CHAPTER - I

THE MULTIVARIATE LACK OF MEMORY PROPERTY

1. INTRODUCTION :

1.1. THE UNIVARIATE LACK OF MEMORY PROPERTY.

Suppose a component, say an electric bulb has the property that the probability that the component will be operative for at least (s+t) units given that it has already run for t units is the same as its initial probability of lasting for at least s units. This means that the future lifetime of the component has the same distribution, no matter how old it is at present. This is called as the "Lack of Memory" property. Formally speaking a continuous random variable X is said to have the lack of memory property (LMP) iff,

$$P(X > s+t \mid X > t) = P(X > s), \text{ for } s,t \geq 0.$$

Or equivalently,

$$S(s + t) = S(s)\, S(t), \text{ for } s,t \geq 0. \tag{1.1}$$

where $S(t) = P(X > t)$ is the survival function of X.

In the following sub section we present a characterization of LMP.

1.2. CHARACTERIZATION OF LMP.

In this section we prove that a continuous random variable X has LMP if and only if, $X \sim e(\lambda)$, for some $\lambda > 0$ (Henceforth $X \sim e(\lambda)$ means X has exponential distribution with parameter λ). This characterization follows from the two lemmas proved below.

Lemma 1.1 : The exponential distribution with parameter λ for some $\lambda > 0$, satisfies LMP.

Proof : Suppose $X \sim e(\lambda)$. Then, $S(t) = P(X > t) = \exp(-\lambda t)$., $t \geq 0$. Therefore,

$$\begin{aligned} S(s + t) &= P(X > s + t) \\ &= \exp(-\lambda(s + t)) \\ &= \exp(-\lambda s) \exp(-\lambda t) \\ &= S(s)\, S(t) \quad \forall \; s,t \geq 0. \end{aligned}$$

Thus from (1.1) it follows that X has LMP. ∎

In fact, the exponential distribution is the only continuous distribution which has LMP. This is proved in the following lemma.

Lemma 1.2 : Let the survival probability S of a non-degenerate continuous random variable satisfies (1.1). Then the underlying distribution is exponential with parameter λ, for some $\lambda > 0$.

Proof : Let $c > 0$ and m and n be positive integers. Applying (1.1) repeatedly, we get,

$$S(c+c+c+\ldots n\text{-times } c) = S(c)\, S(c) \ldots\ldots n\text{-times } S(c)$$

$$\Rightarrow \quad S(nc) = (S(c))^n \quad \text{for all } n \in I^+ \ \forall\ c > 0, \tag{1.2}$$

$$\Rightarrow \quad S(c) = (S(c/m))^m \quad \text{for all } m \in I^+. \tag{1.3}$$

We now claim that $0 < S(1) < 1$.

If $S(1) = 1$, then putting $c = 1$ in (1.2), we get,

$$S(n) = (S(1))^n \quad \forall\ n$$

$$\Rightarrow S(n) = 1 \quad \forall\ n$$

Letting $n \to \infty$ this gives $S(\infty) = 1$, which contradicts the fact that $S(\infty) = 0$. This implies that,

$$S(1) \neq 1 \tag{1.4}$$

Further if $S(1) = 0$, then putting $c = 1$ in (1.3) we get,

$$S(1) = (S(1/m))^m \quad \forall\ m$$

$$\Rightarrow S(1/m) = (S(1))^{1/m} \quad \forall\ m$$

$$\Rightarrow S(1/m) = 0 \quad \forall\ m$$

Letting $m \to \infty$ this gives $S(0) = 0$, which contradicts the fact that $S(0) = 1$. Therefore,

$$S(1) \neq 0 \tag{1.5}$$

From (1.4) and (1.5) it follows that, $0 < S(1) < 1$.

Suppose $S(1) = e^{-\lambda}$, $0 < \lambda < \infty$, $\Rightarrow \lambda = -\log(S(1))$. Putting $c = 1$ in (1.3) it follows that,

$$S(1/m) = (S(1))^{1/m}$$
$$= e^{-\lambda/m} \quad \forall\ m \in I^+$$

Therefore taking $c=1/m$ in equation (1.2), we get

$$S(n/m) = e^{-n\lambda/m} \qquad \forall\ n,m \in I^+.$$

i.e. $S(y) = e(-\lambda y)$, for every positive rational number y. (1.6)

Next, let x be a positive irrational number. Then, since x is a limit point of the set of all rational numbers, there exists a sequence $\{ y_n \}$ of rational numbers which decreases to x. That is $\{ y_n \} \downarrow x$. Since the function S is right continuous, it follows that,

$$\lim_{n\to\infty} S(y_n) = S(x) \tag{1.7}$$

Since y_n are rational numbers, from (1.6) we have,

$$S(y_n) = e(-\lambda y_n) \qquad \forall\ n,$$

Since $y_n \to x$, as $n \to \infty$,

$$\lim_{n\to\infty} S(y_n) = \lim_{n\to\infty} e(-\lambda y_n)$$

$$= e^{-\lambda x}$$

Using equation (1.7), this gives

$$S(x) = \exp(-\lambda x)$$

for all positive irrational numbers x. Thus it follows that,

$$S(x) = \exp(-\lambda x) \qquad \forall\ x \geq 0,$$

which is survival function of an exponential distribution with parameter λ. Thus the lemma follows. ■

In many situations the component lifetimes will be dependent and will have some joint probability distribution. For example the failure of paired airplane engines, the registration of an event on two adjacent geiger counters, and the failure of paired organs such as lungs, kidneys and eyes. In order to study such situations it is important to extend univariate LMP to higher dimensions.

In section 2, we discuss some extensions of LMP for Bivariate and Multivariate case and its interpretation. In section 3, we present characterizations of BLMP given by Marshall and Olkin (1967), Block and Basu (1974) and Kulkarni and Prasad (1996). In section 4, we discuss characterizations of MLMP given by Ghurye and Marshall (1984) and Kulkarni (1998). In section 5, some supplementary results are presented. It is shown that the only distribution having BLMP with exponential marginals is BVE given by Marshall and Olkin (1967). Conditions on marginals of a bivariate distribution having BLMP are also discussed and some distributions having BLMP are presented.

In the next section we discuss some extensions of LMP for bivariate and multivariate case.

2. EXTENSION OF LMP TO BIVARIATE AND MULTIVARIATE CASE.

2.1 A NATURAL EXTENSION OF LMP:

For bivariate case a straightforward extension of LMP (1.1) is,

$$S(s_1+ t_1, s_2+ t_2) = S(s_1, s_2)\ S(t_1, t_2),\ \forall\ s_1, s_2, t_1, t_2 \geq 0. \qquad (1.8)$$

However the following lemma shows that the only distributions satisfying (1.8) are those having independent exponential marginals, which are not of any use in modeling joint behavior of dependent components.

