C H A P T E R – IV

CURVATURE OF A FAMILY OF DISTRIBUTIONS

4.1 INTRODUCTION :

This chapter is basically based on a part of Efron's (1975) paper on 'Curvature'. Curvature is used to measure closeness between one parameter family of distributions and one parameter E.F.D. In section 4.2 we study concept of curvature with it's role in statistics. We have shown that curvature of one parameter E.F.D. is zero. Also curvatures of non exponential models are calculated. In section 4.3, curved E.F.D.'s are introduced; curvature of a family of Normal distributions with mean θ and variance $g(\theta)$ where $g(\theta)$ is twice differentiable function of θ is calculated.

4.2 CURVATURE AND IT'S ROLE IN STATISTICS :

Let L be a curve in $\mathbb{R}^2$ as shown in fig. 1. A measure of the rapidity with which a curve in $\mathbb{R}^2$, or it's tangent line, is turning at a particular point on the curve is given by the rate of change of the angle made by the tangent line with some fixed direction (which can be taken for convenience to be that of coordinate axis) with respect to the arc length, measured from some

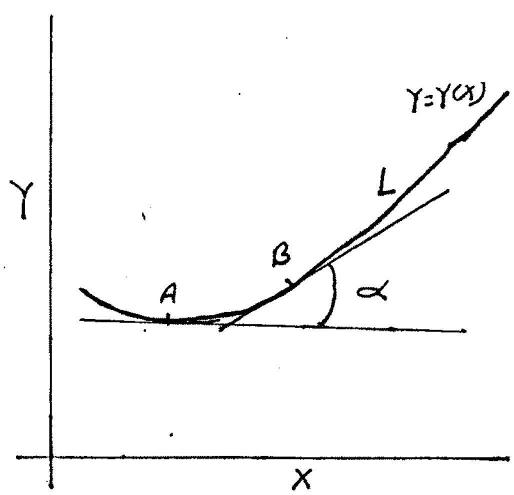


Fig 1

fixed point on the curve. The absolute value of this rate of change (if it exists) is called the curvature of curve at the particular point. From fig.1, curvature of the curve L at point A is given by

$$\text{Curvature} = \left| \frac{d\alpha}{ds} \right|$$

where s is arc length of the curve L between points A and B and α is angle between tangents to the curve at the points A and B. Suppose that Y = Y(x) is the equation of the curve, then the curvature of the curve evaluated at x_o is given by

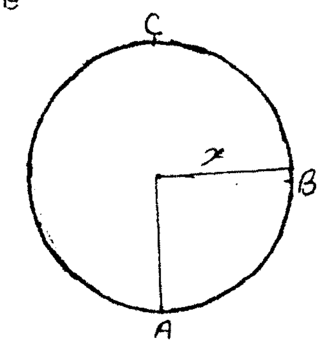
$$\gamma_{xo} = \frac{|\ddot{Y}|}{(1+\dot{Y}^2)^{3/2}}$$

That is

$$\gamma_{xo} = \left[\frac{\ddot{Y}^2}{(1+\dot{Y}^2)^3} \right]^{1/2} \qquad (1)$$

where · denotes differntiation w.r.t. x and derivatives are evaluated at x_o. Reciprocal of the curvature is called the rediøus of the curvature at that point. For further details we refer John (1970).

<u>Example 1</u> : Curvature of the circle



at point B is given by

$$\nu_B = \frac{\pi/2}{2\pi r/4} = \frac{1}{r}$$

and curvature at C is given by

$$\nu_C = \frac{\pi}{2\pi r/2} = \frac{1}{r}$$

Equation of the circle (whose centre is origin and rediøus is r, as in fig.2) is given by

$$x^2+y^2 = r^2$$

so we get

$$\dot{Y} = -\frac{x}{y} \text{ and } \ddot{Y} = -\frac{1+\dot{Y}^2}{Y}$$

By putting these values in (1) we get,

$$\nu_x = \frac{1}{r}$$

Thus the curvature of the circle is constant and the rediøus of curvature of the circle at that point is the redious of circle. Note that the curvature is independent of origin. It is easy to check that curvature of straight line is zero.

