CHAPTER II

BAYES TEST PROCEDURE FOR VECTOR VALUED PARAMETER AND MULTIPLE HYPOTHESIS TESTS

2.0 Introduction :

Let X be a random variable (may be vector) with density function $f_w(x)$, $w \in \Omega$ where $\Omega \subset R^k$, $k \geq 2$. Let H_1: $w \in W_1$ and H_2: $w \in W_2$, $W_2 = \Omega - W_1$. Let d_i denote the decision of accepting the hypothesis H_i, $i = 1,2$, When $w \in W_i$, if the decision taken is d_i, then there is no error in the decision taken. However if $w \in W_i$ and the decision taken is d_j, $j \neq i$ then there is an error associated with the decision. As $\Omega \subset R^k$ there are different ways to quantify the loss associated with a decision rule. For this We essentially ~~we~~ have to define norms on R^k, and based on the norm defined the loss function can be defined. Some commonly used norms are ;

(1) $$||x|| = (\Sigma x_i^2)^{1/2}$$

(2) $$||x|| = \max |x_i|$$

(3) $$||x|| = \Sigma |x_i|$$

Let $A \subset R^k$ we define the distance of A from w, $d(w,A) = \inf\limits_{y \subset A} ||w-y||$. A general form of a loss function is given by

$$L_1 = L(w,d_1) = \begin{cases} 0, & \text{if } w \in W_1 \\ h_1[d(w,W_1)], & \text{if } w \in W_2. \end{cases}$$

$$L_2 = L(w,d_2) = \begin{cases} 0, & \text{if } w \in W_2 \\ h_2[d(w,W_2)], & \text{if } w \in W_1. \end{cases}$$

where h_i, $i = 1,2$, is non-negative increasing function defined on $[0,\infty)$

$$d(w, A) = 0 \quad \text{if } w \in A.$$

The above loss function can be written as

$$L(w,d_i) = h_i[d(w,W_i)], \; i = 1,2 \quad \text{with } h(0) = 0$$

In remainder of this chapter we consider the testing of hypothesis problem concerned with mean θ of normal distribution and for this we need to refer result (Degroot 176). A random sample from multivariate normal distribution with unknown value of the mean vector M and a specified precision matrix (inverse of variance co-variance matrix) r. Suppose also that the distribution of M is a multivariate normal distribution with mean vector μ and precision matrix τ such that $\mu \in R^k$ and τ is a symmetric positive definite matrix. Then the posterior distribution of M when $X_i = x_i (i=1,2,..,n)$ is a multivariate normal distribution with mean vector μ^* and precision matrix $\tau + nr$, where $\mu^* = (\tau + nr)^{-1} (\tau\mu + nr\,\bar{x})'$.

In Section 2.1 we introduce ______ model (2.1.1) by choosing suitable norms and the h-functions. The problem of multiple hypotheses testing is described in Section 2.2. A finite partition of the parameter space Ω is specified as

$\left\{ W_1, W_2, \ldots, W_m \right\}$. The statistician has to decide to which one of the m subsets w belongs. Here the decision $w \in W_j$ is interpreted as the acceptance of the hypothesis $H_j: w \in W_j$ (i = 1,2,...,m) and rejection of all other (m-1) hypothesis. In this problem we notice that Bayes procedure against any prior is not necessarily unique and does not requires randomisation.

2.1 Models :

2.1.1 : Model :

Hypothesis concerning mean of bivariate normal distribution.

Let X,Y be random variables having bi-variate normal distribution with mean (θ_1, θ_2) and the precision matrix I_2. The prior distribution be normal with mean (0, 0) and precision matrix I_2.

(A) $H_1: \theta \in \Theta_1 = \left\{ (\theta_1, \theta_2) : \theta_1 \geq 0, \theta_2 \geq 0 \right\}$.

$H_2: \theta \in \Theta_2, \quad \Theta_2 = \Theta - \Theta_1, \quad \Theta = R^2$.