Lemma 1.3 : If (1.8) holds then,

$$S(s_1,s_2) = \exp(-(\theta_1 s_1+\theta_2 s_2)) \qquad \forall \quad \theta_1 > 0,\ \theta_2 > 0.$$

Proof : Suppose (1.8) holds. Setting $s_2 = t_2 = 0$, in equation (1.8) yields,

$$S_1(s_1+ t_1) = S_1(s_1)\ S_1(t_1) \qquad \forall \quad s_1,t_1 > 0.$$

where S_1 is the survival function of marginal distribution of first component. Thus S_1 satisfies univariate LMP. Hence by Lemma 1.2 we have,

$$S_1(x) = e^{-\theta_1 x}, \quad \text{for some } \theta_1 > 0, \quad \forall\ x \geq 0.$$

On similar lines it follows that,

$$S_2(y) = e^{-\theta_2 y}, \quad \text{for some } \theta_2 > 0, \quad \forall\ y \geq 0,$$

where S_2 is survival function of marginal distribution of second component.

By choosing $t_1 = s_2 = 0$ in equation (1.8), we obtain

$$\begin{aligned} S(s_1,t_2) &= S_1(s_1)\ S_2(t_2) && \forall\ s_1,t_2 \geq 0 \\ &= e^{-\theta_1 s_1} e^{-\theta_2 t_2} && \forall\ s_1,t_2 \geq 0. \end{aligned}$$

which proves the lemma. ■

Thus under (1.8), the joint survival function is the product of marginal survival functions. This means that the requirement of (1.8) is too stringent to yield a useful version of Bivariate Lack of Memory Property and needs to be weakened so that some bivariate distribution having meaningful dependence structure would satisfy the new modified definition.

In the next subsection we discuss a weaker form of bivariate LMP proposed by Marshall and Olkin (1967). Its extension to multivariate case is also given. Henceforth we refer to this version of bivariate LMP as bivariate lack of memory property

already used

(BLMP), and its multivariate extension as Multivariate Lack of Memory property (MLMP).

2.2. THE BIVARIATE AND MULTIVARIAE LACK OF MEMORY PROPERTY:

Let (X,Y) be bivariate random variable having survival function S. Then (X,Y) is said to have bivariate LMP if and only if,

$$S(s_1+ t, s_2+ t) = S(s_1, s_2)\, S(t,t), \ \forall\ s_1, s_2, t \geq 0 \qquad (1.9a)$$

Note that, This is same as,

$$P(X > s_1+t, Y > s_2+t \mid X > t, Y > t) = P(X > s_1, Y > s_2), \ \forall\ s_1, s_2, t \geq 0.$$

Therefore the condition (1.9a) can be interpreted as the conditional probability of both components each of the same age t, surviving an additional time $\underline{s} = (s_1, s_2)$ is equal to the probability of survival of $\underline{s} = (s_1, s_2)$ time units of new components. We further note that (1.9a) is also same as,

$$P(X>s_1+t, Y>s_2+t \mid X > s_1, Y > s_2) = P(X > t, Y > t), \quad \forall\ s_1,\ s_2,\ t \geq 0.$$

Thus (1.9a) can also be interpreted in an another way as conditional probability of two components of ages $\underline{s} = (s_1, s_2)$, both surviving an additional time t is equal to the probability that they survive t time units starting at the origin.

2.3. MULTIVARIATE EXTENSION:

Let $\underline{X} = (X_1, X_2, \ldots, X_k)$ be multivariate random variable having survival function S. Then $\underline{X}$ is said to have multivariate LMP if, MLMP and only if,

$$S(s_1+t, \ldots s_k+t) = S(s_1, s_2, \ldots s_k)\ S(t, t, \ldots t), \ \forall\ s_1, s_2, \ldots s_k,\ t \geq 0 \tag{1.9b}$$

$$\text{i.e. } S(\underline{s} + t\underline{e}) = S(\underline{s})\ S(t\underline{e}) \qquad \forall\ \underline{s}, t \geq 0. \tag{1.9c}$$

where $\underline{e} = (1, \ldots, 1)$. & $\underline{S} = (s_1, s_2, \ldots, s_k)$

Interpretations similar to those given in (1.9a) for bivariate case also hold for multivarite case.

In the next section we present some characterizations of BLMP.

3. CHARACTERIZATIONS OF BLMP.

3.1. CHARACTERIZATION DUE TO MARSHALL AND OLKIN (1967):

Marshall and Olkin (1967) give the following characterization of bivariate distributions possessing bivariate lack of memory property (BLMP).

Theorem 1.1: BLMP (1.9a) holds iff,

$$S(x,y)= \begin{cases} e^{-\theta y} \quad S_1(x-y) \quad \text{if} \quad x \ge y \ge 0 \\ e^{-\theta x} \quad S_2(y-x) \quad \text{if} \quad y \ge x \ge 0 \end{cases} \tag{1.10}$$

where the marginal survival function $S(t,0)$ and $S(0,t)$ are denoted by $S_1(t)$ and $S_2(t)$ respectively.

Proof : Suppose (1.9a) holds. Setting $s_1 = s_2 = s$ in (1.9a) yields,

$$S(s+t,s+t) = S(s,s)\ S(t,t) \quad \forall\ s,t \ge 0.$$

Therefore, since $S_Z(t) = S(t,t)$, $t \ge 0$ is the survival function of the univariate random variable $Z = \min(X,Y)$, by Lemma 1.2, we must have,

$$S(s,s) = \exp(-\theta s), \quad \text{for some } \theta > 0, \quad \forall\ s \ge 0.$$

Now putting $s_2 = 0$ in (1.9a), we get,

$$\begin{aligned} S(s_1+ t,t) &= S(s_1,0)\ S(t,t) \\ &= S_1(s_1)\ \exp(-\theta t),\ \theta > 0 \quad \forall\ s_1,t \ge 0. \end{aligned}$$

letting $x = s_1+t$, $y = t$, this gives

$$S(x,y) = e^{-\theta y}\ S_1(x-y), \qquad \text{for } x \ge y \ge 0. \tag{1.11}$$

