Chapter 3

TOLERANCE LIMITS FOR IFR CLASS OF DISTRIBUTIONS

3.1 Introduction:

Shaked and Shanthikumar (1994) have discussed various stochastic orders and their applications. They have compared random variables or their distribution by various stochastic orders with some results. Many researchers ~~are~~ have pointed out the utility of stochastic orders in different areas like Finance, Economics, Bio-statistics, Reliability, etc. They have proved useful results that are related to the various stochastic orders. Kijima and Ohnishi (1999) have given stochastic orders and their application in financial optimization. Cheng and Zhou (2005) have derived some applications of stochastic orders to actuarial science.

Let $X_1, X_2, \ldots, X_n$ be a random sample obtained from an IFR distribution with distribution function F. Consider two statistic $L(\underline{X})\ and\ U(\underline{X})$ based on this random sample, such that $L(\underline{X}) < U(\underline{X})$. Suppose $L(\underline{X}) = L(X_1, \ldots, X_n)$ is such that

$$P_F\left[1 - F\left(L(\underline{X})\right) \geq 1 - q\right] \geq 1 - \alpha,$$

where $0<q<1$ and $0<\alpha<1$. Then $(L(\underline{X}),\infty)$ is called as $(1-\alpha)$ level $(1-q)$ content lower tolerance limit for F. Similarly $U(\underline{X})=U(X_1,\ldots,X_n)$ is said to be $(1-\alpha)$ level $(1-q)$ content upper tolerance limit for F if

$$P_F\left[\ F\left(U(\underline{X})\right)\geq 1-q\ \right]\geq 1-\alpha.$$

Goodman and Madansky (1966) have reported parameter free and nonparametric tolerance limits. Many researchers have provided parametric tolerance limits for various continuous and discrete distributions. Patel (1986) has given an extensive review of tolerance limits.

Distribution free tolerance limits based on order statistics, which suffer from the problem of sample size. If sample size is not enough, the corresponding tolerance interval does not give the desired coverage. Instead of obtaining distribution free tolerance limits for an arbitrary distribution F, it is possible to construct tolerance interval for F, when F belongs to an IFR class of distribution. Hanson and Koopmans (1964) have provided such tolerance interval for IFR distribution for which log F is concave. Barlow and Proschan (1966) have provided tolerance limits for F, when F is an IFR and restricted to be positive on only non-negative values of corresponding random variable.

This chapter is devoted to application of the usual stochastic ordering. The section 3.2 contains prerequisite. Some results related to star-shaped function, convex function, IFR and DFR distributions have been discussed. These results are necessary to obtain tolerance limits. We have summarized these results from Barlow and Proschan (1966). In section 3.3 we discuss lower tolerance results for an IFR class of distributions. Section 3.4 is devoted to the lower tolerance limits for Weibull and Exponentiated exponential distributions.

3.2 Prerequisite:

Definition 3.1 Star-shaped function:

(a) A function ϕ is Star-shaped on $[0, b), for\ 0 < b \leq \infty$,

if $\phi(\alpha x) \leq \alpha\ \phi(x),\ \ for\ \ 0 \leq \alpha \leq 1, 0 \leq x < b,$

or $\dfrac{\phi(x)}{x}$ is increasing $for\ x\ \ in\ \ [0, b)$. ... (3.1)

(b) Let F and G be continuous distribution, G be strictly increasing on its support, and F(0) = 0 = G(0). Then F is star-shaped with respect to G if $G^{-1}F(x)$ is star-shaped. That is $\dfrac{G^{-1}F(x)}{x}$ is increasing $for\ x \geq 0$.

Example 3.1: Let $\phi(x) = x^2$.

We have $\dfrac{\phi(x)}{x} = x$. ... (3.2)

By taking derivative of (3.2) with respect to x, we get

$$\frac{d}{dx}\left(\frac{\phi(x)}{x}\right) = 1 \quad > 0.$$

Hence $\dfrac{\phi(x)}{x}$ is increasing in x.

Therefore $\phi(x)$ is Star-shaped function.

Definition 3.2 Convex function:

(a) A function ϕ is convex on (a, b), if for $0 \le \alpha \le 1$,

$-\infty \le a < b \le \infty, \quad a < x, \quad y < b$. Then

$$\phi\left[\alpha x + (1-\alpha)y\right] \le \alpha\, \phi(x) + (1-\alpha)\phi(y). \qquad \text{... (3.3)}$$

(b) Let F and G be continuous distributions, G be strictly increasing on its support, and F(0) = 0 = G(0). Then F is convex with respect to G if $G^{-1}F(x)$ is a convex function in x on the support of F.