Role of Curvature :

For non-exponential families the M.L.E. is not, in general, a sufficient statistic. The information lossed (as compared with information in the sample) by M.L.E. can be expressed in terms of curvature. This is done by Fisher (1925). Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with p.d.f. $f(x|\theta)$; we denote, $I_n(\theta)$ as Fisher information about θ, contained in the whole sample, $I_n(\hat{\theta})$ be Fisher information in the M.L.E. $\hat{\theta}(X_1, X_2, \ldots, X_n)$ and i_θ be Fisher information contained in single observation. According to Fisher (1925),

$$\lim_{n\to\infty} \left[I_n(\theta)-I_n(\hat{\theta})\right] = i_\theta\left[\frac{\mu_{02}-2\mu_{21}+\mu_{40}}{i_\theta^2} -1- \frac{\mu_{11}^2+\mu_{30}^2-2\mu_{11}\mu_{30}}{i_\theta^3}\right], \quad (2)$$

where

$$\mu_{hj} = E_\theta\left[\frac{\dot{f}(x|\theta)}{f(x|\theta)}\right]^h \left[\frac{\ddot{f}(x|\theta)}{f(x|\theta)}\right]^j , \quad (3)$$

the dot $\cdot$ indicating differentiation w.r.t. θ. Later on we will show that the bracketed quantity on R.H.S. of (2) equals to γ_θ^2 where γ_θ is curvaturre of the family of distributions $f(x|\theta)$. Also it is observed that (proved later on) curvature of the one parameter exponential family is zero. Thus if, the bracketed quantity is zero then the distribution from which sample is drawn, forms exponential family, otherwise it does not forms exponential family. Thus originally, bracketed quantity (this quantity being called curvature by Efrom) is basic tool to measure departurness of one parameter family form one parameter exponential family. Efron has discussed the fact that, if curvature of the family of distributions is small (that is near zero) then that family has good statistical properties as exponential family has. For example, locally most powerful test has poor characteristic, if it is based on the sample from distribution having larger curvature. One parameter exponential families have very nice properties for estimation, testing, and other inference problems. Fundamentally this is because they can be considered to be 'straight lines' through the space of all possible probability distributions on the sample space.

Now we discuss, curvature of the curve in K>2 dinensional space, we refer Efron (1975). Let $L = \{\eta_\theta, \theta \epsilon \Theta \subseteq \mathbb{R}\}$ be a curved line in $\mathbb{R}^k$ that is a locus of points $\eta_\theta, \theta \epsilon \Theta \subseteq \mathbb{R}$. For each θ,

η_θ is a vector in $\mathbb{R}^k$ and we denote $\dot{\eta}_\theta = \frac{\partial}{\partial\theta}\eta_\theta$, $\ddot{\eta}_\theta = \frac{\partial^2}{\partial\theta^2}\eta_\theta$.

We assume that these derivatives are exist in a neighbarhood of a value of θ, where we wish to define curvature. Let Σ_θ be K × K symmetrie non-negative definite matrix, defined continuously in θ. Let M_θ be the 2 × 2 matrix, with entries denoted by $\nu_{20}(\theta)$, $\nu_{11}(\theta)$ and $\nu_{02}(\theta)$ as below

$$M_\theta = \begin{bmatrix} \nu_{20}(\theta) & \nu_{11}(\theta) \\ \nu_{11}(\theta) & \nu_{02}(\theta) \end{bmatrix} = \begin{bmatrix} \dot{\eta}_\theta' \Sigma_\theta \dot{\eta}_\theta & \dot{\eta}_\theta' \Sigma_\theta \ddot{\eta}_\theta \\ \ddot{\eta}_\theta' \Sigma_\theta \dot{\eta}_\theta & \ddot{\eta}_\theta' \Sigma_\theta \ddot{\eta}_\theta \end{bmatrix} \tag{3}$$

and let

$$\gamma_\theta = \left[\frac{|M_\theta|}{\nu_{20}^3(\theta)}\right]^{1/2} \tag{4}$$

Then γ_θ is 'the curvature of L at θ with respect to the inner product Σ_θ. For further details we refer John M.H.O. (1972).