Let $||.||$ be any norm and

$$h(t) = \begin{cases} 1, & \text{if } t > 0 \\ 0, & \text{if } t = 0 \end{cases}$$

In this case $h_1(t) = h_2(t) = 1$ for $t > 0$. This is zero-one loss function.

Using the result given in section 2.0 . the posterior distribution of θ will be bivariate normal (BN) with mean vector $(\bar{x}/2, \bar{y}/2)'$ and variance-co-variance matrix

$$1/2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Following the notations in (1.3.1) and (1.3.2) we get

$$R_1 = \int_{\theta \in \text{Ⓗ}_2} \int 1.BN_{\theta_1\theta_2}(\bar{x}/2, \bar{y}/2, 1/2, 1/2, 0)\, d\underline{\theta}$$

where $BN_{\theta_1,\theta_2}(\bar{x}/2, \bar{y}/2, 1/2, 1/2, 0)$ represents the posterior distribution of θ_1,θ_2 given $x = x_1,\ldots,x_n$ and $y = y_1,\ldots,y_n$ and is bivariate normal with parameters $\bar{x}/2$, $\bar{y}/2$, 1/2, 1/2, 0.

$$R_2 = \int_{\theta \in \text{Ⓗ}_1} \int 1.BN_{\theta_1,\theta_2}(\bar{x}/2, \bar{y}/2, 1/2, 1/2, 0)\, d\underline{\theta}$$

By the criteria given in (1.3.3)

Accept the H_1 if $R_1 < R_2$. But in this case we get

$R_1 = 1 - R_2$.

Therefore accept H_1 if $R_2 > 1/2$ or $R_1 < 1/2$ (2.1.1)

$$R_2 = \int_0^\infty \int_0^\infty \frac{1}{\sqrt{2\pi}\sqrt{1/2}} \exp - \frac{1}{2.1/2}(\theta_1 - \bar{x}/2)^2\, d\theta_1 . \frac{1}{\sqrt{2\pi}\sqrt{1/2}} \exp - \frac{1}{2.1/2}(\theta_2 - \bar{y}/2)^2\, d\theta_2.$$

$$= P[\theta_1 > 0].\ P[\theta_2 > 0]$$

where $\theta_1 \backsim N(\bar{x}/2, 1/2)$ and $\theta_2 \backsim N(\bar{y}/2, 1/2)$.

$$= P[(\theta_1 - \bar{x}/2)\sqrt{2} > (0 - \bar{x}/2)\sqrt{2}].\ P\ [(\theta_2 - \bar{y}/2)\sqrt{2} > -\bar{y}/2\ \sqrt{2}]$$

$$= \Phi(\bar{x}/\sqrt{2})\ .\ \Phi(\bar{y}/\sqrt{2}).$$

Thus the acceptance region is given by

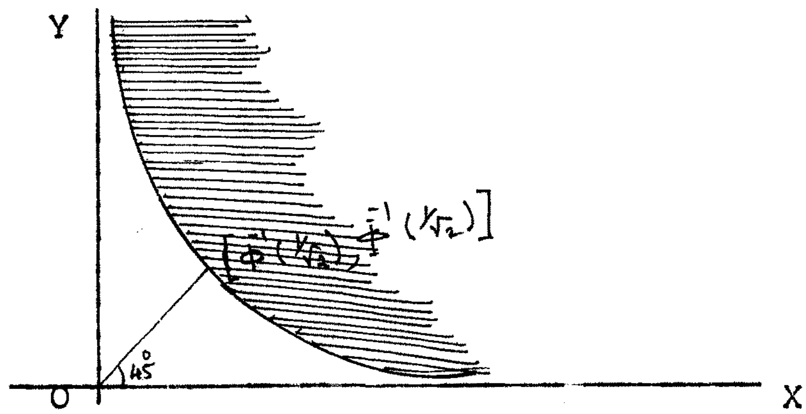
$\left\{(x,y) : \Phi(\bar{x}/\sqrt{2}) . \Phi(\bar{y}/\sqrt{2}) > 1/2\right\}$, That is

$\left\{(x,y) : \Phi(X = \bar{x}/\sqrt{2}) . \Phi(Y = \bar{y}/\sqrt{2}) > 1/2\right\}$.