Example 3.2: Let $f(x) = x^3$. ... (3.4)

Now

$$(\alpha\, x + (1-\alpha)\, y)^3 = \alpha^3 x^3 + (1-\alpha)^3\, y^3 + 3\,(\alpha\, x)^2\, (1-\alpha)\, y + 3\,(\alpha\, x)\,(1-\alpha)^2\, y^2$$

$$\leq \alpha^3 x^3 + (1-\alpha)^3 y^3$$

$$= \alpha^3 f(x) + (1-\alpha)^3 f(y)$$

$$\leq \alpha f(x) + (1-\alpha) f(y).$$

Hence by definition 3.2, $f(x)$ is convex function.

Let $X_1, X_2, \ldots\ldots X_n$ be a random sample of size n observed from F and $Y_1, Y_2, \ldots\ldots Y_n$ be a random sample of size n observed from G. We assume that F is continuous, F(0) = 0 = G(0) and let $G(x) = 1 - e^{-x}$ for $x \geq 0$.

We further assume that F is a member of IFR class of distributions. We prove following result.

Lemma 3.1: If F is IFR then

$G^{-1}(F(t)) = -\log_e(1 - F(t))$ is convex when finite.

Proof: In order to prove $G^{-1}(F(t))$ is convex, it is enough to show that

$$\frac{\partial^2}{\partial t^2}\left[G^{-1}(F(t))\right] > 0 \quad \forall\ t \geq 0.$$

Now

$$\frac{\partial}{\partial t}\left[G^{-1}(F(t))\right] = \frac{\partial}{\partial t}\left[-\log\left(1 - F(t)\right)\right]$$

$$= \frac{f(t)}{[1 - F(t)]} \text{ and}$$

$$\frac{\partial^2}{\partial t^2}\left[G^{-1}(F(t))\right] = \frac{\partial}{\partial t}\left(\frac{f(t)}{[1 - F(t)]}\right) > 0 \quad \forall\, t \geq 0.$$

Hence the proof.

Similarly one can write following Lemma.

Lemma 3.2: If F is DFR then $G^{-1}(F(t)) = -\log_e(1-F(t))$ is concave.

3.3 Lower tolerance limits for an IFR distributions:

Suppose $X_1,\ldots,X_n$ and $Y_1,\ldots,Y_n$ are the order statistics corresponding to X-sample from F and Y-sample from G respectively. Following Barlow and Proschan (1966), define a statistic

$$T(\underline{Y}) = \frac{1}{n}\sum_{i=1}^{n}(n-i+1)(Y_i - Y_{i-1}).$$

Therefore

$$n\,T(\underline{Y}) = \sum_{i=1}^{n}(n-i+1)(Y_i - Y_{i-1}).$$

Since $Y_1,\ldots,Y_n$ are the order statistics from an exponential distribution with mean 1. We have the following Lemma immediately.

Lemma 3.3: $2\,n\,T(\underline{Y})$ has Chi-square distribution with 2n degrees of freedom.

Proof: Omitted for brevity.

Let $\chi^2_{1-\alpha}(2n)$ be the $100\,(1-\alpha)^{th}$ percentage of Chi-square distribution with 2n degrees of freedom. That is

$$P\left(\chi^2(2n) \le \chi^2_{1-\alpha}(2n)\right) = 1-\alpha .$$

Define

$$L(\underline{Y}) = \frac{-2\, n \log_e(1-q)\ T(\underline{Y})}{\chi^2_{1-\alpha}(2n)}$$

$$= \frac{-2\, n \sum_{i=1}^{n} (n-i+1)\left(Y_i - Y_{i-1}\right) \log_e(1-q)}{n\, \chi^2_{1-\alpha}(2n)}$$

$$= \frac{-2 \log_e(1-q) \sum_{i=1}^{n} (n-i+1)\left(Y_i - Y_{i-1}\right)}{\chi^2_{1-\alpha}(2n)} .$$

Lemma 3.4: $L(\underline{Y})$ is $(1-\alpha)$ level $(1-q)$ content lower tolerance limit for exponential distribution with mean 1.

Proof: Consider

$$P_G\left[1 - G\left(L(\underline{Y})\right) \ge 1-q\right]$$

$$= P\left[\left(1-\left(1-e^{-L(\underline{Y})}\right)\right) \ge 1-q\right]$$

$$= P\left[e^{-L(\underline{Y})} \ge 1-q\right]$$

$$= P\left(-L(\underline{Y}) \ge \log_e(1-q)\right)$$

$$= P\left(2\sum_{i=1}^{n}(n-i+1)\left(Y_i - Y_{i-1}\right) \le \chi^2_{1-\alpha}(2n)\right).$$

Hence by Lemma (3.3) we write

$$= P\left(\chi^2(2n) \le \chi^2_{1-\alpha}(2n)\right)$$

$$= 1-\alpha .$$

To obtain lower tolerance limit for F we need following results.