Similarly putting $s_1 = 0$ in (1.9a), we get,

$$\begin{aligned} S(t,s_2+ t) &= S(0,s_2)\ S(t,t) \\ &= S_2(s_2)\ \exp(-\theta t),\ \theta > 0\ \forall\ s_2,t \ge 0. \end{aligned}$$

letting $y = s_2 + t$, $x = t$ this gives,

$$S(x,y) = e^{-\theta x} S_2(y-x) \quad \forall\, y \ge x \ge 0 \tag{1.12}$$

Hence from (1.11) and (1.12) we have,

$$S(x,y) = \begin{cases} e^{-\theta y} S_1(x-y) & \text{for } x \ge y \ge 0. \\ e^{-\theta x} S_2(y-x) & \text{for } y \ge x \ge 0. \end{cases} \tag{1.13}$$

Conversely, let (1.10) holds. Then

$$S(s_1+t, s_2+t) = \begin{cases} e^{-\theta(s_2+t)} S_1(s_1-s_2) & \text{for } s_1 \ge s_2,\ t \ge 0 \\ e^{-\theta(s_1+t)} S_2(s_2-s_1) & \text{for } s_2 \ge s_1,\ t \ge 0 \end{cases} \tag{1.14}$$

Putting t=0 this gives,

$$S(s_1,s_2) = \begin{cases} e^{-\theta s_2} S_1(s_1-s_2) & \text{for } s_1 \ge s_2 \ge 0 \\ e^{-\theta s_1} S_2(s_2-s_1) & \text{for } s_2 \ge s_1 \ge 0 \end{cases}$$

Also, $S(t,t) = e^{-\theta t}$, for $t \ge 0$. Therefore,

$$S(s_1,s_2)\, S(t,t) = \begin{cases} e^{-\theta(s_2+t)} S_1(s_1-s_2), & \text{for } s_1 \ge s_2,\ t \ge 0 \\ e^{-\theta(s_1+t)} S_2(s_2-s_1), & \text{for } s_2 \ge s_1,\ t \ge 0 \end{cases} \tag{1.15}$$

From (1.14) and (1.15), we have,

$$S(s_1+t, s_2+t) = S(s_1,s_2)\, S(t,t) \qquad s_1, s_2, t \ge 0.$$

Thus S has BLMP. ■

Some more results related to Marshall and Olkin's characterization of BLMP are discussed in section 5.

3.2. CHARACTERIZATION OF BLMP GIVEN BY BLOCK AND BASU (1974):

An another characterization of BLMP established by Block and Basu (1974) is presented it in this section. The main result is we give given in Theorem 1.2. Before the proof of the Theorem, we ~~prove~~ give two relevant lemmas namely lemma (1.4) and (1.5).

Lemma 1.4 : Let (X,Y) have non-negative bivariate distribution with marginal densities which are absolutely continuous on $(-\infty,\infty)$ and suppose (X,Y) have BLMP. Then there exists a positive number $\theta > 0$ such that,

a) $\min(X,Y) \sim e(\theta)$

b) $$P(X-Y \le t) = \begin{cases} F_1(t) + \theta^{-1} f_1(t) & \text{if } t \ge 0, \\ 1 - F_2(-t) - \theta^{-1} f_2(-t) & \text{if } t < 0, \end{cases}$$

where F_i and f_i $i = 1,2$ are respectively the marginal distribution functions and marginal densities of X and Y.

Proof: Let S be the survival function of (X,Y). Since (X,Y) have BLMP, we have from (1.9a),

$$S(s_1+ t,s_2+ t) = S(s_1,s_2)\, S(t,t) \qquad \forall\ s_1,s_2,t \ge 0.$$

Letting $s_1 = s_2 = s$, we get

$$S(s + t,s + t) = S(s,s)\, S(t,t) \qquad \forall\ s,t \ge 0$$

$$\Rightarrow \qquad S_z(s+t) = S_Z(s)\, S_Z(t) \qquad \forall\ s,t \geq 0,$$

where S_z is the survival function of the random variable $Z = \min(X,Y)$. Now by Lemma 1.2 it follows that there exist $\theta > 0$, such that $Z \sim \exp(\theta)$. Thus part (a) is proved.

Now, Since (X,Y) have BLMP, by Theorem 1.1, (1.10) holds. For $x \neq y$, since f'_1 and f'_2 exist, (Note that by hypothesis f_1 and f_2 are absolutely continuous which means that f'_1 and f'_2 exists, integrable and $f_j(z) = \int_{-\infty}^{z} f'_j(t)\,dt$, j=1,2.) differentiating (1.10) partially with respect to x and y, we get,

$$\frac{\partial^2 S(x,y)}{\partial x\ \partial y} = f(x,y)$$

$$= \begin{cases} \exp(-\theta y)\ [f'_1(x-y) + \theta\ f_1(x-y)] & \text{if } x \geq y \geq 0, \\ \exp(-\theta x)\ [f'_2(y-x) + \theta\ f_2(y-x)] & \text{if } y \geq x \geq 0, \end{cases} \tag{1.16}$$

In order to prove part (b), first we obtain joint distribution of min(X,Y) and (X-Y) and then obtain the marginal distribution of (X-Y), by integrating over the range of min(X,Y). We have,

$$\begin{aligned} P[\min(X,Y) \leq s,\ X-Y \leq t) &= P[\min(X-Y) \leq s,\ X-Y \leq t,\ X > Y\ ] \\ &+ P[\min(X-Y) \leq s,\ X-Y \leq t,\ X = Y\ ] \\ &+ P[\min(X-Y) \leq s,\ X-Y \leq t,\ Y > X\ ] \end{aligned} \tag{1.17}$$

To obtain $P(\min(X,Y) \leq s,\ X-Y \leq t,\ X > Y)$, we integrate (1.16) over the region, $0 \leq y \leq s$ and $y \leq x \leq y+t$. We get,

$P(\min(X,Y) \le s, X-Y \le t, X > Y)$

$$= \begin{cases} \displaystyle\int_0^s \int_y^{y+t} e^{-\theta y}\left[f_1'(x-y) + \theta f_1(x-y)\right]dx\,dy & t \ge 0 \\ 0 & t < 0 \end{cases}$$

$$= \begin{cases} \displaystyle\int_0^s e^{-\theta y}\left[f_1(t) + \theta F_1(t) - f_1(0) - \theta F_1(0)\right]dy & t \ge 0, \\ 0 & t < 0, \end{cases}$$

$$= \begin{cases} (1 - e^{-\theta s})\,\theta^{-1}\left[f_1(t) + \theta F_1(t) - f_1(0)\right] & t \ge 0, \\ 0 & t < 0. \end{cases} \qquad (1.18)$$