<u>Note 1</u> : If we take K = 2, $\theta = X$, $\eta_\theta = (X, Y(X))'$ and $\Sigma_\theta = I$ in (3) and (4) we get

$$\dot{\eta}_\theta = (1, \dot{Y})', \quad \ddot{\eta}_\theta = (0, \ddot{Y})'$$

and

$$M_\theta = \begin{bmatrix} 1+\dot{Y}^2 & \dot{Y}\ddot{Y} \\ \dot{Y}\ddot{Y} & \dot{Y}^2 \end{bmatrix}$$

Hence we have

$$\gamma_\theta = \left[\ddot{Y}^2 \,|\, (1+\dot{Y}^2)^3\right]^{1/2}$$

For straight line, $\eta_\theta = (\theta,\ a+b\theta)$ and for circle with centre

at origin and r be radious, $\eta_\theta = \left[\theta,\ \sqrt{r^2-\theta^2}\ \text{where}\ 0<\theta<r\right]$.

Quantities in (3), can be expressed in terms op $f(x|\theta)$ as below (we refer Efron [1975]).

Let $\ell_\theta(x) = \log\left[f(x|\theta)\right]$ and denote

$$\dot{\ell}_\theta(x) = \frac{\partial}{\partial\theta}\left[\ell_\theta(x)\right],\quad \ddot{\ell}_\theta(x) = \frac{\partial^2}{\partial\theta^2}\left[\ell_\theta(x)\right]$$

(for convenience we will supress random element x in much of the subsequent notations)

It can be shown that (refer Kendall and Stuart [1973])

$$E_\theta(\dot{\ell}_\theta) = 0 \text{ and } E_\theta(\dot{\ell}_\theta^2) = -\,E_\theta(\ddot{\ell}_\theta) = i_\theta.$$

We take

$$M_\theta = \text{Covariance maatrix of } (\dot{\ell}_\theta, \ddot{\ell}_\theta)$$

$$= \begin{bmatrix} i_\theta = E_\theta(\dot{\ell}^2_\theta) & E_\theta(\dot{\ell}_\theta \ddot{\ell}_\theta) \\ E_\theta(\dot{\ell}_\theta \ddot{\ell}_\theta) & E_\theta(\ddot{\ell}_\theta^{\,2}) - i^2_\theta \end{bmatrix} \tag{5}$$

From (3) and (5) we can write

$$\left.\begin{aligned} \nu_{20}(\theta) &= E_\theta(\dot{\ell}^2_\theta) = i_\theta \\ \nu_{11}(\theta) &= E_\theta(\dot{\ell}_\theta \ddot{\ell}_\theta) = \text{Cov.}(\dot{\ell}_\theta, \ddot{\ell}_\theta) \\ \nu_{02}(\theta) &= E_\theta(\ddot{\ell}_\theta^{\,2}) - i^2_\theta = \text{Var.}(\ddot{\ell}_\theta) \end{aligned}\right\} \tag{6}$$

Using above notations, (4) can be written as

$$\gamma^2_\theta = \frac{1}{i^2_\theta}\left[\nu_{02}(\theta) - \frac{\nu^2_{11}(\theta)}{i_\theta}\right] \tag{7}$$

<u>Note</u> : Equation (2) can be written as

$$\lim_{n\to\infty}\left[I_n(\theta) - I_n(\hat{\theta})\right] = i_\theta\, \gamma^2_\theta \tag{8}$$

From (7) we have

$$\gamma^2_\theta = \frac{\nu_{02}(\theta)}{i^2_\theta} - \frac{\nu^2_{11}(\theta)}{i^3_\theta} \tag{9}$$

By definition

$$\ell_\theta = \log f(x|\theta)$$

Hence

$$\dot{\ell}_\theta = \frac{\dot{f}(x|\theta)}{f(x|\theta)} \quad \text{and} \quad \ddot{\ell}_\theta = \frac{\ddot{f}(x|\theta)}{f(x|\theta)} - \left[\frac{\dot{f}(x|\theta)}{f(x|\theta)}\right]^2$$

Using (3) and (6) we have

$$\nu_{02}(\theta) = E(\dot{\ell}_\theta^{\,2}) - i_\theta^2$$

$$= \mu_{02} + \mu_{40} - 2\mu_{21} - i_\theta^2$$

and

$$\nu_{11}(\theta) = E(\dot{\ell}_\theta \dot{\ell}_\theta)$$

$$= \mu_{11} - \mu_{30}$$

Hence from (9) we get

$$\gamma_\theta^2 = \frac{\mu_{02}+\mu_{40}-2\mu_{21}}{i_\theta^2} - 1 - \frac{\mu_{11}^2+\mu_{30}^2-2\mu_{11}\mu_{30}}{i_\theta^3} \tag{10}$$

Hence result follows from (2), (8) and (10).

