's t-distribution with (n-1) d.f. and by Basu's Theorem it is independent of S. So that the right hand side of (2.3.3) is equivalent to,

$$P_\sigma\left[T' \leq (n-1)^{1/2}\,t_o/\,(n-t_o^2)^{1/2}\right]$$

$$= F_{t_{n-1}}\left(\frac{(n-1)^{1/2}t_o}{(n-t_o^2)^{1/2}}\right) \qquad (2.3.7)$$

where $t_o < \sqrt{n}$ and $F_{t_{n-1}}(x)$ is the distribution function of t-variate with (n-1) d.f. Hence, the MVUE of Θ is

$$\hat{\Theta} = F_{t_{n-1}}\left(\frac{(n-1)^{1/2}\left(\frac{L-\mu}{S}\right)}{\left(n-\left(\frac{L-\mu}{S}\right)^2\right)^{1/2}}\right) \qquad (2.3.8)$$

If $L > \mu$ then $\Theta > .5$. But from section (1.1) $0<\Theta<.5$, then obviosuly $L<\mu$. Using the estimator $\hat{\Theta}$, the criteria for accepting or rejecting the lot is as follows.

Accept the lot if $\hat{\Theta} \leq \Theta_o$ otherwise reject the lot, But

$$\hat{\Theta} \leq \Theta_o \quad \text{iff} \quad F_{t_{n-1}} \left[\frac{(n-1)^{1/2}(\frac{L-\mu}{s})}{(n-(\frac{L-\mu}{s})^2\ )^{1/2}} \right] \leq \Theta_o$$

which implies that,

$$\frac{(n-1)^{1/2}(\frac{L-\mu}{s})}{[n-(\frac{L-\mu}{s})^2\ ]^{1/2}} \leq - F^{-1}_{t_{n-1}} (\Theta_o)$$

That is,

$$\frac{(n-1)^{1/2}(\frac{L-\mu}{s})}{(n- (\frac{L-\mu}{s})^2\ )^{1/2}} \leq - k' \qquad (2.3.9)$$

where $k' = - F^{-1}_{t_{n-1}} (\Theta_o)$. Solving inequality (2.3.9) we get,

$$s^2 \leq (\frac{n-1}{k'^2}+1) \ (\frac{(L-\mu)^2}{n}) \qquad (2.3.10)$$

using (2.3.10) the OC function can be computed as follows. For this we note that Θ is a strictly decreasing function of σ . Thus we write the OC function of σ. Let Z_p denote the lower p-th quantile of the standard Normal distribution then, $Z_\Theta = \frac{L-\mu}{\sigma}$. So that

$$\begin{aligned} L(\sigma) &= P_\sigma \text{ [Accepting the lot]} \\ &= P_\sigma \left[S^2 \leq (\frac{n-1}{k'^2}+1) \frac{(L-\mu)^2}{n} \right] \\ &= P_\sigma \left[\frac{(n-1)S^2}{\sigma^2} \leq \frac{(n-1)}{n} (\frac{n-1}{k'^2}+1) (\frac{L-\mu}{\sigma})^2 \right] \\ &= P_\sigma \left[\chi^2_n \leq k\, Z^2_\Theta \right] \\ &= Q_n(k\, Z^2_\Theta) \qquad (2.3.11) \end{aligned}$$

where $k = (\frac{n-1}{n})(\frac{n-1}{k^2}+1)$ and Q_n is distribution function of chi-square random variable with n d.f. We find n and k so that the resulting plan has OC function passing through the producer's risk point $(\theta_1, 1-\alpha)$ and consumer's risk point (θ_2, β). Using (2.3.11) we get the following two equations.

$$k\, Z^2_{\theta_1} = Q_n^{-1}(1-\alpha) = \chi^2_{n,}(1-\alpha) \qquad (2.3.12)$$

and

$$k\, Z^2_{\theta_2} = Q_n^{-1}(\beta) = \chi^2_{n}, \beta \qquad (2.3.13)$$

Dividing (2.3.13) by (2.3.12) which gives

$$\frac{Z^2_{\theta_2}}{Z^2_{\theta_1}} = \frac{\chi^2_{n,\beta}}{\chi^2_{n,(1-\alpha)}} \qquad (2.3.14)$$

From equation (2.3.12) we get

$$k = \frac{\chi^2_{n,1-\alpha}}{Z^2_{\theta_1}} \qquad (2.3.15)$$

and from equation (2.3.13) we get

$$k = \frac{\chi^2_{n,\beta}}{Z^2_{\theta_2}} \qquad (2.3.16)$$

It is found that from (2.3.15) the OC function of the plan passes through the producer's risk point $(\theta_1, 1-\alpha)$ and if it is found from (2.3.16) it passes through the consumer's risk point (θ_2, β).

Case (b) : Upper specification limit U is given :

For the upper specification limit Θ is,

$$\Theta = P_{\sigma} \; (X > U),$$

$$\Theta = P_{\sigma} \left(\frac{X-\mu}{\sigma} > \frac{U-\mu}{\sigma} \right)$$

$$= 1 - \Phi\left(\frac{U-\mu}{\sigma}\right) \qquad (2.3.17)$$

In order to obtain minimum variance unbiased estimate of Θ define,

$$T_2 = \begin{cases} 1 & \text{if } X_1 > U \\ 0 & \text{otherwise.} \end{cases}$$

Clearly T_2 is unbiased for Θ. Proceeding similarly as per case (a), we will get the MVUE of Θ as follows :

$$\hat{\Theta} = 1 - F_{t_{n-1}} \left(\frac{(n-1)^{1/2} \left(\frac{U-\mu}{s}\right)}{\left(\frac{U-\mu}{s}\right)^2} \right) \qquad (2.3.18)$$