Next, $P(\min(X,Y) \le s, X-Y \le t, Y > X)$ is obtained by integrating (1.16) over the region $0 < x < s$, $x < y < \infty$ for $t \ge 0$ and the region $0 < x < s$, $x-t < y < \infty$, for $t < 0$. This gives,

$P(\min(X,Y) \le s, X-Y \le t, Y > X)$

$$= \begin{cases} \displaystyle\int_0^s \int_x^{\infty} e^{-\theta x}\left[f_2'(y-x)+\theta f_2(y-x)\right] dy\,dx & t \ge 0, \\ \displaystyle\int_0^s \int_{x-t}^{\infty} e^{-\theta x}\left[f_2'(y-x)+\theta f_2(y-x)\right]dy\,dx & t < 0, \end{cases}$$

$$= \begin{cases} \int_0^s e^{-\theta x} [\theta - f_2(0)]dx & \text{if } t \geq 0, \\ \int_0^s e^{-\theta x} [\theta - f_2(-t)-\theta F_2(-t)]dx & \text{if } t < 0, \end{cases}$$

$$= \begin{cases} (1-\exp(-\theta s))\ \theta^{-1}[\theta-f_2(0)] & \text{if } t \geq 0. \\ (1-\exp(-\theta s))\ \theta^{-1}[\theta-f_2(-t)-\theta F_2(-t)] & \text{if } t < 0. \end{cases} \quad (1.19)$$

Further it is easy to observe that,

$$P(\min(X,Y)\leq s, X-Y\leq t, Y=X) = \begin{cases} P(X \leq s, X = Y) & \text{if } t\geq 0, \\ 0 & \text{if } t<0, \end{cases} \quad (1.20)$$

From (1.17),(1.18),(1.19) and (1.20), We get,

$P[\min(X,Y) \leq s, X-Y \leq t)$

$$= \begin{cases} \left[1-e^{-\theta s}\right]\left[F_1(t)+\theta^{-1}f_1(t)+1-\frac{1}{\theta}\{f_1(0)+f_1(0)\}\right]+P(X=Y\leq s) & t\geq 0, \\ \left[1-e^{-\theta s}\right]\left[1-F_2(-t) - \frac{1}{\theta}\ f_2(-t)\ \right] & t<0, \end{cases} \quad (1.21)$$

Letting $s \to \infty$ and noting from the proof of Marshall and Olkin (1967) Theorem 5.1 that, $P(X=Y) = \theta^{-1}[f_1(0) + f_2(0)] - 1$, we get

$$P(X-Y \leq t) = \begin{cases} F_1(t) + \theta^{-1}\ f_1(t) & \text{if } t \geq 0. \\ 1 - F_2(-t) - \theta^{-1}f_2(-t) & \text{if } t < 0. \end{cases}$$

Thus (b) is proved. ■

REMARK 1.1: From equation (1.65) and (1.69) (cf section 5.2), it is known that $P(X = Y) = \theta^{-1}\ [f_1(0) + f_2(0)] - 1$. Therefore, if P(X =

Y) = 0 then $\theta^{-1}[f_1(0) + f_2(0)] = 1$ and $P(X = Y \le s) = 0$. Then (1.21) becomes,

$P[\min(X,Y) \le s, X-Y \le t)$

$$= \begin{cases} [1-\exp(-\theta s)]\ [F_1(t) + \theta^{-1} f_1(t)] & \text{if } t \ge 0, \\ [1-\exp(-\theta s)]\ [1 - F_2(-t) - \theta^{-1} f_2(-t)] & \text{if } t < 0, \end{cases}$$

$$= P(\min(X,Y) \le s]\ P[X-Y \le t] \qquad \text{(This follows from (a) and (b))}$$

Hence $U = \min(X,Y)$ and $V = X-Y$ are independently distributed. Conversely, if U and V are independently distributed, then exactly reversing the above steps, it follows that $P(X = Y) = 0$. Moverover from the discussion page no. 1035 of Block and Basu (1974), it is clear that the condition $P(X = Y) = 0$ is equivalent to (X,Y) being absolutely continuous. From above discussion it is clear that U and V are independently distributed if and only if (X,Y) are jointly absolutely continuous. ■

Lemma 1.5: Let (X,Y) have a non-negative bivariate distribution with continuous marginal densities and such that $U = \min(X,Y)$ and $V = X-Y$ satisfy the following conditions: there is $\theta > 0$ such that,

1) U and V are independent. (1.22)

2) $U \sim e(\theta)$ (1.23)

$$3)\ P(V \le t) = \begin{cases} F_1(t) + \theta^{-1} f_1(t) & \text{if } t \ge 0, \\ 1 - F_2(-t) - \theta^{-1} f_2(-t) & \text{if } t < 0, \end{cases} \quad (1.24)$$

Then (X,Y) has bivariate lack of memory property (BLMP).

Proof : By Theorem 1.1, it is enough to prove that the joint survival function of (X,Y) has the form

$$S(x,y) = \begin{cases} e^{-\theta y} \; S_1(x-y), & \text{if } x \ge y \ge 0 \\ e^{-\theta x} \; S_2(y-x), & \text{if } y \ge x \ge 0 \end{cases} \quad (1.25)$$

which is equivalent to BLMP.

First we consider the case $0 \le x \le y$. We have,

$$\begin{aligned} S(x,y) &= P(X > x, Y > y) \\ &= P(X > x, Y > y, X \ge Y) + P(X > x, Y > y, X < Y) \\ &= P(U+V > x, U > y, V \ge 0) + P(U > x, U-V > y, V < 0) \quad (1.26) \end{aligned}$$

Note that the last step follows since $X \ge Y \Rightarrow U = \min(X,Y) = Y$ and $V = X-Y \Rightarrow X = Y+V = U+V$. Also, when $X < Y$ then $U = \min(X,Y) = X$ and $Y = X-V = U-V$.