Lemma 3.5: Let $\overline{A}_i = \sum_{j=i}^{n} a_j$, then

$$\sum_{i=1}^{n} a_i x_i = \sum_{i=1}^{n} \overline{A}_i (x_i - x_{i-1}), \quad where \quad x_0 = 0. \qquad \ldots (3.5)$$

Proof: Consider R.H.S. of equation (3.5).

$$\sum_{i=1}^{n} \overline{A}_i (x_i - x_{i-1}) = \sum_{i=1}^{n} \overline{A}_i x_i - \sum_{i=1}^{n} \overline{A}_i x_{i-1}$$

$$= \left(\overline{A}_1 x_1 + \overline{A}_2 x_2 + \ldots\ldots + \overline{A}_n x_n\right) - \left(\overline{A}_1 x_0 + \overline{A}_2 x_1 + \ldots\ldots + \overline{A}_n x_{n-1}\right)$$

$$= \left(\overline{A}_1 - \overline{A}_2\right) x_1 + \left(\overline{A}_2 - \overline{A}_3\right) x_2 + \ldots\ldots + \left(\overline{A}_{n-1} - \overline{A}_n\right) x_{n-1} + \overline{A}_n x_n .$$

$$\ldots (3.6)$$

Since $\overline{A}_i = \sum_{j=i}^{n} a_j$, therefore $\overline{A}_1 - \overline{A}_2 = \sum_{j=1}^{n} a_j - \sum_{j=2}^{n} a_j = a_1$.

Hence equation (3.6) becomes

$$\sum_{i=1}^{n} \overline{A}_i (x_i - x_{i-1}) = a_1 x_1 + a_2 x_2 + \ldots\ldots + a_n x_n$$

$$= \sum_{i=1}^{n} a_i x_i .$$

Therefore $\sum_{i=1}^{n} a_i x_i = \sum_{i=1}^{n} \overline{A}_i (x_i - x_{i-1})$.

Theorem 3.1: Let $\phi\left(\sum_{i=1}^{n} a_i x_i\right) \le \sum_{i-1}^{n} a_i \phi\left(x_i\right)$. ... (3.7)

for all star-shaped ϕ $on\,[a,b)$ and all $0 \le x_1 \le x_2 \le ... \le x_n < b$ for which $0 \le \sum_{i=1}^{n} a_i x_i < b$, if and only if there exist $k\,(1 \le k \le n)$ such that

$0 \le \overline{A}_1 \le ...... \le \overline{A}_k \le 1$ *and when* $k < n$, $\overline{A}_{k+1} = ... = \overline{A}_n = 0$.

Proof: <u>Sufficiency Part:</u>

Assume $0 \le \overline{A}_1 \le ...... \le \overline{A}_k \le 1$ and when

$k < n$, $\overline{A}_{k+1} = ... = \overline{A}_n = 0$.

Then

$a_i \le 0$ $for\, i = 1,2,.....k-1$, $0 \le a_k \le 1$ and when

$k < n$, $a_i = 0$ for $i = k+1,...n.$.

Using the above Lemma 3.5, we have

$$\sum_{i=1}^{n} a_i x_i = \sum_{i=1}^{n} \overline{A}_i\left(x_i - x_{i-1}\right), \quad where \quad x_0 = 0.$$

We conclude that

$0 \le \sum_{i=1}^{n} a_i x_i \le x_k$, Thus

$\frac{\phi\left(x_k\right)}{x_k} \ge \frac{\phi\left(x_i\right)}{x_i}$, $for\ i = 1,2,......k-1$ and

$$\frac{\phi\ (x_k)}{x_k} \geq \frac{\phi\left(\sum_{i=1}^{k} a_i x_i\right)}{\sum_{i=1}^{n} a_i x_i} .$$

Hence,

$$\frac{\left\{\sum_{i=1}^{k-1}(-a_i)x_i + \sum_{i=1}^{n} a_i x_i\right\}\phi(x_k)}{x_k} \geq \sum_{i=1}^{k-1}(-a_i)\,\phi\ (x_i) + \phi\ \left(\sum a_i x_i\right)$$

$$\Rightarrow \frac{\left\{\sum_{i=1}^{k-1}(-a_i)x_i + a_k x_k + \sum_{i=1}^{k-1} a_i x_i + \sum_{i=k+1}^{n} a_i x_i\right\}\phi(x_k)}{x_k}$$

$$\geq \sum_{i=1}^{k-1}(-a_i)\phi(x_i) + \phi\left(\sum a_i x_i\right) ,$$

since $a_i = 0\ for\ i = k+1,...n$,

$$\Rightarrow a_k\, \phi(x_k) \geq \sum_{i=1}^{k-1}(-a_i)\phi(x_i) + \phi \sum (a_i x_i)$$

$$\Rightarrow \phi \sum (a_i x_i) \leq a_k\, \phi\ (x_k) + \sum_{i=1}^{k-1} a_i \phi\ (x_i)$$

$$\Rightarrow\ \phi\left(\sum_{i=1}^{k} a_i x_i\right) \leq \sum_{i-1}^{k} a_i \phi(x_i) .$$

Therefore $\phi\left(\sum_{i=1}^{n} a_i x_i\right) \leq \sum_{i-1}^{n} a_i \phi(x_i)$,

since $a_i = 0\ , for\ i = k+1,...n$.