In the following we sketch this region of acceptance. Note that both X and Y should be positive (otherwise the condition will not be satisfied). For a given X > 0 we choose the value of Y such that the equality in (2.1.1) holds. Different values of X and Y are tabulated in the table (2.1.9).

Table : (2.1.9)

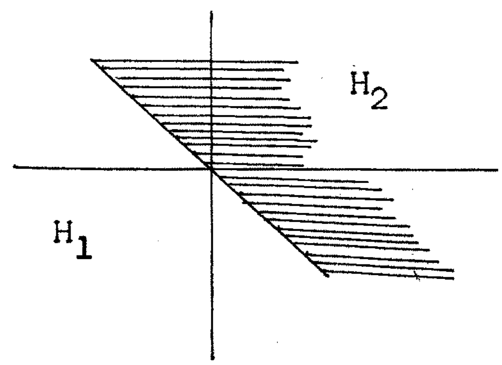
X	$\Phi(X)$	$\Phi(Y)$	Y
0.5	0.6915	0.7230	0.59
0.6	0.7257	0.6890	0.49
0.7	0.7580	0.6596	0.41
0.8	0.7881	0.6344	0.34
0.9	0.8159	0.6128	0.29
1.0	0.8413	0.5943	0.24
1.1	0.8643	0.5785	0.20
1.2	0.8849	0.5650	0.16
1.5	0.9332	0.5358	0.09
1.8	0.9641	0.5186	0.05
2.0	0.9772	0.5117	0.03
2.5	0.9938	0.5031	0.01



B) Let us study the same problem by changing the hypothesis of test and corresponding loss functions.

$H_1 : \theta_1 + \theta_2 \leq 0$

$H_2 : \theta_1 + \theta_2 > 0$



$$L_1 = \begin{cases} \frac{\theta_1 + \theta_2}{\sqrt{2}} & , \text{ if } \theta \in H_2 \\ 0 & , \text{ if } \theta \in H_1 \end{cases}$$

$$L_2 = \begin{cases} 0 & , \text{ if } \theta \in H_2 \\ -\left(\frac{\theta_1 + \theta_2}{\sqrt{2}} \right) & , \text{ if } \theta \in H_1 \end{cases}$$

This is equivalent to choosing $||x|| = |x_1|+|x_2|$ and $h(t) = t/\sqrt{2}$ for $t > 0$.

$$R_1 = \int_{-\infty}^{\infty} \int_{-\theta_1}^{\infty} \frac{\theta_1 + \theta_2}{\sqrt{2}} \cdot BN_{\theta_1,\theta_2}(\bar{x}/2, \bar{y}/2, 1/2, 1/2, 0)\, d\underline{\theta}$$

$$= \int_{-\infty}^{\infty} \Big[\int_{-\infty}^{\infty} \frac{\theta_1 + \theta_2}{\sqrt{2}} \cdot BN_{\theta_1,\theta_2}(\bar{x}/2, \bar{y}/2, 1/2, 1/2, 0)\, d\underline{\theta}$$

$$- \int_{-\infty}^{-\theta_1} \frac{\theta_1 + \theta_2}{\sqrt{2}} \cdot BN_{\theta_1,\theta_2}(\bar{x}/2, \bar{y}/2, 1/2, 1/2, 0)\, d\underline{\theta} \Big].$$

$$R_1 = E\left(\frac{\theta_1+\theta_2}{\sqrt{2}} \right) + R_2$$

Accept H_1 if $R_1 - R_2 < 0$

That is if $E\left(\frac{\theta_1+\theta_2}{\sqrt{2}} \right) < 0.$

gives, accept H_1 if $\bar{x} + \bar{y} < 0$.