First consider the first term in the R.H.S. of (1.26). Since U and V are independently distributed (by (1.22)) and for $y \ge x$, the event $U + V > x$ is implied by the other two, we have,

$$\begin{aligned} P(U+V > x, U > y, V \ge 0) \\ = P(U > y, V \ge 0) \end{aligned}$$

$$= P(U > y)\ P(V \geq 0)$$

$$= \int_y^{\infty} \theta\ e^{-\theta t} dt\ (1 - P(V < 0\ )) \qquad \text{(by (1.23))}$$

$$= e^{-\theta y} \left[1 - \lim_{t \to 0} [1 - F_2(-t) - \theta^{-1} f_2(-t)] \right] \qquad \text{(from 1.24))}$$

$$= e^{-\theta y} \left[1 - 1 + F_2(0) + \theta^{-1} f_2(0)\ \right]$$

$$= e^{-\theta y}\ \theta^{-1} f_2(0) \quad \text{if } y \geq 0. \quad (F_2(0) = 0) \qquad (1.27)$$

Next consider the second term in the R.H.S. of (1.26). For $y \geq x$ we have,

$$P(U > x, U-V > y, V < 0) = P(U>x, U-V>y, x-y<V<0) + P(U>x, U-V>y, V \leq x-y)$$

$$= P(U > y+V, x-y < V < 0) + P(U > x, V \leq x-y) \qquad (1.28a)$$

Consider,

$$P(U > y+V,\ x-y < V < 0) = \int_{x-y}^{0} [\int_{y+v}^{\infty} dP(U \leq u)]\ dP(V \leq v)$$

$$= \int_{x-y}^{0} [\int_{y+v}^{\infty} \theta\ e^{-\theta u} du]\ dP(V \leq v)$$

$$= \int_{x-y}^{0} e^{-\theta (y+v)} [f_2(-v) + \theta^{-1} f_2'(-v)]\ dv$$

$$= e^{-\theta y} [\theta^{-1} e^{-\theta (x-y)} f_2(y-x) - \theta^{-1} f_2(0)]$$

$$= \theta^{-1} e^{-\theta x} f_2(y-x) - \theta^{-1} e^{-\theta y} f_2(0) \qquad (1.28b)$$

Next, for $y > x$, since U and V are independently distributed (by (1.22)), we have,

$$P(U > x, V \le x-y) = P(U > x)\, P(V \le x-y)$$

$$= \int_x^\infty \theta\, e^{-\theta u} du \int_{-\infty}^{x-y} dP(V \le v)$$

$$= e^{-\theta x} \int_{-\infty}^{x-y} [f_2(-v) + \theta^{-1} f_2'(-v)]\, dv$$

$$= e^{-\theta x}\Big[1 - F_2(-v) - \theta^{-1} f_2(-v)\Big]_{-\infty}^{x-y}$$

$$= e^{-\theta x}\Big[1 - F_2(y-x) - \theta^{-1} f_2(y-x)\Big] \qquad (1.28c)$$

Therefore, from equation (1.28a),(1.28b) and (1.28c), we get

$$P(U > x, U-V > y, V < 0) = e^{-\theta x}\Big[1 - F_2(y-x)\Big] - \theta^{-1} e^{-\theta y} f_2(0) \qquad (1.29)$$

From (1.26), (1.27) and (1.29), we get

$$S(x,y) = e^{-\theta x}[1-F_2(y-x)] \qquad \text{if } y \ge x \ge 0$$

$$= e^{-\theta x}\ S_2(y-x) \qquad \text{if } y \ge x \ge 0 \qquad (1.30)$$

Next, We consider the case $0 \le y \le x$. Recalling equation (1.26) we have,

$$S(x,y) = P(U + V > x, U > y, V \ge 0) + P(U > x, U-V > y, V < 0) \qquad (1.31)$$

Consider, the first term in the R.H.S. of (1.31).

Since $0 \le y \le x$, we have

$$P(U+V>x, U>y, V \ge 0) = P(U+V>x, y<U\le x, V \ge 0) + P(U+V>x, U>x, V \ge 0)$$

$$= P(y < U \le x, x-U \le V < \infty) + P(U > x, V \ge 0) \qquad (1.32)$$

Since U and V are independently distributed, we have,

$$P(U > x, V \ge 0) = P(U > x)\, P(V \ge 0)$$

$$= \int_x^\infty \theta\, e^{-\theta u} du\, [1 - P(V < 0)] \qquad \text{(from (1.23))}$$

$$= e^{-\theta x}[1 - \lim_{t\to 0}[1-F_2(-t) - \theta^{-1} f_2(-t)]] \qquad \text{(from (1.24))}$$

$$= e^{-\theta x}[1 - 1 + F_2(0) + \theta^{-1} f_2(0)]$$

$$= e^{-\theta x}\, \theta^{-1} f_2(0) \qquad \text{(Since } F_2(0)=0\text{)} \qquad (1.33)$$

and

$$P(y<U\le x, x-U<V<\infty) = \int_y^x dP(U \le u) \int_{x-u}^\infty dP(V \le v)$$

$$= \int_y^x \theta\, e^{-\theta u}[\int_{x-u}^\infty [f_1(v) + \theta^{-1} f_1'(v)]\, dv]\, du$$

$$= \int_y^x \theta\, e^{-\theta u}[1 - F_1(x-u) - \theta^{-1} f_1(x-u)]\, du$$

$$= e^{-\theta y} - e^{-\theta x} - [- e^{-\theta x} F_1(0) + e^{-\theta y} F_1(x-y)]$$

$$= e^{-\theta y}\, [1 - F_1(x - y)] - e^{-\theta x} \qquad (1.34)$$

From (1.32) to (1.34) we get,

$$P(U+V>x, U>y, V\geq 0) = e^{-\theta x}\,\theta^{-1} f_2(0) + e^{-\theta y}[1 - F_1(x-y)] - e^{-\theta x} \quad (1.35)$$

Next consider the second term in the R.H.S. of (1.31).

Since U and V are independently distributed and $y \leq x$ we have,

$$P(U > x, U-V > y, V < 0) = P(U > x, V < 0)$$

$$= P(U > x) \;\; P(V < 0)$$

$$= e^{-\theta x}\,[1 - \theta^{-1} f_2(0)] \qquad \text{(from (1.23) \& (1.24))} \quad (1.36)$$

From (1.31),(1.35) and (1.36) we get,

$$S(x,y) = e^{-\theta y}[1 - F_1(x - y)]$$

$$= e^{-\theta y} S_1(x-y), \qquad \text{if } x \geq y \geq 0 \quad (1.37)$$

Thus from (1.30) and (1.37), we have

$$S(x,y) = \begin{cases} e^{-\theta y} \;\; S_1(x-y), & \text{if } \;\; x \geq y \geq 0 \\ e^{-\theta x} \;\; S_2(y-x), & \text{if } \;\; y \geq x \geq 0 \end{cases}$$

Hence it follows that, (X,Y) has BLMP, and the lemma is proved. ■

The main result is proved in the Theorem 1.2 below.

Theorem 1.2: Let (X,Y) have a non-negative bivariate distribution which is absolutely continuous. Then the BLMP holds if and only if for $U = \min(X,Y)$ and $V = X-Y$, there is a $\theta > 0$ such that,

1) U and V are independent.