<u>Necessity part:</u>

Let

$\phi(x) = x^2, \quad 0 = x_1 = x_2 = ... = x_{i-1}$ *and* $x_i = ... = x_n = b' < b$.

Then from sufficiency part, we have

$$\left(\sum_{i=1}^{n} a_i x_i\right)^2 \le \sum a_i x_i^{\,2}$$

$$\Rightarrow \left(\sum_{j=1}^{i-1} a_j x_j + \sum_{j=i}^{n} a_j x_j\right)^2 \le \left(\sum_{j=1}^{i-1} a_j x_j^{\,2} + \sum_{j=i}^{n} a_j x_j^{\,2}\right)$$

$$\Rightarrow \left(0 + \sum_{j=i}^{n} a_j b'\right)^2 \le \left(0 + \sum_{j=i}^{n} a_j b'^2\right)$$

$\Rightarrow \left(\sum_{j=i}^{n} a_j\right)^2 \le \sum_{j=i}^{n} a_j$, so that $0 \le \overline{A}_i \le 1$, *for* $i = 1,2,......, n.$

Next we shall show that $\overline{A}_j > 0$, *implies* $\overline{A}_{j-1} \le \overline{A}_j$.

To see this let,

$$0 = x_1 = ... = x_{j-2} < x_{j-1} < x_j = x_{j+1} = ... = x_n < b.$$

Then Lemma 3.5, we write

$$\sum_{i=1}^{n} \overline{A}_i (x_i - x_{i-1})$$

$$= \sum_{i=1}^{j-2} \overline{A}_i (x_i - x_{i-1}) + \overline{A}_{j-1}(x_{j-1} - x_{j-2}) + \overline{A}_j (x_j - x_{j-1}) + \sum_{i=j+1}^{n} \overline{A}_i (x_i - x_{i-1})$$

Therefore

$$\sum_{i=1}^{n} a_i x_i = \overline{A}_{j-1} x_{j-1} + \overline{A}_j (x_j - x_{j-1}).$$

Fix x_j and choose x_{j-1} and z sufficiently small so that

$x_{j-1} < z < x_j$ and $\sum_{i=1}^{n} a_i x_i > z$.

Let,

$$\phi_z(x) = \begin{cases} 0, & x < z \\ x, & x \geq z \end{cases}$$ a star-shaped function.

From inequality (3.7), we get

$$\phi_z\left(\sum_{i=1}^{n} a_i x_i\right) = \overline{A}_{j-1} x_{j-1} + \overline{A}_j (x_j - x_{j-1})$$

$$\Rightarrow \overline{A}_{j-1} x_{j-1} + \overline{A}_j x_j - \overline{A}_j x_{j-1} \leq \overline{A}_j x_j$$

$$\Rightarrow \overline{A}_{j-1} \leq \overline{A}_j .$$

If each $\overline{A}_i$ is zero, the proof is complete. If not, let k denote the largest subscript i such that $\overline{A}_i > 0$, Assume that $\overline{A}_{j+1} = 0 \quad for \quad j < k-1$. We shall show that this implies $\overline{A}_i = 0, for\ i \leq j$.

Let $x_j < z \leq x_{j+1}\ and\ x_k$ be so large that,

$$\sum_{i=1}^{n} a_i x_i = \sum_{i=1}^{k} \overline{A}_i (x_i - x_{i-1}) > z .$$

Then,

$$\phi_z\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i x_i$$

$$\leq \sum_{i=j+1}^{n} a_i x_i .$$

This implies,

$$\sum_{i=1}^{j} a_i x_i = \sum_{i=1}^{j} \overline{A}_i (x_i - x_{i-1}) \leq \overline{A}_{j+1}(x_{j+1} - x_j).$$

Therefore

$$\sum_{i=1}^{j} a_i x_i = \sum_{i=1}^{j} \overline{A}_i (x_i - x_{i-1}) + \overline{A}_{j+1} x_j \leq \overline{A}_{j+1} x_{j+1} = 0 .$$

This implies,

$\overline{A}_i = 0$ *for* $i = 1,2,......,j.$ since $\overline{A}_{j+1} = 0$ *and* $0 \leq \overline{A}_i \leq 1$.

