CHAPTER - I

INTRODUCTION AND HISTORY

1.1. INTRODUCTION

Extensions of univariate distributions such as Binomial, Poisson, Normal and Gamma to bivariate or multivariate situations have been of considerable interest. In particular, the extension of univariate exponential distribution to bivariate or multivariate case was of much interest, An illustration, consider the following example,

EXAMPLE : Suppose that some components of two similar plants share a certain common risk of breakdown because components are produced by the same manufacturer or because components face the same exogenous risk, where as other remaining components are rather plant specific and hence their life lengths are independent. We assume that, every plant is operating under a serial structure, which functions only when both component functions and both the components act independently within a plant. If a plant fails due to failure of common component; then there is a more chance that the other plant will fail; where as if a plant fails due to failure of specific component; then there is no new information regarding the failure time of the other plant. When we only know that a plant failed at a point in time without identifying the source of failure; whether from the common component or from the plant specific one, the failure time of a plant still conveys useful information to the plant manager regarding the failure time of the other plant. So far as there is any likelihood that the plant failure was caused by a failure of common component. A way of quantifying this information is to specify the dependence between the two failure times through dependence between the common components' failure times.

Exponential distribution plays a central role in reliability theory as well as in building models for data arising from life tests. However there does not exists a unique natural extensions of the univariate exponential distribution to the bivariate or multivariate case. A number of bivariate or multivariate extensions of univariate exponential distribution are available in literature. Important bivariate exponential distributions (BVED) were introduced by Gumbel(1960), Freund(1961), Marshall-Olkin(1967), Downton(1970), Hawkes(1972), Paulson(1973), Block-Basu(1974), Proschan-Sullo(1974), Sarakar(1987) and Keunkwan Ryu(1993) as models for the distribution of (X, Y) the failure times of dependent components (C_1, C_2).

1.2. History of some standard Bivariate Exponential Distributions

Freund(1961) proposed a bivariate exponential distribution (BVED) as a model for a system with the life times (X, Y) of the two components (C_1, C_2) operating in the following manner. Initially X and Y are independent exponential with parameters θ_1 and θ_2 respectively, where $\theta_1, \theta_2 \geq 0$. The interdependence of the component is such that failure of a component C_1 (or C_2) changes the failure rate of C_2 (or C_1) from θ_2 to θ_2' (or θ_1 to θ_1'). This is due to failure of one component which results an extra load on the other component as in several biometrical applications like survival of Kidneys, lungs, eyes, ears or any paired organs in human body.

The p.d.f. of bivariate random vector (X, Y) following a Freund's model is given by

$$f(x, y) = \begin{cases} \theta_1\theta_2' \exp(-\theta_2' y - (\theta_1 + \theta_2 - \theta_2')x) ; & x < y \\ \theta_2\theta_1' \exp(-\theta_1' x - (\theta_1 + \theta_2 - \theta_1')y) ; & x > y \end{cases},$$

with P[X = Y]= 0. Freund model leads to a regular exponential family with a four dimensional orthogonal parameters. However we may have some situations where the probability of simultaneous failure is positive. Typically this can happen when both components fail at the same time due to an external shock. Then BVED of Marshall-Olkin(M-O) is the most important applicable exponential distribution. Its importance is mainly due to the fact that itsapplicable in varietyofsituations.

Marshall-Olkin(1967) derived a bivariate exponential distribution under a rather simplified scenario of the physical environments facing two plants: A plant-specific shock causes the plant receiving the shock to breakdown. A common shock breaks both plants. Shocks arrive according to exponential waiting time distribution. Shocks to different components occur independently. In fact, any single shock need not be fatal to derive the bivariate exponential distribution in so far as shocks that are bypassed without causing failures have no after effects; that is, once a plant survives a shock, then its physical condition is virtually the same as that of a new plant that has not experienced a shock. This assumption, coupled with the exponential waiting time distribution. There are two types of memoryless properties. First, marginal lack of memory property (MLMP) means that one dimensional marginal failure times are exponentially distributed and thus are memoryless. Second, joint lack of memory property (JLMP) means that, given both plants survive up to a certain age, the remaining joint life distribution is same as that of a new pair of plants.