Example 1 : If $X \sim b(n,\theta)$ then we have

$$\nu_{20}(\theta) = \frac{n}{\theta(1-\theta)}, \quad \nu_{02}(\theta) = \frac{n(1-2\theta)^2}{\theta^3(1-\theta)^3}$$

$$\nu_{11}(\theta) = \frac{n(1-2\theta)}{\theta^2(1-\theta)^2}$$

By putting these values in (7) we get

$$\gamma_\theta = 0$$

Example 2 : If $X \sim P(\theta)$ then we have

$$\nu_{20}(\theta) = \frac{1}{\lambda}, \qquad \nu_{02}(\theta) = \frac{1}{\lambda^3}$$

$$\nu_{11}(\theta) = -\frac{1}{\lambda^2}$$

By putting these values in (7) we get

$$\gamma_\theta = 0$$

Now we show that Curvature of one Parameter Exponential Family is Zero.

<u>Theorem 1</u> : Curvature of one Parameter E.F.D. is Zero.

<u>Proof</u> : Let $f(x|\theta)$ be p.d.f. of X which forms one parameter E.F.D. then $f(x|\theta)$ is given by

$$f(x|\theta) = h(x)\ \exp\left\{Q(\theta)T(x)+d(\theta)\right\}$$

In terms of above notations, we have

$$\ell_\theta = Q\ T(x)+d,$$

where Q and d are functions of θ only.

Hence we have

$$\dot{\ell}_\theta = \dot{Q}\ T(x)+\dot{d}$$

and

$$\ddot{\ell}_\theta = \ddot{Q}\ T(x)+\ddot{d}$$

Now we consider

$$\dot{\ell}_\theta - \frac{\dot{Q}}{\ddot{Q}}\ \ddot{\ell}_\theta = \frac{\dot{d}}{1} - \frac{\dot{Q}\ \ddot{d}}{\ddot{Q}}$$

$$= D(\theta) \text{ say}$$

Hence we can write,

$$\dot{\ell}_\theta(x) = D(\theta)+D_1(\theta)\ddot{\ell}_\theta(x),$$

where $$D_1(\theta) = \frac{\dot{Q}(\theta)}{\ddot{Q}(\theta)}$$

Hence, for fix θ, $\dot{\ell}_\theta(x)$ and $\ddot{\ell}_\theta(x)$ are linearly related.

Therefore, covariance matrix of $(\dot{\ell}_\theta, \ddot{\ell}_\theta)$ is singular.

That is $|M_\theta| = 0$

Hence, from (4)

$$\gamma_\theta = 0$$

|||

<u>Aliter</u> : We find

$$\nu_{20}(\theta) = \frac{\ddot{Q}\,\dot{d}-\dot{Q}\,\ddot{d}}{\dot{Q}}$$

$$\nu_{02}(\theta) = \frac{\ddot{Q}^3\,\dot{d}-\dot{Q}\,\ddot{Q}^2\,\ddot{d}}{\dot{Q}^3}$$

$$\nu_{11}(\theta) = \frac{\ddot{Q}^2\,\dot{d}-\dot{Q}\,\ddot{Q}\,\ddot{d}}{\dot{Q}^2}$$

By putting above values in (7) we get

$$\gamma_\theta = 0$$

|||

Now we obtain curvature of some non-exponential families of distributions.

Example 3 : Let X has student's t distribution with n degrees of freedom as below

$$f(x|\theta) = \begin{cases} \frac{\Gamma(n+1)}{\Gamma(n/2)\sqrt{n\pi}} \left[1 + \frac{(x-\theta)^2}{n}\right]^{-(\frac{n+1}{2})} & , \ -\infty < x, \theta < \infty \\ & \quad n \geq 1, \text{ integer} \\ 0 & , \ \text{other wise.} \end{cases}$$

We observe that differentiation w.r.t. x and w.r.t. θ are same (except for negative sign) and moreover curvature is independent of change of origin. Thus if we put $\theta = 0$ then differentiation w.r.t. x and w.r.t. θ are same.