Theorem 3.2: (Barlow and Proschan, (1966)) Let $G^{-1}F$ be star-shaped on the support of F, F(0) = 0 = G(0). If there exists $k\,(1 \leq k \leq n)$ such that

$0 \leq \overline{A}_1 \leq ...... \leq \overline{A}_k \leq 1$, and when $k < n$, $\overline{A}_{k+1} = ... = \overline{A}_n = 0$.

Then

$$F\left(\sum_{i=1}^{n} a_i X_i\right) \leq_{st} G\left(\sum_{i=1}^{n} a_i Y_i\right). \qquad \text{... (3.8)}$$

Proof: Let $G^{-1}F$ be star-shaped, then by Theorem 3.1,

$$G^{-1}F\left(\sum_{i=1}^{n} a_i X_i\right) \le \left(\sum_{i=1}^{n} a_i G^{-1}F(X_i)\right).$$

Let $G^{-1}F(X_i) = Y_i$.

Therefore

$$G^{-1}F\left(\sum_{i=1}^{n} a_i X_i\right) \le \left(\sum_{i=1}^{n} a_i Y_i\right)$$

$$\Rightarrow F\left(\sum_{i=1}^{n} a_i X_i\right) \le G\left(\sum_{i=1}^{n} a_i Y_i\right).$$

Hence $F\left(\sum_{i=1}^{n} a_i X_i\right) \le_{st} G\left(\sum_{i=1}^{n} a_i Y_i\right)$.

The Theorem 3.2 is used to obtain conservative lower tolerance limits for F.

Theorem 3.3: If F is IFR $F(0) = 0$, $F(\xi_q) = q$, then

$$P_F\left\{1 - F[C_{1-a,q,n}\, ,\ T(\underline{X})] \ge 1 - q\right\} \ge 1 - \alpha \qquad \text{... (3.9)}$$

or equivalently,

$$P_F\left\{\xi_q \ge C_{1-a,q,n}\, T(\underline{X}) \ge 1 - q\ \right\} \ge 1 - \alpha\,, \qquad \text{... (3.10)}$$

where $C_{1-a,q,n} = \min\,(B_{1-a,q,n}\, , 1)$.

Proof: Since (3.9) and (3.10) are equivalent, to show (3.9), we have Lemma 3.5,

$$\sum_{i=1}^{r} a_i X_i = \sum_{i=1}^{r} \overline{A}_i (X_i - X_{i-1}), \quad where \quad \overline{A}_i = \sum_{j=1}^{r} a_j\ .$$

By Theorem 3.2, we have

$$F\left(\sum_{i=1}^{r}\overline{A}_i(X_i - X_{i-1})\right) \le_{st} G\left(\sum_{i=1}^{r}\overline{A}_i(Y_i - Y_{i-1})\right).$$

When $0 \le \overline{A}_i \le 1 \; for \; i = 1,2,\ldots\ldots,r$.

Choosing $\overline{A}_i = -2\log(1-q)(n-i+1)/\chi_{1-\alpha}{}^2(2n)$.

We get

$$F\left[\frac{-2\log(1-q)}{\chi_{1-\alpha}{}^2(2n)}\sum_{i=1}^{r}(n-i+1)(X_i - X_{i-1})\right] \le_{st}$$

$$G\left[\frac{-2\log(1-q)}{\chi_{1-\alpha}{}^2(2n)}\sum_{i=1}^{r}(n-i+1)(Y_i - Y_{i-1})\right].$$

When $-2n\log(1-q)/\chi_{1-\alpha}{}^2(2n) \le 1$.

It follows that in this case

$$P_F\{1 - F\,[B_{1-\alpha,q,n}\,T(\underline{X})] \ge 1-q\} \ge 1-\alpha. \qquad \ldots (3.11)$$

If $-2n\log(1-q)/\chi_{1-\alpha}{}^2(2n) > 1$. Then let $\overline{A}_i = (n-i+1)/n$.

So that

$$F\left(\sum_{i=1}^{r}\frac{(n-i+1)}{n}(X_i - X_{i-1})\right) \le_{st} G\left(\sum_{i=1}^{r}\frac{(n-i+1)}{n}(Y_i - Y_{i-1})\right).$$

Similarly,

$$P_G\left[1 - G\left(\sum_{i=1}^{r}\frac{(n-i+1)}{n}(Y_i - Y_{i-1})\right) \ge 1-q\right] \ge P_G[1 - G[T(\underline{Y})] \ge 1-q] = 1-\alpha$$

.

$$\ldots (3.12)$$

So that

$$P_F\left\{1-F\ [C_{1-a,q,n}T(\underline{X})]\geq 1-q\ \right\}\geq 1-\alpha,$$

where $C_{1-a,q,n}=\min(B_{1-a,q,n},1)$.