Using the estimator $\hat{\Theta}$, the criteria for accepting or rejecting the lot is as follows. Accept the lot if $\hat{\Theta} \lessgtr \Theta_0$ otherwise reject the lot. Knowing that $U > \mu$. Then,

$$\hat{\Theta} \lesseqgtr \Theta_0 \text{ iff } 1 - F_{t_{n-1}} \left(\frac{(n-1)^{1/2} \left(\frac{U-\mu}{s}\right)}{\left(n-\left(\frac{U-\mu}{s}\right)^2\right)^{1/2}} \right) \lesseqgtr \Theta_0$$

which implies that

$$\frac{(n-1)^{1/2}(\frac{U-\mu}{s})}{(n-(\frac{U-\mu}{s})^2)^{1/2}} \geq + F^{-1}_{t_{n-1}}(1-\theta_o)$$

That is

$$\frac{(n-1)^{1/2}(\frac{U-\mu}{s})}{(n-(\frac{U-\mu}{s})^2)^{1/2}} \geq + k' \qquad (2.3.19)$$

where $k' = + F^{-1}_{t_{n-1}}(1-\theta_o)$. Solving (2.3.19) we get

$$s^2 \not\leq (\frac{n-1}{k'^2}+1)\frac{(U-\mu)^2}{n} \qquad (2.3.20)$$

Using (2.3.20) the OC function can be computed as follows:

$$L(\sigma) = P_\sigma \text{ (Accepting the lot)}$$

$$= P_\sigma\left(s^2 \not\leq (\frac{n-1}{k'^2}+1)\frac{(U-\mu)^2}{n}\right)$$

$$= 1-P(\chi^2_n \leq kZ^2_\theta) \qquad (2.3.21)$$

$$= 1-Q_n(kZ^2_\theta)$$

where $Z_\theta = \frac{U-\mu}{\sigma}$, $k = \frac{(n-1)}{n}(\frac{n-1}{k'^2}+1)$ and Q_n is distribution function of chi-square random variable with n d.f. We find n and k so that the resulting plan has OC function passing through the producer's risk point $(\theta_1, 1-\alpha)$ and consumer's risk point (θ_2, β). Using (2.3.21) we get the following two equations.

$$kZ^2_{\theta_1} = Q^{-1}_n(1-\alpha)$$

$$= \chi^2_{n,1-\alpha} \qquad (2.3.22)$$

and

$$k Z_{\theta_2}^2 = Q_n^{-1}(1-\beta)$$

$$= \chi^2_{n,\, 1-\beta} \qquad (2.3.23)$$

From equation (2.3.22) and (2.3.23) we get

$$\frac{Z_{\theta_2}^2}{Z_{\theta_1}^2} = \frac{\chi^2_{n,\, 1-\beta}}{\chi^2_{n,\alpha}} \qquad (2.3.24)$$

from equation (2.3.22) we have

$$k = \frac{\chi^2_{n,1-\alpha}}{Z_{\theta_1}^2} \qquad (2.3.25)$$

and from equation (2.3.23) we get

$$k = \frac{\chi^2_{n,1-\beta}}{Z_{\theta_2}^2} \qquad (2.3.26)$$

It is found that (2.3.25) the resulting OC function of the plan passes through the producer's risk point $(\theta_1, 1-\alpha)$ and if it is found from (2.3.26) it passes through the consumers risk point (θ_2, β).

Example 2.2 :

Suppose that we are given the following quantities. $\alpha = .1$, $\beta = .1$, $\theta_1 = .01$, and $\theta_2 = .0383$.
Then by using equation (2.3.14) we will get the value of n.

By using chi-square distribution table we have n = 44. From equation (2.3.15) we will get the value of k that is

$$k = 10.4190$$

and from equation (2.3.16) we get

$$k = 10.3696$$

Taking the average of k we have k = 10.3942.

In the following tables V to VIII the different values of α, β and θ_2 the n and k is computed. These values are compared with the attribute plan parameter with the same quantities.

TABLE -V

$\alpha = .1, \quad \beta = .1, \quad \theta_1 = .01$

θ_2	Attribute plan parameters		Variable plan parameters	
	n	c	n	k
.03883	174	3	44	10.3942
.0329	243	4	60	13.7368

TABLE-VI

$\alpha = .01, \quad \beta = .05, \quad \theta_1 = .01$

θ_2	Attribute plan parameters		Variable plan parameters	
	n	c	n	k
.0942	82	3	26	8.7741
.0716	127	4	40	12.0199

TABLE-VII

$\alpha = .05, \quad \beta = .05, \quad \theta_1 = .01$

θ_2	Attribute plan parameters		Variable plan parameters	
	n	c	n	k
.1335	35	1	10	3.2907
.0770	47	2	23	6.4966
.0465	197	4	52	12.9086

TABLE-VIII

$\alpha = .1, \quad \beta = .05, \quad \theta_1 = .01$

θ_2	Attribute plan parameters		Variable plan parameters	
	n	c	n	k
.0376	243	4	60	13.6910
.0334	314	5	70	15.6289

2.4 Variable plan when both mean and variance are unknown :

Case(a): Lower specification limit L is given :

Let θ be the probability of an item being defective, then

$$\begin{aligned}\theta &= P_{\mu,\sigma}(X \leq L) \\ &= P_{\mu,\sigma}\left(\frac{X-\mu}{\sigma} \leq \frac{L-\mu}{\sigma}\right) \\ &= \Phi\left(\frac{L-\mu}{\sigma}\right) \qquad (2.4.1)\end{aligned}$$

Let $X_1, X_2, \ldots, X_n$ be the measurements on the n items chosen at random from the lot so that $X_1, X_2, \ldots, X_n$ are i.i.d. normal with mean μ and variance σ^2. In order to obtain minimum variance unbiased estimate of θ define,

$$T_1 = \begin{cases} 1 & \text{if } X_1 \leq L \\ 0 & \text{otherwise.} \end{cases}$$