C) For the same hypothesis as in (B) consider the loss function as given below :

$$L_1 = \begin{cases} 0 , & \text{if } \theta \in H_1 \\ (\theta_1+\theta_2)^2 , & \text{if } \theta \in H_2 \end{cases}$$

$$L_2 = \begin{cases} 0 , & \text{if } \theta \in H_2 \\ (\theta_1+\theta_2)^2 , & \text{if } \theta \in H_1 \end{cases}$$

$h(t) = t^2$.

$$R_1 = \int_{-\infty}^{\infty} \int_{-\theta_1}^{\infty} (\theta_1+\theta_2)^2\, BN_{\theta_1,\theta_2}(\bar{x}/2, \bar{y}/2, 1/2, 1/2, 0)\, d\underline{\theta}$$

$$= \int_{-\infty}^{\infty} [\int_{-\infty}^{\infty} (\theta_1+\theta_2)^2 \, BN_{\theta_1,\theta_2}(\bar{x}/2, y/2, 1/2, 1/2, 0) \, d\underline{\theta}$$

$$- \int_{\infty}^{-\theta_1} (\theta_1+\theta_2)^2 \, BN_{\theta_1,\theta_2}(\bar{x}/2, \bar{y}/2, 1/2, 1/2, 0) d\underline{\theta}]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\theta_1+\theta_2)^2 \, BN_{\theta_1,\theta_2}(\bar{x}/2, \bar{y}/2, 1/2, 1/2, 0) \, d\underline{\theta}$$

$$- \int_{-\infty}^{\infty} \int_{-\infty}^{-\theta_1} (\theta_1+\theta_2)^2 \, BN_{\theta_1,\theta_2}(\bar{x}/2, \bar{y}/2, 1/2, 1/2, 0) \, d\underline{\theta}$$

$$= E\,(\theta_1 + \theta_2)^2 + R_2$$

Therefore $R_1 - R_2 = E(\theta_1 + \theta_2)^2$

$$= E(\theta_1^2 + 2\theta_1\theta_2 + \theta_2^2)$$

$$= 1/2 + \bar{x}^2/4 + 1/2 + \bar{y}^2/4 + 2\,\bar{x}/2.\bar{y}/2.$$

Therefore $R_1 - R_2 = 1 + (\bar{x}^2+\bar{y}^2)/4 + (\bar{x}\bar{y})/2$

$$= 4 + (\bar{x} + \bar{y})^2$$

We have accept H_1 if $R_1 - R_2 < 0$

Therefore accept H_1 if $(\bar{x} + \bar{y})^2 < -4$

D) Now for the same bivariate distribution let us consider different hypothesis.

$$H_\partial : \theta_1^2 + \theta_2^2 \leq \partial$$

$$k_\partial : \theta_1^2 + \theta_2^2 > \partial.$$

corresponding loss functions defined are .

$$L_H = \begin{cases} 0 & , \quad \text{if } \theta_1^2 + \theta_2^2 \leq \partial \\ \theta_1^2+\theta_2^2-\partial & , \quad \text{if } \theta_1^2 + \theta_2^2 > \partial \end{cases}$$

$$L_k = \begin{cases} 0 & , \quad \text{if } \theta_1^2 + \theta_2^2 > \partial \\ \partial-(\theta_1^2+\theta_2^2) & , \quad \text{if } \theta_1^2 + \theta_2^2 \leq \partial \end{cases}$$

$||x|| = (x^2+y^2)^{1/2}$ and $h(t) = t$.

R_H = Risk in accepting $H\partial$.