Thus above Theorem 3.3 gives us $(1-\alpha)$ level, $(1-q)$ content lower tolerance limit for an IFR distribution F. In the following we obtain lower tolerance limits for two IFR distributions namely Weibull and Exponentiated exponential distribution.

3.4 Application:

Weibull Distribution:

Let $X_1, X_2, \ldots\ldots X_n$ be a random sample from Weibull distribution with parameter λ and δ. We know that Weibull distribution belongs to IFR class. Now $(1-\alpha)$ level $(1-q)$ content lower tolerance limits are given by $I_1=(L(\underline{X}),\ \infty)$. In the following we obtain the tolerance limits and compute the same with different values of n and α.

Let

$$f(x)=\begin{cases}\lambda\ \delta\ x^{\delta-1}\ \exp\left(-\lambda\ x^{\delta}\right), & x\geq 0, \quad \lambda,\delta>0\\ 0, & o.w.\end{cases}.$$

The distribution function is given by,

$$F_X(x)=1-\exp\left(-\lambda\, x^{\delta}\right).$$

The Hazard rate is

$$r(t) = \frac{f(t)}{\overline{F}(t)}$$

$$= \lambda\, \delta\, t^{\delta-1} \uparrow in\ t, \text{ if } \delta > 1, \text{ that is F is IFR.}$$

Following graph shows the IFR distribution for different values of δ, taking $\lambda = 1$.

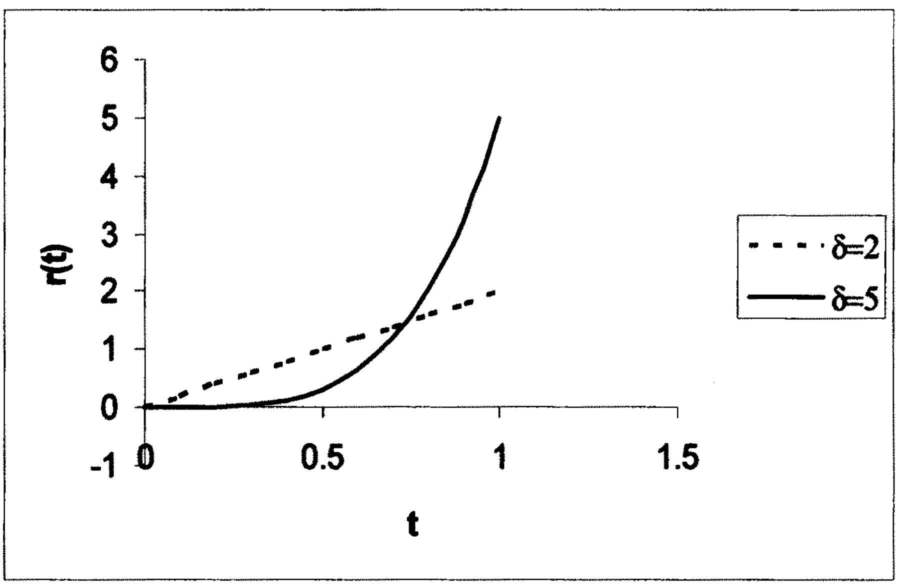

Fig. (3.1): ***IFR distribution for*** **δ = 2 and 5.**

Let $u = 1 - \exp\left(-\lambda\, t^{\delta}\right)$

Hence $t = \left[-\frac{1}{\lambda}\ \log(1-u)\right]^{1/\delta}$.

Now

$$T(\underline{X}) = \sum_{i=1}^{n} (n-i+1) n^{-1} (X_i - X_{i-1}),$$

Hence

$$L(\underline{X}) = \frac{-2n \log(1-q)}{\chi_{1-\alpha}^{2}(2n)} T(\underline{X}).$$

Let $n = 25, \quad \alpha = 0.05, \quad q = 0.05$.

Therefore

$$L(\underline{X}) = \frac{-50 \log(0.95)}{\chi_{0.95}^{2}(50)} T(\underline{X}).$$

Simulation study has been carried out to estimate $L(\underline{X})$. We obtain $L(\underline{X})$ for selected values of α, δ, n and q based on 1000 simulated samples. Average of these limits is computed and denoted by $E\left(L(\underline{X})\right)$.

Lower Tolerance Limits:

Table 3.1: Tolerance limits for different values of n and δ.

α	δ	n	E $(L(\underline{X}))$	Coverage
0.05	2	10	0.083024	0.99966
	2	25	0.065743	0.99978
	2	50	0.058413	0.99983
	2	100	0.053965	0.99985
	5	10	0.086986	1
	5	25	0.067798	1
	5	50	0.060336	1
	5	100	0.055912	1
0.01	2	10	0.110945	0.99988
	2	25	0.076642	0.99994
	2	50	0.064919	0.99996
	2	100	0.058073	0.99997
	5	10	0.114537	1
	5	25	0.07917	1
	5	50	0.067269	1
	5	100	0.060226	1

In the following we find tolerance limits for different values of θ and δ of Exponentiated exponential model introduced by Gupta (1998).