When μ and σ^2 are both unknown then $(\bar{X}, S^2)$ is sufficient and complete statistic, where $\bar{X} = \frac{1}{n}\sum_1^n X_i$ and $S^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i-\bar{X})^2$. Then by Rao-Blackwell-Lehmann - Scheffe Theorem we get the MVUE of θ as,

$$E(T_1|\bar{X},S^2) = P_{\mu,\sigma}[X_1 \leq L|\bar{X}=x, S^2=s^2] \qquad (2,4.2)$$

We need to evaluate the right hand side of (2.4.2) we need to know the conditional density of X_1 given that $\bar{X} = x$, $S^2 = s$. Consider,

$$P_{\mu,}\ [X_1 \leq L \mid \bar{X} = x,\ S^2 = s^2]$$

$$= P_{\mu,\sigma}\left[\frac{X_1-\bar{X}}{S} \leq \frac{L-\bar{x}}{s} \mid \bar{X} = \bar{x},\ S^2 = s^2\right]$$

$$= P_{\mu,\sigma}\left[\frac{\sqrt{n-1}\ (X_1-\bar{X})}{[\sum_{i=1}^{n}(X_i-\bar{X})^2]^{1/2}} \leq \frac{L-\bar{x}}{s} \mid \bar{X}=\bar{x},\ S^2=s^2\right]$$

$$= P_{\mu,\sigma}\left[\frac{(X_1-\bar{X})}{[\sum_{i=1}^{n}(X_i-\bar{X})^2]^{1/2}} \leq \frac{L-\bar{x}}{s.(n-1)^{1/2}} \mid \bar{X}=\bar{x},\ S^2=s^2\right]$$

$$= P_{\mu,\sigma}[\ T \leq t_o \mid \bar{X} = \bar{x},\ S^2 = s^2\ ] \qquad (2.4.3)$$

where

$$T = \frac{(X_1-\bar{X})}{[\sum_{1}^{n}(X_i-\bar{X})^2]^{1/2}} \text{ and}$$

$$t_o = L-\bar{x}\ /s\sqrt{n-1} \qquad (2.4.4)$$

Now consider,

$$T = \frac{(X_1-\bar{X})}{[\sum_{1}^{n}(X_i-\bar{X})^2]^{1/2}} \qquad (2.4.5)$$

Define $U_i = (X_i-\bar{X})(\frac{n}{n-1})^{1/2}$, $i = 1,2,\ldots n$.

Then $U_2,U_3,\ldots,U_n$ have a symmetric $(n-1)$ variate normal distribution with $E(U_i) = 0$ for all i, $i = 1,2,\ldots,n$ and their dispersion matrix is of the form

$$\Sigma = \sigma^2 \begin{bmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ \cdot & & & \\ \rho & & \cdots & 1 \end{bmatrix}_{(n-1)x(n-1)}$$

where $\rho = -\frac{1}{n-1}$, then T becomes

$$T = \frac{U_1}{[\sum_{i=1}^{n} U_i^2]^{1/2}}$$

$$= \frac{U_1}{[\sum_2^n U_i^2 + U_1^2]^{1/2}}$$

$$= \frac{U_1}{[\sum_{i=2}^{n}(U_i-\bar{U})+(n-1)\bar{U}^2+U_1^2]^{1/2}} \qquad (2.4.6)$$

where

$$\bar{U} = \frac{1}{n-1} \sum_{i=2}^{n} U_i$$

$$= -\frac{1}{(n-1)}$$

and

$$V = \frac{\sum_{i=2}^{n} (U_i - \bar{U})^2}{(1-\rho)(n-2)} \qquad (2.4.7)$$

so, right hand side of (2.4.5) becomes,

$$\frac{U_1}{[(1-\rho)(n-2).V+U_1^2 + (U_1^2/(n-1))]^{1/2}}$$

$$= \frac{U_1}{[(1-\rho)(n-2)V + (\frac{n}{n-1})\ U_1^2\ ]^{1/2}}$$

$$= \frac{U_1}{[(\frac{n}{n-1})(n-2)V + (\frac{n}{n-1})\ U_1^2\ ]^{1/2}}$$

$$= \frac{U_1}{[b_n . a_n\ V + b_n\ U_1^2\ ]^{1/2}}$$

$$= \frac{(U_1\ /\ \sqrt{V}}{[b_n\ a_n + b_n\ U_1^2\ /\ V]^{1/2}} \qquad (2.4.8)$$

where $b_n = n/n-1$ and $a_n = (n-2)$. In the following lemma we prove the independence of U_1 and V and obtain their distribution.

Lemma 2.1 :

Let $U_2, U_3, \ldots, U_n$ have a symmetric (n-1) variate normal distribution with mean $E(U_i) = 0$, for $i = 1,2,\ldots,n$ and the dispersion matrix as

$$\Sigma = \sigma^2 \begin{bmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ \vdots & \vdots & & \vdots \\ \rho & \rho & \cdots & 1 \end{bmatrix} \quad (n-1)\times(n-1).$$

Where $\rho = \frac{-1}{n-1}$, then $\bar{U} = \sum_{2}^{n} U_i / (n-1)$ and the variance $S_U^2 = \sum_{i=2}^{n} (U_i-\bar{U})^2 / n-2$ are independently distributed, U_1 is normal variate with mean zero and variance σ^2 and $S_U^2 / \sigma^2(1-\rho)$ has chi-square distribution with (n-2) d.f.