Hence if $\theta = 0$ then,

$$f(x) = \frac{\Gamma(n+1)}{\Gamma(n/2)\sqrt{n\pi}} \left[1 + \frac{x^2}{n}\right]^{-(\frac{n+1}{2})}$$

We obtain

$$\ell_\theta = K - \frac{n+1}{2} \log \left(1+\frac{x^2}{n}\right) ,$$

Where $K = \log \frac{\Gamma(n+1)}{\Gamma(n/2)\sqrt{n\pi}}$

Hence we have

$$\dot{l}_\theta = -\left[\frac{n+1}{n}\right] \quad x\left[1 + \frac{x^2}{n}\right]^{-1}$$

and

$$\dot{l}'_\theta = -\left[\frac{n+1}{2}\right] \left[1 - \frac{x^2}{n}\right] \left[1 + \frac{x^2}{n}\right]^{-2}$$

Also we obtain

$$\nu_{20}(\theta) = \frac{n+1}{n+3}$$

$$E(\dot{l}'^{\,2}_\theta) = \frac{(n+1)(n+2)(n^2+18n+19)}{n(n+3)(n+5)(n+7)}$$

$$\nu_{02}(\theta) = \frac{n+1}{n+3}\left[\frac{(n+2(n^2+18n+19)}{n(n+5)(n+7)} - \frac{n+1}{n+3}\right]$$

$$\nu_{11}(\theta) = E(\dot{l}_\theta \dot{l}'_\theta)$$

$$= E\left[\left[\frac{n+1}{n}\right]^2 \frac{(1 - \frac{X^2}{n})}{(1 + \frac{X^2}{n})^2} \; X\right]$$

$= 0$ since integrant becomes odd function of X

By putting above values in (7) we get,

$$\gamma^2_\theta = \frac{\nu_{02}(\theta)}{i^2_\theta}$$

$$= \frac{6(3n^2+18n+19)}{n(n+1)(n+5)(n+7)}$$

If $n = 1$ then X has standard Cauchy distribution and $\gamma_\theta^2 = 5/2$

Example 4 : For Gamma translation family

$$f(x|\theta) = \begin{cases} \dfrac{(x-\theta)^{a-1}\, e^{-(x-\theta)}}{\Gamma a} , & x \geq \theta,\ \theta > 0 \\ 0 , & \text{other wise} \end{cases}$$

we obtain

$$\ell_\theta = -\log \Gamma a + (a-1)\log(x-\theta) - x+\theta$$

$$\dot{\ell}_\theta = -\frac{a-1}{x-\theta} + 1$$

$$\ddot{\ell}_\theta = -\frac{a-1}{(x-\theta)^2}$$

$$\nu_{20}(\theta) = \frac{1}{a-2}$$

$$\nu_{02}(\theta) = \frac{4a-10}{(a-2)^2(a-3)(a-4)}$$

$$\nu_{11}(\theta) = \frac{2}{(a-2)(a-3)}$$

and

$$\gamma_\theta^2 = \frac{2}{(a-3)^2}\,\frac{a-1}{a-4} \qquad a > 4, \text{ fixed constant.}$$

4.3 CURVED E.F.D. : Curved E.F.D. is one parameter family of distributions for which there does not exists one dimensional sufficient statistics and is obtained from any K dimensional E.F.D. by converting K parameters to one parameter using some intermediate relations in single parameter. For example, if $X \sim N(\mu,\sigma^2)$ then it forms two parameter $\mathbb{E}.\mathbb{F}.\mathbb{D}.$ while if we take $\mu = \theta$ and $\sigma^2 = \theta^2$ $(\theta>0)$ then $X \sim N(\theta,\theta^2)$ and it form curved $\mathbb{E}.\mathbb{F}.\mathbb{D}.$

Let $f(\underline{x}|\underline{\eta})$ be p.d.f. of $\underline{X}$ which form k-parameter $\mathbb{E}.\mathbb{F}.\mathbb{D}.$, then $f(\underline{x}|\underline{\eta})$ is given by

$$f(\underline{x}|\underline{\eta}) = h(\underline{x}) \exp\left\{\Sigma\ \eta_i\ T_i(\underline{x})+d(\eta)\right\}, \tag{1}$$

where $\underline{\eta}$ is natural parameter and $\eta \in N \subseteq \mathbb{R}^k$.