Exponentiated exponential model:

$$f(x) = \begin{cases} \dfrac{\delta}{\theta}\; e^{-x/\theta}\left[1 - e^{-x/\theta}\right]^{\delta-1}, & x \geq 0, \quad \delta, \theta > 0 \\ 0, \quad o.w. & \end{cases}$$

The distribution function is given by,

$$F_X(x) = \left(1 - e^{-x/\theta}\right)^{\delta}.$$

Hence F is IFR for $\delta > 1$.

Following graph shows the IFR distribution for different values of δ, taking θ = 1.

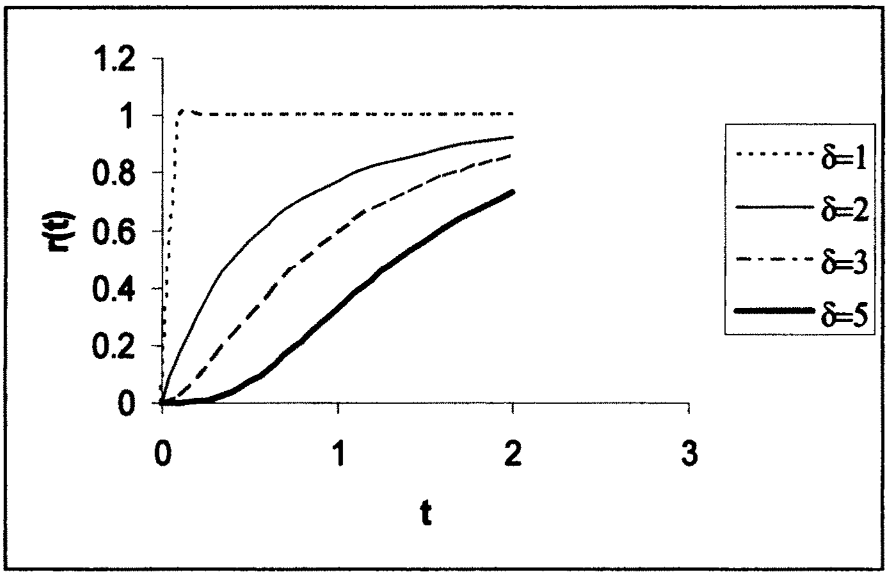

Fig. (3.2): ***IFR distribution for*** **δ = 1, 2, 3 and 5.**

Simulation study has been carried out to estimate $L(\underline{X})$. The result obtained by different values of θ and δ are given below.

Lower Tolerance Limits: ($\alpha = 0.05$)

Table 3.2: Coverage for different values of θ and δ.

$\theta = 1$

δ	n	$E(L(\underline{X}))$	Coverage
1	10	0.094211	0.910091
1	25	0.073779	0.928877
1	50	0.066118	0.93602
1	100	0.061062	0.940765
2	10	0.142797	0.982292
2	25	0.110759	0.989007
2	50	0.098959	0.991123
2	100	0.090955	0.992441
3	10	0.173794	0.99594
3	25	0.135288	0.997974
3	50	0.120809	0.998526
3	100	0.111884	0.998814
5	10	0.216378	0.999721
5	25	0.168524	0.99991
5	50	0.150554	0.999947
5	100	0.139432	0.999963
10	10	0.275386	0.999999
10	25	0.21554	1
10	50	0.19293	1
10	100	0.178407	1

$\theta = 2$

δ	n	$E(L(\underline{X}))$	Coverage
1	10	0.191055	0.908893
1	25	0.146894	0.929185
1	50	0.131916	0.93617
1	100	0.122256	0.940703
2	10	0.283959	0.982481
2	25	0.219503	0.989196
2	50	0.198444	0.991078
2	100	0.18218	0.99242
3	10	0.347339	0.995948
3	25	0.270461	0.997976
3	50	0.241643	0.998526
3	100	0.223497	0.998818
5	10	0.431059	0.999726
5	25	0.337606	0.99991
5	50	0.299775	0.999948
5	100	0.278067	0.999963
10	10	0.553165	0.999999
10	25	0.433855	1
10	50	0.385358	1
10	100	0.356625	1