R_k = Risk in accepting $k\partial$.

$$R_k = \int_{H_\partial}\int [\partial-(\theta_1^2+\theta_2^2)]\ BN_{\theta_1,\theta_2}(\bar{x}/2,\ \bar{y}/2,\ 1/2,\ 1/2,\ 0)d\underline{\theta}$$

$$= \int_{\textcircled{H}}\int [\partial-(\theta_1^2+\theta_2^2)]\ BN_{\theta_1,\theta_2}(\bar{x}/2, \bar{y}/2,\ 1/2,\ 1/2,\ 0)\ d\underline{\theta} + \int\int_{k_\partial}(\theta_1^2+\theta_2^2-\partial)BN_{\theta_1,\theta_2}(\bar{x}/2,\bar{y}/2,\tfrac{1}{2},\tfrac{1}{2},0)d\underline{\theta}$$

That is $R_k - R_H = \partial - E(\ \theta_1^2 + \theta_2^2\ )$.

$$= \partial - 1 - \frac{\bar{x}^2 + \bar{y}^2}{4}$$

We have reject H_∂ if $R_k < R_H$.

Therefor for $\partial \leq 1$ reject H_∂.

If $\partial > 1$, then reject H_∂ if

$$\bar{x}^2 + \bar{y}^2 > 4(\partial-1).$$

E) In the following discussion take zero-one loss function for the same problem. as,

$$L_H = \begin{cases} 1 & , \quad \text{if} \quad \theta_1^2 + \theta_2^2 \geq \partial. \\ 0 & , \quad \text{if} \quad \theta_1^2 + \theta_2^2 < \partial. \end{cases}$$

$$L_k = \begin{cases} 1, & \text{if } e_1^2 + e_2^2 < \partial \\ 0, & \text{if } e_1^2 + e_2^2 \geq \partial \end{cases}$$

Accept H_∂ if $R_k - R_H > 0$.

But $R_k + R_H = 1$

Therefore accept H_∂ if $R_k > 1/2$.

$$R_k = P[e_1^2 + e_2^2 \leq \partial \,/\, (\underline{e} \sim BN(\bar{x}/2, \bar{y}/2, 1/2, 1/2, 0)]$$

$$= P[\chi_2^2(\lambda) \leq \partial]$$

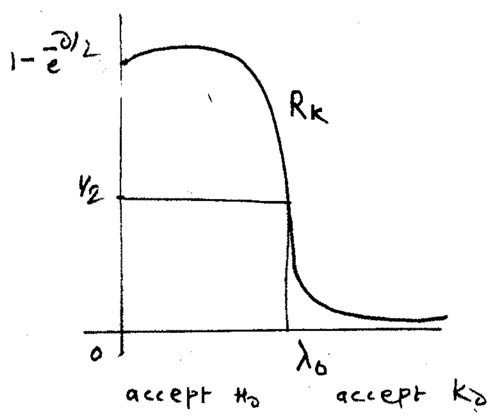
$e_1^2 + e_2^2$ follows non-central χ^2 distribution with 2 d.f. the non-centrality parameter being $\lambda = \bar{x}^2 + \bar{y}^2$.

$R_k = g(\lambda)$ is a decreasing function of $\lambda = \bar{x}^2 + \bar{y}^2$.

or $\lambda = 0$, $\quad g(0) = P[\chi_2^2 \leq \partial]$

$$= \int_0^\partial 1/2 \; e^{-\chi^2/2} \, (\chi^2)^{\frac{2}{2}-1} \, d\chi^2$$

$$= 1 - e^{-\partial/2}$$



Reject H_∂ if $R_k < R_H$.