The Marshall-Olkin model satisfies the lack of memory property and has exponential marginals but it is not absolutely continuous with respect to Lebesgue measure in $\mathbb{R}^2$ and contains singular component on the diagonal (X = Y) arising out of possible simultaneous failures. The BVED of Marshall-Olkin(M-O) is defined as follows; The Life times (X, Y) of the components (C_1, C_2) are said to follow a BVED of M-O if

$$\bar{F}_M(x,y) = P[\ X > x,\ Y > y\ ]$$

$$= \exp\{-\lambda_1 x - \lambda_2 y - \lambda_3 \max(x,\ y)\ \},\quad x,\ y \geq 0,$$

$$\lambda_1,\ \lambda_2,\ \lambda_3 \geq 0,$$

Where λ_1, λ_2 and λ_3 are parameters. The M-O model involving three parameters leads to a conditional exponential family. The alternative model to Marshall-Olkin proposed by Block-Basu(1974).

Block-Basu noted that Marshall-Olkin model is not absolutely continuous with respect to Lebesgue measure in $\mathbb{R}^2$ and hence considered the problem of developing an absolutely continuous BVED which satisfies both marginal and joint lack of memory property respectively. A reasonable alternative to the assumption of exponential marginals is that, marginals are weighted combination of exponential distributions. Using the marginals of this type and retaining the absolute continuity and the joint lack of memory property, Block-Basu(1974) obtained this BVED. In Block-Basu model, the survival function of life times (X, Y) of the two components C_1 and C_2 is given by

$$\bar{F}_B(x, y) = \lambda/(\lambda_1 + \lambda_2)\bar{F}_M(x, y) - \lambda_3/(\lambda_1 + \lambda_2) \exp[-\lambda \max(x, y)],$$

where $\lambda_1, \lambda_2, \lambda_3 \ge 0$; $\lambda = \lambda_1 + \lambda_2 + \lambda_3$; $x, y \ge 0$ and $\bar{F}_M(x, y)$ is the survival function of M-O model. In this case marginals of X and Y are not exponential when $\lambda_3 > 0$ but are weighted combination of two exponentials with parameters $(\lambda_i + \lambda_3)$ and λ $(i = 1, 2)$ respectively.

Proschan-Sullo(1974) obtained BVED or modified Freund model which is a combination of both M-O and Freund model. Initially both components are in operation, the life time (X, Y) follows BVED of M-O as

$$\bar{F}_M(x, y) = \exp\{-\theta_1 x - \theta_2 y - \theta_3 \max(x, y)\}, \quad x, y \ge 0,$$
$$\theta_1, \theta_2, \theta_3 \ge 0$$

when the component C_1 fails, the parameter of the component C_2 changes from $\eta_2 = \theta_2 + \theta_3$ to $\eta_2' = \theta_2' + \theta_3$. when the component C_2 fails the parameter of the component C_1 changes from $\eta_1 = \theta_1 + \theta_3$ to $\eta_1' = \theta_1' + \theta_3$. There are two types of dependence between X and Y of which first one is due to simultaneous failures of components and second is due to failure of one component which results in an extra load on the other component.

Hence the p.d.f. of BVED of Proshan Sullo is given by

$$f(x,\ y) = \begin{cases} \theta_1\eta_2'\exp(-\ \eta_2'y\ -\ (\theta\ -\ \eta_2')\ x\ )\ ;\ \text{if } x < y \\ \theta_2\eta_1'\exp(-\ \eta_1'x\ -\ (\theta\ -\ \eta_1')\ y\ )\ ;\ \text{if } x > y \\ \theta_3\exp(-\ \theta\ x\ )\ ;\ \text{if } x = y = x. \end{cases}$$