$\theta = 5$

δ	n	$E(L(\underline{X}))$	Coverage
1	10	0.47497	0.909378
1	25	0.370331	0.92861
1	50	0.328015	0.936503
1	100	0.306675	0.940508
2	10	0.703803	0.98276
2	25	0.555553	0.988941
2	50	0.49489	0.991119
2	100	0.45581	0.992408
3	10	0.868203	0.99595
3	25	0.67934	0.99795
3	50	0.603457	0.99853
3	100	0.559731	0.998812
5	10	1.075857	0.999728
5	25	0.843108	0.99991
5	50	0.74719	0.999948
5	100	0.696601	0.999963
10	10	1.382087	0.999999
10	25	1.077171	1
10	50	0.963917	1
10	100	0.894425	1

$\theta = 25$

δ	n	$E(L(\underline{X}))$	Coverage
1	10	2.339031	0.910682
1	25	1.82684	0.929532
1	50	1.625841	0.937036
1	100	1.527368	0.940734
2	10	3.53772	0.982589
2	25	2.77243	0.988981
2	50	2.459582	0.991221
2	100	2.286905	0.992358
3	10	4.325392	0.99599
3	25	3.405534	0.997935
3	50	3.013377	0.998536
3	100	2.790355	0.998822
5	10	5.427977	0.999717
5	25	4.221202	0.999909
5	50	3.74667	0.999948
5	100	3.474102	0.999963
10	10	6.932767	0.999999
10	25	5.424854	1
10	50	4.826886	1
10	100	4.464567	1

Tolerance Limits: ($\alpha = 0.01$)

Table 3.3: Coverage for different values of θ and δ.

θ = 1

δ	n	$E(L(\underline{X}))$	Coverage
1	10	0.124221	0.883185
1	25	0.08564	0.917925
1	50	0.073687	0.928962
1	100	0.065247	0.936836
2	10	0.186392	0.971082
2	25	0.129581	0.985229
2	50	0.110347	0.989085
2	100	0.098382	0.991221
3	10	0.229286	0.991398
3	25	0.158462	0.996853
3	50	0.134238	0.998018
3	100	0.120254	0.998545
5	10	0.281907	0.999105
5	25	0.196615	0.999819
5	50	0.167088	0.999914
5	100	0.149831	0.999948
10	10	0.364243	0.999993
10	25	0.252142	1
10	50	0.214244	1
10	100	0.192165	1

θ = 2

δ	n	$E(L(\underline{X}))$	Coverage
1	10	0.245414	0.884523
1	25	0.175403	0.916034
1	50	0.146212	0.929502
1	100	0.131736	0.936254
2	10	0.37623	0.970594
2	25	0.259555	0.985187
2	50	0.218702	0.98927
2	100	0.195463	0.991331
3	10	0.449613	0.991839
3	25	0.318829	0.9968
3	50	0.268837	0.99801
3	100	0.240279	0.998549
5	10	0.561758	0.999119
5	25	0.393291	0.999819
5	50	0.333263	0.999915
5	100	0.300526	0.999947
10	10	0.728722	0.999993
10	25	0.505039	1
10	50	0.428952	1
10	100	0.38337	1

$\theta = 5$

δ	n	$E(L(\underline{X}))$	Coverage
1	10	0.623201	0.882814
1	25	0.433705	0.916915
1	50	0.367653	0.929108
1	100	0.327075	0.936679
2	10	0.923869	0.971538
2	25	0.648581	0.9852
2	50	0.549567	0.989166
2	100	0.490906	0.991255
3	10	1.143256	0.991461
3	25	0.785354	0.996929
3	50	0.669615	0.998031
3	100	0.601213	0.998546
5	10	1.412857	0.999096
5	25	0.991942	0.999811
5	50	0.837999	0.999913
5	100	0.747247	0.999948
10	10	1.822117	0.999993
10	25	1.260024	1
10	50	1.07115	1
10	100	0.958727	1

$\theta = 25$

δ	n	$E(L(\underline{X}))$	Coverage
1	10	3.154387	0.88146
1	25	2.154815	0.917418
1	50	1.834157	0.92926
1	100	1.641816	0.936437
2	10	4.698876	0.97064
2	25	3.215739	0.985431
2	50	2.750967	0.989142
2	100	2.457846	0.991232
3	10	5.722683	0.991435
3	25	3.961175	0.996854
3	50	3.345733	0.998035
3	100	2.995126	0.998561
5	10	7.071211	0.999092
5	25	4.879539	0.999825
5	50	4.184859	0.999913
5	100	3.752178	0.999947
10	10	9.123084	0.999993
10	25	6.316585	1
10	50	5.351944	1
10	100	4.792191	1

Conclusion: a) As sample size increases lower tolerance limit approaches to the actual limit.

b) As α increases lower tolerance limit decreases, which results in increase in the coverage probability for the fixed value of α.

c) Coverage probability decreases as θ increases.

d) Coverage probability increases as δ increases.

Thus tolerance limits using stochastic relation of IFR with exponential distribution perform satisfactory.