$\Leftrightarrow R_k < 1/2$

$\Leftrightarrow g(0) < 1/2$

$\Leftrightarrow 1 - e^{-\partial/2} < 1/2$

$\Leftrightarrow \quad e^{-\partial/2} > 1/2.$

$\Leftrightarrow \; - \partial/2 > \log e^{1/2}.$

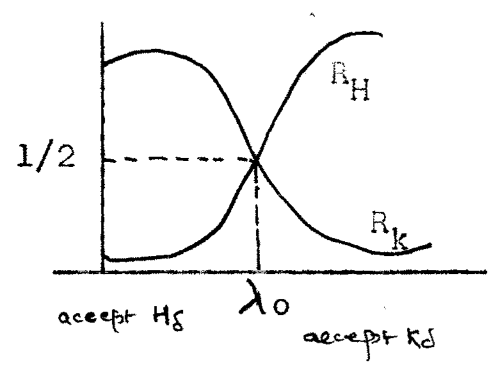
$\Leftrightarrow \; - \partial > 2 \log \quad 1/2$

$\Leftrightarrow \; \partial < -2 \log \quad 1/2.$

If $\partial \geq -2 \log_e 1/2$ then accept H_∂ whenever

$R_H < R_k$ i.e. $\lambda < \lambda_o$

where $\lambda_o = \bar{x}^2 + \bar{y}^2.$



2.1.2 Model : Multinomial :

Let $X = (X_1, X_2, \ldots, X_k)'$ be a k-dimensional random vector (multinomial) with parameter n, $\underline{\theta} = (\theta_1, \theta_2, \ldots, \theta_k)'$. Its distribution be denoted by $M_k(n, \underline{\theta})$, $0 \leq \theta_i \leq 1$ and $\sum_{i=1}^{k} \theta_i = 1$. The problem is to test the hypothesis

$H_1 : \underline{\theta} = \underline{\theta}_1 = (\theta_{11}, \theta_{12}, \ldots, \theta_{1k})'$ against

$H_2 : \underline{\theta} = \underline{\theta}_2 = (\theta_{21}, \theta_{22}, \ldots, \theta_{2k})'$ based on a single observation x. Let the prior distribution be $P[\underline{\theta} = \underline{\theta}_1] = \xi$ and $P[\underline{\theta} = \underline{\theta}_2] = 1 - \xi$. Consider the loss function zero-one. It is very evident from the discussion of section (1.3) that accept H_1 if $P(\theta_1/x) > P(\theta_2/x)$.

That is accept H_1 if,

$$\left(\frac{\theta_{11}}{\theta_{21}}\right)^{x_1} \cdot \left(\frac{\theta_{12}}{\theta_{22}}\right)^{x_2} \cdots \cdot \left(\frac{\theta_{1k}}{\theta_{2k}}\right)^{x_k} > 1$$

That is $\sum_{i=1}^{k} x_i \log \frac{\theta_{1i}}{\theta_{2i}} > 0.$

Now consider the case where $X \sim M_k(n, \underline{\theta})$ having $H_1 : P < P_o$, where $p = \theta_1 + \theta_2 + --- + \theta_r$, $r < k$ against $H_2 : p \geq p_o$.

With zero-one loss function, the prior distribution of p will be beta (a,b). (The prior distribution of θ be a direchlet distribution with parameter $(m_1, m_2, \ldots, m_k)$).

Note that $y = x_1 + x_2 + \ldots + x_r$ has $B(n,p)$; $p = \theta_1 + \theta_2, + --- + \theta_r$.

Now the problem receives the form exactly equal to example 1.3.3.

2.2 Multiple Hypothesis Testing :

Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables, having a common distribution function. $F_w(x)$, $w \in \Omega$, Ω is a specified interval in a Euclidean k-space, $E^{(k)}$. The sample space n is fixed. Let T designate the minimal sufficient statistic for the family $F = \{F_w, w \in \Omega\}$. T is some r dimensional vector, $1 \leq r \leq n$. The problem of multiple hypothesis testing is described in section (2.0) where the parameter space Ω is partitioned into

$\{W_1, W_2, \ldots, W_m\}$. The decision that $w \in W_j$ is interpreted as the acceptance of hypothesis H_j: $w \in W_j$, $(j = 1,2,\ldots,m)$ and the rejection of the other $(m-1)$ alternative hypotheses. The decision is performed by a randomized test function $\phi(T)$, which is probability vector $\phi(T) = (\phi_1(T), \ldots, \phi_m(T))'$.

$$\phi_j(T) \geq 0, \; j = 1,2,\ldots,m.$$

$$\sum_{j=1}^{m} \phi_j(T) = 1.$$

The class D of all randomized test functions is the $(m-1)$ dimensional simplex of all probability vectors $\phi = (\phi_1, \phi_2, \ldots, \phi_m)'$. Non-negative functions $L_j(w)$, $j = 1,2,\ldots,m$ are defined on Ω. The function $L_j(w)$ designates the loss associated with the acceptance of H_j, when w is the value of the parameter.