Proof:

Let U_i, i = 2,3,..,n have a symmetric (n-1) variate normal distribution. Now, consider the orthogonal transformation

$$W = CU \tag{2.4.9}$$

where C is the orthogonal matrix with first row of C as

$$\left(\frac{1}{\sqrt{n-1}}, \frac{1}{\sqrt{n-1}}, \ldots\ldots, \frac{1}{\sqrt{n-1}}\right).$$

In particular let C be the orthogonal matrix obtained by Helmert's transformation as,

$$c = \begin{bmatrix} \frac{1}{\sqrt{n-1}}, & \frac{1}{\sqrt{n-1}}, & \cdots, & \frac{1}{\sqrt{n-1}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \cdots & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ \frac{1}{\sqrt{(n-1)(n-2)}} & & \cdots & \frac{-(n-1)}{\sqrt{(n-1)(n-2)}} \end{bmatrix} \quad (n-1)\times(n-1)$$

so that,

$$\begin{bmatrix} w_2 \\ w_3 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{n-1}} & \frac{1}{\sqrt{n-1}} & \cdots & \frac{1}{\sqrt{n-1}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\sqrt{(n-1)(n-2)}} & \cdots\cdots & & \frac{-(n-1)}{\sqrt{(n-1)(n-2)}} \end{bmatrix} \begin{bmatrix} U_2 \\ U_3 \\ \vdots \\ U_n \end{bmatrix} \quad \text{.. (2.4.10)}$$

Now,

$$\begin{aligned} E(w) &= E(CU) \\ &= C\,E(U) \\ &= 0 \end{aligned}$$

That is,

$$E(w_i) = 0, \; i = 2, \ldots, n$$

And the dispersion matrix transforms to

$$D = C'\Sigma C$$

$$D = C' \begin{bmatrix} 1 & \rho \cdots & \rho \\ \rho & 1 \cdots & \rho \\ \cdots & \cdots & \cdots \\ \rho & \rho \cdots & 1 \end{bmatrix} . C$$

On simplification we get the, dispersion matrix D as,

$$D = \sigma^2 \begin{bmatrix} 1+(n-2)\rho & 0 & \cdots & 0 \\ 0 & (1-\rho) & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & & (1-\rho) \end{bmatrix} \qquad (2.4.11)$$

from the right hand side of (2.4.11) it follows that

$$V(w_2) = [\,1 + (n-2)\rho\,]\ \sigma^2$$

$$V(w_i) = (1-\rho)\ \sigma^2 \qquad i = 2, \ldots n$$

and

$$\text{Cov}\ (w_i, w_j) = 0, \qquad i \neq j \qquad (2.4.12)$$

That is the transformed variables are uncorrelated.

Now, the distribution of w_2 is normal with mean zero and variance $(1+(n-2)\rho)\ \sigma^2$. But from (2.4.10) we have,

$$w_2 = \frac{1}{\sqrt{n-1}} \sum_{i=2}^{n} U_i = \frac{(n-1)\bar{U}}{\sqrt{n-1}} = \frac{-U_1}{\sqrt{n-1}}$$

So, the distribution of U_1 is normal with mean zero and variance $[1+(n-2)\rho\,]\ (n-1)\ \sigma^2 = \sigma^2$, that is U_1 is normal with mean zero and variance σ^2, and w_i has normal distribution with mean zero and variance $[(1-\rho)\ \sigma^2]$ for $i = 3, \ldots n$ and all are independent. Now, since the transformation (2.4.9) is orthogonal we have

$$W'W = U'U$$

That is,

$$\sum_{2}^{n} W_i^2 = \sum_{2}^{n} U_i^2 = \sum_{2}^{n} (U_i - \bar{U})^2 + (n-1)\,\bar{U}^2$$

$$= (n-2)\, S_u^2 + (m-1)\bar{U}^2$$

$$= (n-2)\, S_u^2 + W_2^2$$

from (2.4.12) it follows that $\bar{U}$ and S_u^2 that is U_1 and S_u^2 are independently distributed. And

$(n-2)\, S_u^2 = -\sum_{2}^{n} (U_i - \bar{U})^2 = (1-\rho).\ \chi^2_{n-2}$ has a chi-square distribution with (n-2) d.f. Where χ^2_{n-2} is chi-square variate with (n-2) d.f. Let

$$V' = \sum_{2}^{n} (U_i - \bar{U})^2 \,/\, 1-\rho \qquad (2.4.13)$$

Now, it follows that V' has chi-square distribution with (n-2) d.f. This proves the lemma. An outline of the proof of this lemma is given on page 136 of Rao (1965).

Let,

$$t_1 = \frac{U_1}{\sqrt{V}} \qquad (2.4.14)$$

Now from (2.4.7) and (2.4.13) it follows that

$$V = \frac{V'}{(n-2)}$$

where V' has chi-square distribution with (n-2) d.f. and is independent of U_1. Hence we get the t_1 has a student's t distribution with (n-2) d.f. Hence the right hand side of (2.4.8) is equivalent to

$$T = t_1 \ / \ (b_n a_n + b_n t_1^2)^{1/2}$$

or

$$T^2 b_n a_n = t_1^2 - b_n t_1^2 T^2$$

That is,

$$t_1^2 = \frac{b_n a_n T^2}{(1-b_n T^2)}, \quad \sqrt{b_n} T < 1$$

That is,

$$t_1 = \frac{(b_n a_n)^{1/2} T}{(1-b_n T^2)^{1/2}}, \quad \sqrt{b_n} T < 1$$

Now, $T \leq t_o$ is equivalent to

$$t_1 \leq \frac{(b_n a_n)^{1/2} t_o}{(1-b_n t_o^2)^{1/2}}, \quad \sqrt{b_n} T < 1$$

also, the distribution of t_1 is independent of μ and σ^2 hence by Basu's Theorem, the right hand side of (2.4.3) is equivalent to

$$P_{\mu,\sigma}[t_1 \leq \frac{(b_n a_n)^{1/2} t_o}{(1-b_n t_o^2)^{1/2}}], \qquad \sqrt{b_n} T < 1$$

$$= F_{t_{n-2}} [\frac{(b_n a_n)^{1/2} t_o}{(1-b_n t_o^2)^{1/2}}] \tag{2.4.15}$$

Hence, MVUE of θ is given by,

$$\hat{\theta} = F_{t_{n-2}} \left[\frac{\left(\frac{n(n-2)}{(n-1)} \right)^{1/2} \frac{L-\bar{x}}{s\sqrt{n-1}}}{\left(1 - \frac{n}{n-1} \left(\frac{L-\bar{x}}{s\sqrt{n-1}} \right)^2 \right)^{1/2}} \right] \qquad (2.4.16)$$