2) $U \sim e(\theta)$.

$$3)\ P(V \le t) = \begin{cases} F_1(t) + \theta^{-1} f_1(t) & \text{if } t \ge 0, \\ 1 - F_2(-t) - \theta^{-1} f_2(-t) & \text{if } t < 0, \end{cases}$$

where for $i = 1,2$ $F_i(t)$ and $f_i(t)$ are respectively the marginal distribution functions and densities of X and Y.

Proof : Let (X,Y) have joint absolute continuous distribution.

IF part: Suppose (X,Y) have BLMP. Then joint absolute continuity of (X,Y) implies (X,Y) have marginal absolute continuity. Therefore Lemma 1.4 is applicable so that (2) and (3) follow from Lemma 1.4.

Next from the joint absolute continuity and Remark 1.1 it follows that U and V are independently distributed.

Only IF part: Suppose (1),(2) and (3) hold. Further the joint absolute continuity of (X,Y) implies that V = X-Y is absolutely continuous which in turn implies the absolute continuity of marginals f_1 and f_2. Then Lemma 1.5 is applicable, which proves that (X,Y) has BLMP. ∎

3.3. CHARACTERIZATION OF BLMP GIVEN BY KULKARNI and Prasad (1994):

Kulkarni (1993) give the following characterization of bivariate lack of memory property (BLMP).

Theorem 1.3: Let S(x,y) be a survival function corresponding to a bivariate random vector (X,Y) for which vector failure rate $\underline{r}$ = (R_1, R_2) exists for all $x,y \ge 0$. Then S has BLMP if and only if $R_1(x,y) + R_2(x,y) \equiv \theta$ for all $x,y \ge 0$, where $\theta > 0$ is a constant.

Proof : Suppose S has BLMP. Then from (1.10), it follows that

$$R(x,y) = -\log S(x,y)$$

$$= \begin{cases} \theta y + R_X(x-y), & \text{if } x \geq y \geq 0 \\ \theta x + R_Y(y-x), & \text{if } y \geq x \geq 0 \end{cases}$$

where $\theta = f_1(0) + f_2(0)$ and R_X and R_Y are marginal cumulative hazard functions.

Differentiating partially R with respect to x and y respectively, we get

$$R_1(x,y) = \begin{cases} r_1(x-y), & \text{if } x \geq y \geq 0 \\ \theta - r_2(y-x) & \text{if } y \geq x \geq 0 \end{cases} \tag{1.38}$$

$$R_2(x,y) = \begin{cases} \theta - r_1(x-y) & \text{if } x \geq y \geq 0 \\ r_2(y-x) & \text{if } y \geq x \geq 0 \end{cases} \tag{1.39}$$

where r_1 and r_2 are failure rates of the marginal distributions of x and y respectively. From (1.38) and (1.39), we have

$$R_1(x,y) + R_2(x,y) = \theta \quad \forall \; x,y \geq 0 \tag{1.40}$$

Conversely suppose for some $\theta > 0$,

$$R_1 + R_2 = \theta \quad \forall \; x,y \geq 0 \tag{1.41}$$

Noting that $R_1 = \dfrac{\partial R}{\partial x}$ and $R_2 = \dfrac{\partial R}{\partial y}$, it follows that the equation (1.41) is a first order partial differential equation (called as Langrange's linear partial differential equation) whose solution is given by $\tau(u,v) = 0$ where $u = x-y$ and $v = \theta x - R(x,y)$ (cf Miller

(1960),pp.87). Assuming that $\tau(u,v) = 0$ can be solved for v as $v = g(u)$ we get,

$$\theta x - R(x,Y) = g(x-y), \text{ so that,}$$

$$\Rightarrow \quad R(x,y) = \theta x - g(x-y) \tag{1.42}$$

where g is some function of $u = x - y$. This gives,

$$S(x,y) = e^{-R}$$

$$= \exp(-\theta x) \exp(g(x-y)).$$

Putting $x = 0$ and $y = 0$ in this equation respectively and noting that $S(x,0)$ and $S(0,y)$ correspond to marginal survival functions we get,

$$S(0,y) = \exp(-R_Y(y)) = \exp(g(-y)), \qquad \forall \; y \geq 0, \text{ and}$$

$$S(x,0) = \exp(-R_X(x)) = \exp(-\theta x + g(x)), \quad \forall \; x \geq 0.$$

This implies that,

$$g(-y) = -R_Y(y), \qquad \forall \; y \geq 0, \text{ and}$$

$$g(x) = \theta x - R_X(x), \qquad \forall \; x \geq 0.$$

Thus,

$$g(z) = \begin{cases} \theta z - R_X(z) & \text{if } z \geq 0. \\ -R_Y(-z) & \text{if } z \leq 0. \end{cases}$$

Hence from (1.42) we get

$$R(x,y) = \begin{cases} \theta y + R_X(x-y) & \text{if } x \geq y \geq 0. \\ \theta x + R_Y(y-x) & \text{if } y \geq x \geq 0. \end{cases}$$

Now, $S = \exp(-R)$

$$S(x,y) = \begin{cases} \exp(-\theta y - R_X(x-y)) & x \ge y \ge 0, \\ \\ \exp(-\theta x - R_Y(y-x)) & y \ge x \ge 0, \end{cases}$$

$$= \begin{cases} e^{-\theta x} \; S_1(x-y) & x \ge y \ge 0, \\ \\ e^{-\theta y} \; S_2(y-x) & y \ge x \ge 0, \end{cases}$$

Hence from (1.10) it follows that S has BLMP. ■

In the next section we discuss some characterizations of MLMP.

4. CHARACTERIZATIONS OF MLMP:

In this section we discuss three characterizations of MLMP. Two of these are given by Ghurye and Marshall (1984) and other one by Kulkarni (1998).

4.1. CHARACTERIZATION OF GHURYE AND MARSHALL (1984):

Ghurye and Marshall (1984) gave following two characterizations of MLMP. One is given in Theorem 1.4 below and the other one follows from Corollary 1.1.

Theorem 1.4.: A random vector $\underline{X}$ satisfies MLMP if and only if the following conditions holds.

i) $\underline{X} = U\underline{e} + \underline{W}$.

ii) U and $\underline{W}$ are independent.

iii) $\min(W_1, \ldots, W_n) = 0$ with probability 1.

iv) either $\underset{\sim}{U}$ has an exponential distribution, say with parameter θ, or $\underset{\sim}{U}$ is degenerate at 0.

where $\underset{\sim}{U} = \min(X_1, \ldots, X_n)$ and $\underset{\sim}{W} = \underset{\sim}{X} - U\underset{\sim}{e}$.