Let $f(t,w)$ the density function of T with respect to a measure $\mu(dt)$ under w. Let $H(w)$ designate a prior distribution on Ω. The risk function associated with a test function ϕ is

$$R(w, \phi) = \sum_{j=1}^{m} L_j(w) \int \phi_j(t) f(t,w) \mu(dt).$$

$$0 \leq R(w, \phi) < \infty \text{ for all } w \in \Omega \text{ since}$$

$$0 \leq \phi_j(t) \leq 1 \text{ for } i = 1,2,\ldots, m.$$

The prior risk associated with $H(w)$ and ϕ is

$$R(H,\phi) = \sum_{j=1}^{m} \int \phi_j(t)\, \mu(dt) \int H(dw)\, L_j(w)\, f(t,w) \quad ..(2.2.1)$$

We assume that, for each $j = 1,2,\ldots,m$.

$$R_j(t) = \int H(dw)\, L_j(w)\, f(t,w) < \infty \text{ a.s.}[\mu] \qquad (2.2.2)$$

$$= f_1(t) \int L_j(w)\, f(w/t)\, dw.$$

$$\int \mu(dt)\, R_j(t) < \infty.$$

This implies that $R(H,\phi) < \infty$ for all $\phi \in D$. A test function ϕ^H is called Bayes against H if it minimizes (2.2.1). Now it is easy to verify that

$$\phi_j^H(T) = \begin{cases} 1, & \text{if } R_j(T) = \min\limits_{i=1,2,\ldots,n} R_i(T) \\ 0, & \text{otherwise.} \end{cases}$$

We notice that a Bayes procedure against any prior distribution is,

i) not necessarily unique — why?

ii) it does not require randomization. — why?

<u>Example 2.2.1</u> :

Suppose that $X_1, X_2, \ldots, X_n$ is a random sample from a normal distribution with an unknown value of mean θ and an unknown value of variance $1/\sigma'$, the prior joint distribution of θ and $1/\sigma'$ is the conditional distribution of θ when $\sigma' = \sigma$ $(\sigma > 0)$ is a normal distribution with mean μ and variance $1/\tau\sigma$ such that $-\infty < \mu < \infty$ and $\tau > 0$ and marginal distribution of $1/\sigma'$ is gamma distribution with parameter α such that $\alpha > 0$. Then the posterior joint distribution of θ and $1/\sigma'$ when $X_i = x_i (i=1,2,\ldots n)$ is the conditional distribution of θ when $1/\sigma' = 1/\sigma$ is a normal distribution with mean μ' and variance $\frac{1}{(\tau+n)\sigma}$ where $\mu' = \frac{\tau\mu+n\bar{x}}{\tau+n}$. And marginal distribution of $1/\sigma$ is a gamma distribution with parameter α. In particular $\tau = 1$. The marginal posterior density of θ, $f(\theta/x)$ follows 't' distribution with 2α d.f. with location parameter μ' and scale parameter $\frac{1}{\alpha(n+1)}$ (Ref. example 1.3.2).