Miss Surekha N. Kulkarni proved the results (2.2.3), (2.3.7) and (2.4.15) in her M.Phil dissertation (1986). Now, using the estimator $\hat{\theta}$, the criteria for accepting or rejecting the lot is as follows.

Accept the lot if $\hat{\theta} \leq \theta_o$ otherwise reject the lot.

Then

$$\hat{\theta} \leq \theta_o \text{ iff } F_{t_{n-2}} \left[\frac{\left(\frac{n(n-2)}{n-1} \right)^{1/2} \frac{L-\bar{x}}{s\sqrt{n-1}}}{\left(1 - \frac{n}{n-1} \left(\frac{L-\bar{x}}{s\sqrt{n-1}} \right)^2 \right)^{1/2}} \right] \leq \theta_o$$

which implies that

$$\frac{\left(\frac{n(n-2)}{n-1} \right)^{1/2} \frac{L-\bar{x}}{s\sqrt{n-1}}}{\left(1 - \frac{n}{n-1} \left(\frac{L-\bar{x}}{s\sqrt{n-1}} \right)^2 \right)^{1/2}} \leq F^{-1}_{t_{n-2}} (\theta_o)$$

That is

$$\frac{\left(\frac{n(n-2)}{n-1} \right)^{1/2} \frac{L-\bar{x}}{s\sqrt{n-1}}}{\left(1 - \frac{n}{n-1} \left(\frac{L-\bar{x}}{s\sqrt{n-1}} \right)^2 \right)^{1/2}} \leq k' \qquad (2.4.17)$$

Where $k' = F^{-1}_{t_{n-2}}(\theta_o)$. Solving (2.4.17)

we get

$$\frac{n(n-2)\ (L-\bar{x})^2}{s^2(n-1)^2 - n(L-\bar{x})^2} \le k'^2$$

$$\frac{n(n-2)\ (L-\bar{x})^2}{k'^2} \le s^2(n-1)^2 - n(L-\bar{x})^2$$

$$\frac{n}{(n-1)^2}\left(\frac{n-2}{k'^2} + 1\right) \le \frac{s^2}{(L-\bar{x})^2}$$

$$\frac{1}{\sqrt{\frac{n}{(n-1)^2}\left(\frac{n-2}{k'^2} + 1\right)}} \ge \frac{L-\bar{x}}{s}$$

which gives

$$\frac{L-\bar{x}}{s} \le k \qquad (2.4.18)$$

where $k = \dfrac{1}{\sqrt{\frac{n}{(n-1)^2}\left(\frac{n-2}{k'^2} + 1\right)}}$

using (2.4.18) the OC function can be computed.

Approximated method for finding n and k :

Using (2.4.18) the OC function can be written as,

$$\begin{aligned} L(\mu,\sigma) &= P_{\mu,\sigma}(\text{Accepting the lot}) \\ &= P_{\mu,\sigma}\left(\frac{L-\bar{X}}{s} \le k\right) \\ &= P_{\mu,\sigma}(\bar{X} + ks \ge L) \end{aligned}$$

In order to obtain the distribution of $\bar{X} + ks$ we shall consider the following Theorem.

Theorem 2.4.1 : (Walker [1]) :

If the density functions $f(x;\theta)$, $\theta \in \Theta \subset E^{(K)}$, satisfy the Cramer-Rao regularity conditions ; if T_n is a sequence of asymptotically normal estimators, with an asymptotic covariance matrix $n^{-1} \Sigma(\theta)$; and if, for each $\theta \in \Theta$, (a) the sequence $\left[E_\theta |\sqrt{n}\,(T_{i,n} - \theta_i)|^{2+\sigma}\right]$, $i = 1,\ldots,k$ are all bounded, and (b) $(\partial/\partial\theta_j)E_\theta\left[T_{i,n} - \theta_i\right] \to 0$ for all i, $j = 1,2,\ldots,k$, then $\Sigma(\theta) - I^{-1}(\theta)$ is non-negative definite, for each $\theta \in \Theta$ (The matrix $I(\theta)$ is the fisher information matrix).

It is known that $(\bar{X}, S^2)'$ is the maximum likelihood estimate for $(\mu, \sigma^2)'$. Also the Cramer-Rao regularity conditions will be satisfied in this case. Then by above Theorem it follows that

$\sqrt{n}(\bar{X} - \mu, S^2 - \sigma^2)$ converges in distribution to bivariate normal with mean vector $(0, 0)'$ and variance covariance matrix I^{-1} where

$$I^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 2\sigma^4 \end{bmatrix}$$

Define $h(\mu, \sigma^2) = \mu + k\sigma$ which is a continuous function of μ and ∂ and also $\frac{\partial h}{\partial \mu}$ and $\frac{\partial h}{\partial \sigma^2}$ does not vanish.