Proof : If part: Suppose $\underset{\sim}{X}$ has MLMP.

To prove (i):

Let $U = \min(X_1, \ldots, X_n)$ and $\underset{\sim}{W} = \underset{\sim}{X} - U\underset{\sim}{e}$. Then,

$$U\underset{\sim}{e} + \underset{\sim}{W} = U\underset{\sim}{e} + \underset{\sim}{X} - Ue$$

$$= \underset{\sim}{X}$$

Thus (i) holds.

To prove (iii):

Consider

$$\begin{aligned} \min(W_1, \ldots, W_n) &= \min(X_1 - \underset{\sim}{U}, \ldots, X_n - \underset{\sim}{U}) \\ &= \min(X_1, \ldots, X_n) - \underset{\sim}{U} \\ &= U - U \\ &= 0 \text{ with probability 1.} \end{aligned}$$

Thus (iii) holds.

To prove (iv):

Putting in (1.9c) $\underset{\sim}{s} = x\underset{\sim}{e}$, we get

$$S(x\underset{\sim}{e} + t\underset{\sim}{e}) = S(x, x, \ldots, x)\, S(t, t, \ldots, t)$$

$$\Rightarrow \quad S((x + t)\underset{\sim}{e}) = S(x\underset{\sim}{e})\, S(t\underset{\sim}{e})$$

$$\Rightarrow \quad S_U(x + t) = S_U(s)\, S_U(t)$$

where $U = \min(X_1, \ldots, X_n)$. Thus U satisfies univariate LMP. Hence from characterization of univariate LMP, U has exponential distribution with parameter θ, for some $\theta > 0$.

To prove (ii):

For real vectors $\underline{a} = (a_1, a_2, \ldots, a_n)$ and $\underline{b} = (b_1, b_2, \ldots, b_n)$. let $\underline{a} \vee \underline{b} = (\max(a_1, b_1), \ldots, \max(a_n, b_n))$. For any $u > 0$ and any $\underline{w}$, let

$$\Omega_k = \bigcup_{j=0}^{\infty} \left\{ \underline{X} > \underline{w} + (u+j2^{-k})\underline{e},\ (u+j2^{-k}) < U \leq u+(j+1)2^{-k} \right\}.$$

First we prove that $\Omega_{k+1} \subset \Omega_k$.

Let us consider

$$\Omega_k = \bigcup_{j=0}^{\infty} \left\{ \underline{X} > \underline{w} + (u+j2^{-k})\underline{e},\ (u+j2^{-k}) < U \leq u+(j+1)2^{-k} \right\}$$

$$= [\underline{X}>\underline{w}+u\underline{e},\ u<U\leq u+2^{-k}] \cup [\underline{X}>\underline{w}+(u+2^{-k})\underline{e},\ (u+2^{-k})<U\leq u+2^{-k+1}]$$

$$\cup\ [\underline{X}>\underline{w}+(u+2^{-k+1})\underline{e},\ u+2^{-k+1}<U\leq u+3\times 2^{-k}] \cup \ldots\ldots\ldots$$

$$= A_{1k} \cup A_{2k} \cup A_{3k} \cup \ldots\ldots\ldots\ldots$$

where $A_{ik} = \left\{\underline{X} > \underline{w}+(u+(i-1)2^{-k})\underline{e},\ (u+(i-1)2^{-k})<U\leq u+(i)2^{-k} \right\}$ (1.43)

$i = 1, 2, \ldots\ldots$

and

$$\Omega_{k+1} = \bigcup_{j=0}^{\infty} \left\{ \underline{X} > \underline{w}+(u+j2^{-k-1})\underline{e},\ (u+j2^{-k-1}) < U \leq u+(j+1)2^{-k-1} \right\}$$

$$= [\underline{X}>\underline{w}+u\underline{e},\ u<U\leq u+2^{-k-1}] \cup [\underline{X}>\underline{w}+(u+2^{-k-1})\underline{e},\ (u+2^{-k-1})<U\leq u+2^{-k}]$$

$$\cup\ [\underline{X}>\underline{w}+(u+2^{-k})\underline{e},\ u+2^{-k}<U\leq u+3\ 2^{-k-1}] \cup \ldots\ldots$$

$$= B_{1k} \cup B_{2k} \cup B_{3k} \cup \ \ldots\ldots\ldots\ldots$$

where $B_{jk}=\left\{\underline{X} > \underline{w}+(u+\frac{(j-1)}{2^{k+1}})\underline{e},\ (u+\frac{(j-1)}{2^{k+1}}) < U \leq u+\frac{j}{2^{k+1}}\right\}$. (1.44)

$j = 1,2,\ldots\ldots$

From (1.43) and (1.44), we observe that

$$B_{jk} \cup B_{(j+1)k} \subset A_{((j+1)/2)k} \quad \forall\ j = 1,2,\ldots\ldots$$

$$\Rightarrow \bigcup_j B_{jk} \subset \bigcup_j A_{jk} \quad \forall \quad j \in I^+$$

$$\Rightarrow \Omega_{k+1} \subset \Omega_k \ \forall\ k \in I^+$$

$$\Rightarrow \Omega_k \downarrow k$$

Therefore $\lim\limits_{k\to\infty} \Omega k = \bigcap\limits_{k=1}^{\infty} \Omega k.$

$$\Rightarrow \lim_{k\to\infty} P(\Omega k) = P(\lim_{k\to\infty} \Omega k)$$

$$= P(\bigcap_{k=1}^{\infty} \Omega k) \qquad (1.45)$$

Now consider

$$\Omega_k = \bigcup_{j=0}^{\infty} \left\{ \underline{X} > \underline{w}+(u+j2^{-k})\underline{e},\ (u+j2^{-k}) < U \leq u+(j+1)2^{-k} \right\}$$

$$= \bigcup_{j=0}^{\infty} \left\{ \underline{X} > \underline{w}+(u+j2^{-k})\underline{e} \right\} \cap \bigcup_{j=0}^{\infty} \left\{ u+j2^{-k} < U \leq u+(j+1)2^{-k} \right\}$$

$$= \left\{ \underline{X} > \underline{w} + u\underline{e} \right\} \cap \left\{ U > u \right\}$$

$$\Rightarrow \bigcap_{k=1}^{\infty} \Omega_k = \bigcap_{k=1}^{\infty} \left\{ \{ \underline{X} > \underline{w} + u\underline{e} \} \cap \{ U > u \} \right\}$$

$$= \left\{ \underline{X} > \underline{w} + u\underline{e},\ U > u \right\}$$

$$= \left\{ \underline{W} \geq \underline{w},\ U > u \right\}$$

$$\rightarrow \lim_{k\to\infty} P(\Omega_k) = P\{ \underline{W} \geq \underline{w},\ U > u \} \tag{1.46}$$