We convert, $\underline{k}$ natural parameter $n_1,n_2,\ldots,n_k$ to $\eta_\theta = (\eta_1(\theta),\eta_2(\theta),\ldots\eta_k(\theta))$, $\theta \in \Theta \subseteq \mathbb{R}$. Hence (1) can be written as

$$f(\underline{x}|\theta) = h(x) \exp\left\{\Sigma\ \eta_i(\theta)T_i(x)+d(\eta_\theta)\right\} \tag{2}$$

Let $$L = \left\{\underline{\eta}_\theta : \theta \in \Theta \subseteq \mathbb{R}\right\} \tag{3}$$

is one parameter subset in the interior of N, where η_θ is twice differentiable function of θ.

<u>Definition 1</u> : Let $\mathbb{P} = \left\{f(\underline{x}|\theta):\theta\in\Theta\subseteq\mathbb{R},\right.$ and it is given by $\left.(2)\right\}$. Statistical curvature of $\mathbb{P}$ at θ is the geometrical curvature of $L = \left\{\eta_\theta : \theta \in \Theta\right\}$ at θ.

Properties of Curvature : We state invariance properties of curvature, for further details we refer Efron (1975).

(i) Statistical curvature is intrinsic property of family $\mathbb{P}$ and does not depend on particular parametrization used to index $\mathbb{P}$.

(ii) statistical curvature is invarient under any mapping to sufficient statistics including of course all one to one mappings of the sample space. That is, if $T = T(X)$ is sufficient for $\mathbb{P} = \{f(x|\theta) : \theta \in \Theta \subseteq \mathbb{R}\}$ and

let $\mathbb{P}_1 = \{f_1(t|\theta) : \theta \in \Theta, \subseteq \mathbb{R},\ f_1$ is p.d.f. of $T\}$ then

$$\gamma_\theta = \gamma_\theta T,$$

where γ_θ and $\gamma_\theta T$ are curvatures of $\mathbb{P}$ and $\mathbb{P}_1$ respectively.

Curvature of Family of Distributions $N(\theta, g(\theta))$:

We obtain curvature of the family of Normal distributions with mean θ and variance $g(\theta)$, $g(\theta)$ is twice differntiable function of θ. Using central limit theorem, asymptotic distribution of sample mean of i.i.d. observations from any distribution $f(x|\theta)$ is normal distribution with mean $= E_\theta(X)=\theta$ and variance $= \dfrac{Var_\theta(X)}{n} = g(\theta)$ say.

The following is new result to the extend that we have proved it independently and is not found in the earlier literature.

Theorem : Let X be a random variable having normal distribution with mean θ and variance $g(\theta)$ where $g(\theta)$ is twice differentiable function of θ, then curvature γ_θ of the family of normal distributions is given by

$$\gamma_\theta^2 = \frac{4g^3\ddot{g}^2}{(\dot{g}^2+2g)^3} , \tag{1}$$

where $\dot{g} = \frac{d}{d\theta} g(\theta)$, $\ddot{g} = \frac{d}{d\theta^2} g(\theta)$.

(we have written g instead of $g(\theta)$).

Proof : p.d.f. of X is given by

$$f(x|\theta) = \begin{cases} \frac{1}{\sqrt{2\pi g}} \exp\left\{-\frac{1}{2g^2}(x-\theta)^2\right\} , & \text{if } -\infty < x, \theta < \infty \\ 0 & , \text{other wise} \end{cases}$$

using definitions and notations in previous section we obtain

$$\ell_\theta = a_1x^2 + b_1x + c_1$$

and

$$\dot{\ell}_\theta = a_2x^2 + b_2x + c_2 ,$$

where $\quad a_1 = \dfrac{\dot{g}}{2g^2}$, $b_1 = \dfrac{g-\theta\dot{g}}{g^2}$, $c_1 = \dfrac{\theta^2\dot{g} - 2\theta g - g\dot{g}}{2g^2}$

$$a_2 = \frac{g\ddot{g} - 2\dot{g}^2}{2g^3} \quad , \quad b_2 = \frac{2\theta\dot{g}^2 - \theta g\ddot{g} - 2g\dot{g}}{g^3}$$

$$c_2 = \frac{\theta^2 g\ddot{g} + 4\theta g\dot{g} - 2g^2 - 2\theta^2\dot{g}^2 - g^2\ddot{g} + g\dot{g}^2}{2g^3}$$

since $X \sim N(\theta, g(\theta))$ we have

$$E_\theta(X) = \theta \quad , \quad E_\theta(X^2) = g+\theta^2$$

$$E_\theta(X^3) = 3\theta g+\theta^3 \quad , \quad E_\theta(X^4) = 3g^2+6\theta^2 g+\theta^4$$