Hence by Theorem (5.5.3) of Zack (1971), it follows that $h(\bar{X}, S^2) = \bar{X} + ks$ is such that $\sqrt{n}[(\bar{X}+ks)-(\mu+k\sigma)]$ converges in distribution to normal with mean 0 and variance $\left(\frac{\partial h}{\partial \mu}, \frac{\partial h}{\partial \sigma^2}\right) I^{-1}\left(\frac{\partial h}{\partial \mu}, \frac{\partial h}{\partial \sigma^2}\right)'$ which is equal to mean $\mu + k\sigma$ and variance $(\sigma^2/n)(1+k^2/2)$. So that

$$= P_{\mu,\sigma}\left[\frac{\bar{X}+ks-(\mu+k\sigma)}{(\sigma/\sqrt{n})(1+k^2/2)^{1/2}} \geq \frac{L-(\mu+k\sigma)}{(\sigma/\sqrt{n})(1+k^2/2)^{1/2}}\right]$$

$$= P_{\mu,\sigma}\left[Z \geq \frac{\sqrt{n}(Z_\theta+k)}{(1+k^2/2)^{1/2}}\right]$$

$$= 1 - P\left[Z \leq \frac{\sqrt{n}(Z_\theta+k)}{(1+k^2/2)^{1/2}}\right] \qquad (2.4.19)$$

where $Z_\theta = \frac{L-\mu}{\sigma}$. We find n and k so that the resulting plan has OC function passing through the producer's risk point $(\theta_1, 1-\alpha)$ and consumer's risk point (θ_2, β). Using (2.4.19) we get the following two equations.

$$P\left[Z \leq \frac{\sqrt{n}(Z_{\theta_1}+k)}{(1+k^2/2)^{1/2}}\right] = \alpha \qquad (2.4.20)$$

and

$$P\left[Z \leq \frac{\sqrt{n}(Z_{\theta_2}+k)}{(1+k^2/2)^{1/2}}\right] = 1-\beta \qquad (2.4.21)$$

equations (2.4.20) and (2.4.21) can be wiritten as

$$\frac{\sqrt{n}\,(Z_{\theta_1} + k)}{(1 + k^2/2)^{1/2}} = Z_\alpha \qquad (2.4.22)$$

and

$$\frac{\sqrt{n}\,(Z_{\theta_2} + k)}{(1 + k^2/2)^{1/2}} = -Z_\beta \qquad (2.4.23)$$

Substracting (2.4.22) from (2.4.23) we get

$$\frac{\sqrt{n}\,(Z_{\theta_2} - Z_{\theta_1})}{(1 + k^2/2)^{1/2}} = -Z_\beta - Z_\alpha$$

that is

$$n = (1 + k^2/2)\left(\frac{Z_\alpha + Z_\beta}{Z_{\theta_1} - Z_{\theta_2}}\right)^2 \qquad (2.4.24)$$

Substituting the value of n in (2.4.22), we get the value of **k**, that is

$$k = -\frac{Z_\alpha Z_{\theta_2} + Z_\beta Z_{\theta_1}}{Z_\alpha + Z_\beta} \qquad (2.4.25)$$

<u>Case-II</u> : <u>Upper specification limit U is given</u> :

For the upper specification limit θ is given by

$$\theta = P_{\mu,\sigma}(X \geq U)$$

$$= P_{\mu,\sigma}\left(\frac{X-\mu}{\sigma} \geq \frac{U-\mu}{\sigma}\right)$$

$$= \Phi\left(\frac{U-\mu}{\sigma}\right)$$

Let $X_1, X_2, \ldots, X_n$ be the measurements on the n items chosen at random from the lot so that $X_1, X_2, \ldots, X_n$ are i.i.d normal with mean μ and variance σ^2. In order to obtain minimum variance unbiased estimate of θ define.

$$T_2 = \begin{cases} 1 & \text{if} \quad X_1 \geq U \\ 0 & \text{otherwise.} \end{cases}$$

When μ and σ^2 are both unknown. Proceeding similarly exactly as per case ([illegible]). Then MVUE is given by

$$\hat{\theta} = F_{t_{n-2}} \left[\frac{\left(\frac{n(n-2)}{(n-1)} \right)^{1/2} \frac{U-\bar{x}}{s\sqrt{n-1}}}{\left(1 - \frac{n}{n-1} \left(\frac{U-\bar{x}}{s\sqrt{n-1}} \right)^2 \right)^{1/2}} \right]$$

Using the estimator $\hat{\theta}$, the criteria for accepting or rejecting the lot is as follows. Accept the lot if $\hat{\theta} \geq \theta_o$ otherwise reject the lot. Then

that is

$$\hat{\theta} \not\leq \theta_o \text{ iff } F_{t_{n-2}} \left[\frac{\left(\frac{n(n-2)}{n-1} \right)^{1/2} \frac{U-\bar{x}}{s\sqrt{n-1}}}{\left(1 - \frac{n}{n-1} \left(\frac{U-\bar{x}}{s\sqrt{n-1}} \right)^2 \right)^{1/2}} \right] \lesseqgtr \theta_o$$

which implies that

$$\frac{\left(\frac{n(n-2)}{n-1} \right)^{1/2} \frac{U-\bar{x}}{s\sqrt{n-1}}}{\left(1 - \frac{n}{n-1} \left(\frac{U-\bar{x}}{s\sqrt{n-1}} \right)^2 \right)^{1/2}} \lesseqgtr F^{-1}_{t_{n-2}} (\theta_o)$$

that is

$$\frac{\left(\frac{n(n-2)}{n-1}\right)^{1/2} \frac{U-\bar{x}}{s\sqrt{n-1}}}{\left(1-\frac{n}{n-1}\left(\frac{U-\bar{x}}{s\sqrt{n-1}}\right)^2\right)^{1/2}} \gtreqless k'$$

where $k' = F^{-1}_{t_{n-2}}(\Theta_o)$. Solving we get

$$\frac{n(n-2)(U-\bar{x})^2}{s^2(n-1)^2 - n(U-\bar{x})^2} \gtreqless k'^2$$

$$\frac{n(n-2)(U-\bar{x})^2}{k'^2} \lesseqgtr s^2(n-1)^2 - n(U-\bar{x})^2$$

$$\frac{n}{(n-1)^2}\left(\frac{n-2}{k'^2}+1\right) \lesseqgtr \frac{s^2}{(U-\bar{x})^2}$$

$$\frac{1}{\sqrt{\frac{n}{(n-1)^2}\left(\frac{n-2}{k'^2}+1\right)}} \gtreqless \frac{(U-\bar{x})}{s}$$

that is

$$\frac{U-\bar{x}}{s} \lesseqgtr k$$

where $k = \dfrac{1}{\sqrt{\frac{n}{(n-1)^2}\left(\frac{n-2}{k'^2}+1\right)}}$ (2.4.26)