with respect to Lebesgue measure in $\mathbb{R}^2$, where $\theta = \theta_1 + \theta_2 + \theta_3$. When $\theta_3 = 0$ then Proschan-Sullo reduces to Freund BVED and when $\theta_1' = \theta_1$ and $\theta_2' = \theta_2$ then Proschan-Sullo reduces to Marshall-Olkin model. Freund model reduces to Block-Basu when $\theta_1 = \lambda_1 + [\lambda_3\lambda_1/(\lambda_1 + \lambda_2)]$, $\theta_2 = \lambda_2 + [\lambda_3\lambda_2/(\lambda_1 + \lambda_2)]$, $\theta_1' = \lambda_1 + \lambda_3$, $\theta_2' = \lambda_2 + \lambda_3$. It is to be noted that Block-Basu model is a proper submodel of Freund as its parameters $(\lambda_1,\ \lambda_2,\ \lambda_3)$ depend on $(\theta_1,\ \theta_2,\ \theta_1',\ \theta_2')$ only through $(\theta_1 + \theta_2)$. The marginals of X and Y are weighted combinations of two exponentials in both these models. It is to be noted that in both these models marginals are exponential only when both the components are independent and satisfy lack of memory property which implies that other two submodels also satisfy lack of memory property. Sarkar(1987) BVED secures absolute continuity by sacrificing joint lack of memory property but marginals are exponentially distributed, which satisfies marginal lack of memory property. Keunkwan Ryu(1993) extended Marshall-Olkin's BVED recently by generalizing its underlying physical setting. The joint distribution of BVED developed by Keunkwan Ryu is not of the exponential type in a strict sense, it does possess both MLMP and JLMP to any degree of approximation, while still keeping the absolute continuity property. The marginal distributions show increasing hazard rates and the joint distribution displays an aging pattern. In case of Downton, Hawkes and Arnold's BVED, the different types of charactristics are observed.

Downton(1970) and Hawkes(1972) have proposed distributions which are derived from the concept of failures due to succesive damage; a generalised property of the univariate exponential distribution. The distribution considered by Downton as a special case of a well known bivariate Gamma distribution. Arnold(1975) has introduced hierarchy of exponential distribuions and Arnold's BVED is derived from a non-fatal shock model and the family of distributions of Block.

In this dissertation a study of the behaviour of distribution of sum of two independent but not identically distributed random varibles is covered which has a great applications in reliability context. Here we have concentrated to obtain only distribution of sum of life times of two components, that is $X_1 + X_2$ in case of Marshall-Olkin, Downton as well as Hawkes model which leads to hypoexponential distrbution. We also study some distributional properties of Hypoexponential distrubution and its applications.

1.3. Chapterwise Summary

This dissertation contains four chapters. A brief account of which is presented below. The first chapter is an introductory and gives quick review of some standard bivariate exponential distributions along with their characteristcs.

In Chapter II BVED's such as Downton(1970) and Hawkes(1972) are discussed in detail with their distributional properties. The distribution of $S_N = \sum_{i=1}^{N} X_i$, is obtained, where N is a geometric random varible and X_i's are i.i.d. exponential variates. The joint p.d.f. of vector $(T_1, T_2)^T$ for Downton's BVED is derived in Section-2.3. The comparison between Downton's BVED and M-O BVED are discussed in the last Section. It is also verified that Downton's model is a particular case of Hawkes model under some constraints.

Chapter III is devoted for hypoexponential distribution and its properties such as definition survival fuction, Laplace transfrom, moments and reproductive property. The detailed proofs are provided wherever necessary. We give some illustrations to describe some situations in which this model arises. The relationship between hypoexponential distribution with Downton's BVED, M-O BVED and Hawkes BVED are discussed respectively. The last Section of this chapter consists of model sampling from hypoexponential distribution and some tables of random numbers from hypoexponential distribution are provided which are obtained by using the computer program in BASIC.

Chapter IV deals with the estimation of the scale parameters of the hypoexponential distribution. we give some standard definitions and important preliminary results which are to be used in later sections of this chapter. The likelihood equations are obtained in Section 4.3 but we observe that these likelihood equations can not be solved analytically and hence it is necessary to obtain the solution of these likelihood equations by different methods such as Newton-Raphson and method of Scoring In last section we provide numerical comparisons of m.l.e.'s of parameters of hypoexponential distribution obtianed by using the Newton-Raphson and method of Scoring.

Appendix-I(A) contains computer program in FORTRAN-77 for implementing Newton-Raphson and Scoring methods and Appendix-I(B) contains computer program in FORTRAN-77 for evaluation of sample mean and theorotical mean from hypoexponential distribution.

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