Using these results we obtain

$\nu_{20}(\theta) = i_\theta =$ Fisher Information in a single observation.

$$= E_\theta(\dot{\ell}_\theta^2)$$

$$= - E_\theta(\ddot{\ell}_\theta)$$

$$= \frac{\dot{g}^2+2g}{2g^2}$$

$$\nu_{11}(\theta) = E_\theta(\dot{\ell}_\theta \ddot{\ell}_\theta)$$

$$= \frac{2g\dot{g}\ddot{g} - 8g\dot{g} + 4\dot{g}^3}{4g^3}$$

and

$$\nu_{02}(\theta) = E(\ddot{\ell}_\theta^2) - i_\theta^2$$

$$= \frac{2g^2\dot{g}^{\cdot 2} + 8\dot{g}^4 + 16g\dot{g}^2 - 8g\dot{g}^2\dot{g}^{\cdot}}{4g^4}$$

curvature γ_θ is given by

$$\gamma_\theta^2 = \frac{1}{i_\theta^2}\left[\nu_{02}(\theta) - \frac{\nu_{11}^2(\theta)}{i_\theta}\right]$$

$$= \frac{4g^3\ \dot{g}^{\cdot 2}}{(\dot{g}^2+2g)^3}$$

|||

Corollory 1 : If we take $g(\theta) = C\theta^2$ then Efron's example 5 (refer Efron (1975),p.p.1200) becomes particular case of above result. In this case $X \sim N(\theta,\ C\theta^2)$.

Hence

$g(\theta) = C\theta^2$ and we get

$E_\theta(X) = \theta$, $E_\theta(X^2) = (C+1)\theta^2$

$E_\theta(X^3) = (3C+1)\theta^3$, $E_\theta(X^4) = (3C^2+6C+1)\theta^4$

Also we obtain

$$\nu_{20}(\theta) = \frac{2C+1}{C\theta^2}$$

$$\nu_{02}(\theta) = \frac{18C+16}{C\theta^4}$$

$$\nu_{11}(\theta) = -\frac{6C+4}{C\theta^3}$$

Hence we obtain

$$\gamma_{\theta}^{2} = \frac{C^{2}}{4(C+1/2)^{3}}$$

<u>Note</u> : If $X \sim N(\theta,\ \theta^{2})$ $\theta > 0$, then curvature γ_{θ} of family of normal distributions $N(\theta,\ \theta^{2})$ is

$$\gamma_{\theta}^{2} = \frac{2}{27} .$$

<u>Corollary 2</u> : Using central limit theorem asymptotic distribution of sample mean of i.i.d. observations $X_1, X_2, \ldots, X_n$ from $b(1,\theta)$, is normal with mean θ and variance $g(\theta) = \frac{\theta(1-\theta)}{n}$. Thus $\bar{X} \sim N(\theta,\ \frac{\theta(1-\theta)}{n})$; curvature of this family of normal distributions is obtained as below.

$$E(X) = \theta,\ E(X^{2}) = \frac{\theta+(n-1)\theta^{2}}{n}$$

$$E(X^{3}) = \frac{3\theta^{2}+(n-3)\theta^{3}}{n}\ ,\ E(X^{4}) = \frac{3\theta^{2}+6\theta^{3}(n-1)+\theta^{4}(n^{2}-6n+3)}{n}$$

$$\nu_{20}(\theta) = \frac{2\theta^{2}(2-n)+2\theta(n-2)+1}{2(\theta-\theta^{2})^{2}}$$

$$\nu_{11}(\theta) = \frac{8\theta^{3}(4-2n-1)+12\theta^{2}(2n-3)+\theta(20-8n)-4}{4(\theta-\theta^{2})^{3}}$$

$$\nu_{02}(\theta) = \frac{2\theta^{4}(9-8n)+4\theta^{3}(8n-9)+5\theta^{2}(6-4n)+4\theta(n-3)+2}{(\theta-\theta^{2})^{4}}$$

$$\gamma_{\theta}^{2} = \frac{16n(\theta-\theta^{2})^{3}}{\left[2\theta^{2}(2-n)+2\theta(n-2)+1\right]^{3}}$$