From(2.4.26) we can compute the OC function of the sampling plan.

Approximate method for finding n and k :

Using (2.4.26),OC functioncan be written as,

$$L(\mu,\sigma) = P_{\mu,\sigma}(\text{ Accepting the lot})$$
$$= P_{\mu,\sigma}\left(\frac{U-\bar{X}}{S} \geq k \right)$$
$$= P_{\mu,\sigma}(\bar{X} + ks \leq U)$$

$\bar{X}+ks$ is approximately normally distributed with mean $\mu+k\sigma$ and variance $(\sigma^2/n)(1+k^2/2)$, so that

$$= P_{\mu,\sigma}\left[\frac{\bar{X}+ks-(\mu+k\sigma)}{(\sigma/\sqrt{n})(1+k^2/2)^{1/2}} \leq \frac{U-(\mu+k\sigma)}{(\sigma/\sqrt{n})(1+k^2/2)^{1/2}} \right]$$

$$= P_{\mu,\sigma}\left[Z \leq \frac{\sqrt{n}(Z_\theta+k)}{(1+k^2/2)^{1/2}} \right]$$

$$= P\left[Z \leq \frac{\sqrt{n}(Z_\theta+k)}{(1+k^2/2)^{1/2}} \right] \qquad (2.4.27)$$

We find n and k so that the resulting plan has OC function passing through the producer's risk point $(\theta_1, 1-\alpha)$ and consumer's risk point (θ_2,β) using (2.4.27) we get the folliwing two equations.

$$P\left[Z \leq \frac{\sqrt{n}(Z_{\theta_1}+k)}{(1+k^2/2)^{1/2}} \right] = 1-\alpha \qquad (2.4.28)$$

and

$$P[\ Z \leq \frac{\sqrt{n}(Z_{\theta_2}+k)}{(1+k^2/2)^{1/2}}\ ] = \beta \qquad (2.4.29)$$

equations (2.4.28) and (2.4.29) can be written as

$$\frac{\sqrt{n}(Z_{\theta_1}+ k)}{(1+ k^2/2)^{1/2}} = - Z_\alpha \qquad (2.4.30)$$

and

$$\frac{\sqrt{n}(Z_{\theta_2} + k)}{(1+ k^2/2)^{1/2}} = Z_\beta \qquad (2.4.31)$$

substracting (2.4.30) from (2.4.31) we get

$$\frac{\sqrt{n}(Z_{\theta_2}- Z_{\theta_1})}{(1+ k^2/2)^{1/2}} = Z_\beta+Z_\alpha$$

that is

$$n = (1+ k^2/2) \left(\frac{Z_\alpha+ Z_\beta}{Z_{\theta_2}- Z_{\theta_1}} \right)^2 \qquad (2.4.32)$$

substituting the value of n in (2.4.30) we get the value of k, that is

$$k = + \frac{Z_\alpha Z_{\theta_2} + Z_\beta Z_{\theta_1}}{Z_\alpha + Z_\beta} \tag{2.4.33}$$

Exact method for finding n and k :

Case I : Lower specification limit L is given :

Let $X_1, X_2, \ldots, X_n$ be the measurements on the n items chosen at random from the lot so that $X_1, X_2, \ldots, X_n$ are i.i.d. normal with unknown mean μ and unknown variance σ^2. The problem is to find a value of k such that

$$P(\bar{X} + ks \geq L) = 1-\alpha \tag{2.4.34}$$

that is

$$P[(\frac{\bar{X}-\mu}{\sigma})\sqrt{n} - (\frac{L-\mu}{\sigma}) \geq - \frac{ks}{\sigma}\sqrt{n}] = 1-\alpha \tag{2.4.35}$$

divide both sides of the inequality (2.4.35) by s/σ and the quantity on the left is of the form of the non-centeral t. Hence we have,

$$P[T_f \geq - k \sqrt{n} | \delta = - (\frac{L-\mu}{\sigma})\sqrt{n}] = 1-\alpha \tag{2.4.36}$$

where T_f is a non-central t variable with f degrees of freedom, where f = n-1 the number of degrees of freedom of S^2 and the non-centrality parameter δ. Then we have

$$P[T_f \geq k \sqrt{n} | \delta = Z_\theta \sqrt{n}] = \alpha \tag{2.4.37}$$

where $Z_\Theta = \frac{L-\mu}{\sigma}$ and T_f has a non-central t-distribution. Hence the quantity k which is desired may be computed from the percentage points of the non-central t-distribution.