Consider the problem of testing

H_1: $\theta \in \Theta_{-1} = (-\infty, -1)$

H_2: $\theta \in \Theta_{0} = [-1, 1]$

H_3: $\theta \in \Theta_{1} = (1,\infty)$

$$L_1 = \begin{cases} 0, & \text{if } \theta < -1 \\ 1, & \text{otherwise} \end{cases}$$

$$L_2 = \begin{cases} 0, & \text{if } |\theta| \leq 1 \\ 1, & \text{otherwise} \end{cases}$$

$$L_3 = \begin{cases} 0, & \text{if } \theta > 1 \\ 1, & \text{otherwise.} \end{cases}$$

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$$R_1 = \int_{-1}^{\infty} \sqrt{\frac{\alpha(n+1)}{2\pi\alpha}} \; \frac{\Gamma\frac{2\alpha+1}{2}}{\Gamma\alpha} \cdot \left[1 + \frac{t^2}{2\alpha} \right]^{-\frac{(2\alpha+1)}{2}} \cdot dt$$

where $t = \frac{(\theta - \mu')}{1/\sqrt{\alpha(n+1)}}$

$$= \int_{-1}^{\infty} \frac{\Gamma\frac{2\alpha+1}{2}}{\Gamma\alpha \, \Gamma 1/2} \; \frac{1}{\sqrt{2\alpha}} \cdot \left(1 + \frac{t^2}{2\alpha}\right)^{-\frac{(2\alpha+1)}{2}} \cdot dt$$

$$= P(\theta \geq -1)$$

That is $P[(\theta - \mu')\sqrt{\alpha(n+1)} \geq -(1 + \mu')\sqrt{\alpha(n+1)}\ ]$

$$R_2 = P[\theta \leq -1] + P[\theta \geq 1]$$

$$= P[(\theta - \mu')\sqrt{\alpha(n+1)} \leq -(1 + \mu')\sqrt{\alpha(n+1)}\ ]$$

$$+ P[(\theta - \mu')\sqrt{\alpha(n+1)} \geq (1 - \mu')\sqrt{\alpha(n+1)}\ ].$$

$$R_3 = P[\theta \leq 1]$$

$$= P[(\theta - \mu')\sqrt{\alpha(n+1)} \leq (1 - \mu')\sqrt{\alpha(n+1)}\ ]$$

Fisher and Yates tables gives the significant values of t, say to corresponding to P_F.

$$P_F = P[\ |t| > t_o] = 1 - P[|t| \leq t_o\ ].$$

$$= 1 - 2\,P[0 \leq t \leq t_o\ ]$$

$$= 2\,(\ 1 - P_s)$$

$$P_s = P[t \leq t_o] = 0.5 + \int_o^{t_o} f(t)\, dt.$$

Therefore

$$R_1 = 1 - P_s[(1+\mu')\sqrt{\alpha(n+1)}]$$

$$R_2 = 1 - P_s[(1+\mu')\sqrt{\alpha(n+1)}] + 1 - P_s[(1-\mu')\sqrt{\alpha(n+1)}]$$

$$= 2 - P_s[(1+\mu')\sqrt{\alpha(n+1)}] - P_s[(1-\mu')\sqrt{\alpha(n+1)}]$$

$$R_3 = P_s[(1-\mu')\sqrt{\alpha(n+1)}]$$

R_2 is symmetric in μ' with minimum at $\mu = 0$

R_2 is monotone decreasing on $(-\infty.0)$ with $\lim_{x \to -\infty} R_2 = 1$

The function R_1 is monotone increasing with $\lim_{x \to -\infty} R_1 = 0$ and $\lim_{x \to \infty} R_1 = 1$

Hence there exists a unique point ξ_{-1} in $(-\infty,0)$ at which $R_1 = R_2$. Symmetrically $\xi_1 = \xi_{-1}$ is the unique point in $(0,\infty)$ is partitioned into three subsets, $(-\infty, \xi_{-1})$; (ξ_{-1}, ξ_1); (ξ_1, ∞).

If $X \in (-\infty, \xi_{-1})$ we accept H_1

If $X \in (\xi_{-1}, \xi_1)$ we accept H_2

If $X \in (\xi_1, \infty)$ we accept H_3.