<u>Case- II</u> : <u>Upper specification limit U is given</u> :

In this case we have to find k such that

$$P(\bar{X} + ks \leq U) = 1-\alpha \qquad (2.4.38)$$

that is

$$P\left[\left(\frac{\bar{X}-\mu}{\sigma}\right)\sqrt{n} - \left(\frac{U-\mu}{\sigma}\right)\sqrt{n} \leq - \frac{ks}{\sigma} \sqrt{n}\right] = 1-\alpha \qquad (2.4.39)$$

divide both sides of the inequality (2.4.39) by S/σ and the quantity on the left is of the form of the non-central t. Hence we have,

$$P\left[T_f \leq - k\sqrt{n} \,\middle|\, \partial = - \left(\frac{U-\mu}{\sigma}\right) \sqrt{n}\right] = 1-\alpha \qquad (2.4.40)$$

where T_f is a non-central t variable with f degrees of freedom and ∂ is a non-centrality parameter. Then we have

$$P[T_f \leq k\sqrt{n} \mid \partial = Z_\Theta\sqrt{n}] = \alpha \qquad (2.4.41)$$

where $Z_\Theta = \frac{U-\mu}{\sigma}$ and T_f has a non-central t-distribution. By using the tables 1 and 2 of Odeh and Owen (1980), we find the value of k, when all of the parameters are given.

In table 1 values of θ = 0.75, 0.90, 0.95, 0.975, 0.99, 0.999, 0.9999 and N = 2(1) 100(2) 180(5) 300(10) 400(25) 650(50), 1000, 1500, 2000, 3000, 5000, 10000 for given α = 0.995, 0.990, 0.975, 0.950, 0.900, 0.750, 0.250, 0.100, 0.050, 0.025, 0.010, 0,0005 and in table 2 sample sample size required when one specification limit is given, when θ_1 = 0.005(0.005)0.05, 0.075, 0.10,

$\theta_2 = 2\theta_1$(0.005)0.10, 0.15, 0.20, 0.30 for given α = 0.01, 0.025, 0.05 and β = 0.05, 0.10, 0.20.

Example 2.3:

Suppose that we are given the following quantities

$\alpha = .05$, $\beta = .05$, and $\theta_1 = .01$ and $\theta_2 = .300$

Using equation (2.4.24) we will get the value of n, that is $n = 7$,

and from equation (2.4.25) we will get the value of k, that is $k = 1.423$.

In the following table IX and X the different values of α, β and θ_2, the n and k is computed and is compared with the exact distribution parameters.

TABLE-IX

$\alpha = .05, \quad \beta = .05, \quad \theta_1 = .01$

θ_2	Approximate distribution n	Approximate distribution k	Exact distribution n	Exact distribution k
.300	7	1.423	8	1.443
.200	12	1.583	12	1.604
.150	16	1.678	17	1.695
.100	26	1.803	27	1.814
.095	28	1.818	29	1.828
.090	30	1.833	31	1.843
.085	33	1.848	33	1.858
.080	35	1.863	36	1.874
.075	39	1.883	39	1.891
.070	42	1.898	43	1.908
.065	47	1.918	48	1.926
.060	52	1.938	54	1.946
.055	59	1.958	61	1.967
.050	69	1.983	70	1.990
.045	81	2.008	83	2.015
.040	97	2.033	102	2.041
.035	128	2.068	130	2.072
.030	175	2.103	176	2.105
.025	252	2.138	267	2.144
.020	483	2.188	496	2.191

TABLE-X

α = .01, β = .05, θ_1 = 01

Q_2	Approximate distribution		Exact distribution	
	n	k	n	k
.300	9	1.2681	10	1.285
.200	15	1.4555	16	1.467
.150	21	1.5668	23	1.578
.100	36	1.7133	37	1.719
.095	39	1.7308	40	1.736
.090	42	1.7484	43	1.753
.085	45	1.7660	46	1.772
.080	48	1.7835	50	1.791
.075	53	1.8070	54	1.810
.070	58	1.8246	60	1.831
.065	65	1.8480	67	1.853
.060	73	1.8714	75	1.877
.055	82	1.8948	85	1.902
.050	96	1.9241	99	1.929
.045	114	1.9534	117	1.958
.040	137	1.9827	144	1.990
.035	181	2.0237	184	2.026
.030	249	2.0647	251	2.066
.025	359	2.1057	382	2.112
.020	692	2.1643	714	2.167

TABLE-XI

$\alpha = .025, \quad \beta = .1, \quad \theta_1 = .01$

θ_2	Approximate distribution		Exact distribution	
	n	k	n	k
.300	6	1.233	7	1.270
.200	10	1.427	11	1.453
.150	14	1.542	15	1.566
.100	24	1.693	25	1.707
.095	26	1.711	27	1.724
.090	27	1.729	29	1.741
.085	30	1.747	31	1.760
.080	32	1.765	33	1.779
.075	35	1.790	36	1.799
.070	38	1.808	40	1.820
.065	43	1.832	44	1.843
.060	48	1.856	50	1.866
.055	54	1.880	57	1.892
.050	64	1.911	66	1.919
.045	75	1.941	78	1.949
.040	90	1.971	96	1.982
.035	120	2.013	123	2.018
.030	165	2.056	167	2.059
.025	238	2.098	254	2.106
.020	459	2.159	475	2.162

TABLE-XII

$\alpha = .025, \quad \beta = .05, \quad \theta_1 = .01$

θ_2	Approximate distribution		Exact distribution	
	n	k	n	k
.300	8	1.342	9	1.361
.200	13	1.517	14	1.533
.150	18	1.620	19	1.637
.100	31	1.756	32	1.765
.095	33	1.772	34	1.781
.090	35	1.789	36	1.797
.085	38	1.805	39	1.814
.080	41	1.821	42	1.831
.075	45	1.843	46	1.849
.070	49	1.860	51	1.869
.065	54	1.881	56	1.889
.060	61	1.903	63	1.911
.055	69	1.925	72	1.934
.050	81	1.952	83	1.959
.045	95	1.979	98	1.986
.040	114	2.007	120	2.015
.035	151	2.045	154	2.048
.030	207	2.083	209	2.085
.025	298	2.121	318	2.128
.020	573	2.175	592	2.179