Consider

$$P(\Omega_k) = P\left\{ \bigcup_{j=0}^{\infty} \left(\underline{X} > \underline{w} + (u+j2^{-k})\underline{e},\ (u+j2^{-k}) < U \leq u + (j+1)2^{-k} \right)\right\}$$

$$= \sum_{j=0}^{\infty} P\left(\underline{X} > \underline{w} + (u+j2^{-k})\underline{e},\ (u+j2^{-k}) < U \leq u+(j+1)2^{-k} \right)$$

(Since the events are disjoint)

$$= \sum_{j=0}^{\infty} \left[P\left(\underline{X}>\underline{w} +(u+j2^{-k})\underline{e},\ U>u+j2^{-k}\right) - P\left(\underline{X}>\underline{w} +(u+j2^{-k})\underline{e},\ U>u+(j+1)2^{-k}\right)\right]$$

where $U = \min(X_1, X_2, \ldots X_n)$.

$$= \sum_{j=0}^{\infty} \left[P\left(\underline{X}>\underline{w} +(u+j2^{-k})\underline{e} \vee (u+j2^{-k})\underline{e}\right)-P\left(\underline{X}>\underline{w} +(u+j2^{-k})\underline{e} \vee (u+(j+1)2^{-k})\underline{e}\right)\right]$$

$$= \sum_{j=o}^{\infty} \left[S\left([\underline{w} + (u+j2^{-k})\underline{e}] \vee (u+j2^{-k})\underline{e} \right] - S\left([\underline{w} + (u+j2^{-k})\underline{e}] \vee (u+(j+1)2^{-k})\underline{e} \right] \right]$$

$$= \sum_{j=o}^{\infty} \left[S[(\underline{w} \vee \underline{0} + (u+j2^{-k})\underline{e})] - S[(\underline{w} \vee 2^{-k}\underline{e} + (u+j2^{-k})\underline{e}] \right]$$

$$= \sum_{j=o}^{\infty} \left[S((u+j2^{-k})\underline{e})\ S(\underline{w} \vee \underline{0}) - S((u+j2^{-k})\underline{e})\ S(\underline{w} \vee 2^{-k}) \right] \quad \text{(from (1.9c))}$$

$$= \sum_{j=o}^{\infty} \left[S_U(u+j2^{-k})\ S(\underline{w} \vee \underline{0}) - S_U(u+j2^{-k})\ S(\underline{w} \vee 2^{-k}) \right]$$

(Since $S((u+j2^{-k})\underline{e}) = S_U(u+j2^{-k})$ where $U = \min(X_1, \ldots, X_n)$ and from (iv), U is exponential with parameter θ)

$$P(\Omega_k) = \sum_{j=o}^{\infty} \left[e^{-\theta(u+j2^{-k})} S(\underline{w} \vee \underline{0}) - e^{-\theta(u+j2^{-k})}\ S(\underline{w} \vee 2^{-k}\underline{e}) \right]$$

$$= e^{-\theta u} \left[S(\underline{w} \vee \underline{0}) - S(\underline{w} \vee 2^{-k}\underline{e}) \right] \sum_{j=o}^{\infty} e^{-\theta(u+j2^{-k})}$$

$$= e^{-\theta u} \left[S(\underline{w} \vee \underline{0}) - S(\underline{w} \vee 2^{-k}\underline{e}) \right] \left[1 + e^{-\theta 2^{-k}} + e^{-2\theta 2^{-k}} + e^{-3\theta 2^{-k}} + \ldots\ldots \right]$$

$$= e^{-\theta u} \left[S(\underline{w} \vee \underline{0}) - S(\underline{w} \vee 2^{-k}\underline{e}) \right] \left[1 - e^{-\theta 2^{-k}} \right]^{-1}$$

$$\lim_{k \to \infty} P(\Omega k) = e^{-\theta u} \lim_{k \to \infty} \left\{ \left[S(\underline{w} \vee \underline{0}) - S(\underline{w} \vee 2^{-k}\underline{e}) \right] \left[1 - e^{-\theta 2^{-k}} \right]^{-1} \right\}$$

$$= P(U>u) \lim_{k \to \infty} \left\{ \left[S(\underline{w} \vee \underline{0}) - S(\underline{w} \vee 2^{-k}\underline{e}) \right] \left[1 - e^{-\theta 2^{-k}} \right]^{-1} \right\}$$

(Since $U \sim \exp(\theta)$.)

Therefore from (1.46), we have

$$P(\underline{W} \geq \underline{w},\ U > u) = P(U>u) \lim_{k \to \infty} \left\{ \left(S(\underline{w} \vee \underline{0}) - S(w \vee 2^{-k}\underline{e}) \right) \left(1 - e^{-\theta 2^{-k}} \right)^{-1} \right\}$$

$$= P(U > u)\ P(\underline{W} \geq \underline{w} \mid U > u)$$

$$= P(U > u)\ P(\underline{W} > \underline{w}) \qquad (1.47)$$

Also if $U<0$, then $P(U > u) = 1$, so that $P(\underline{W} \geq \underline{w},\ U > u) = P(\underline{W} \geq \underline{w})$

$$\Rightarrow \quad P(U > u)\ P(\underline{W} \geq \underline{w}) = P(U > u)\ P(\underline{W} > \underline{w}) \qquad (1.48)$$

Thus from (1.47)and (1.48) it follows that U and $\underline{W}$ are independent.

Hence (ii) is proved.

Thus MLMP implies (i),(ii),(iii) and (iv).

Only IF Part: Suppose conditions (i), (ii), (iii) and (iv) hold.

where $\underline{X} = \underline{w} + U\underline{e}$ and $U = \min(X_1, X_2, \ldots, X_n)$.

Then $\quad S(\underline{x}) = P(\underline{X} > \underline{x})$

$$= P(\underline{W} + U\underline{e} > \underline{x})$$

$$= \int_0^\infty P(\underline{W} > \underline{x} - u\underline{e} \mid U=u)\ f(u)\ du$$

$$= \int_0^\infty P(\underline{W} > \underline{x} - u\underline{e})\ \theta\ e^{-\theta u} du \qquad \text{(by (iv) and (ii))}$$

$$= \int_0^\infty S_{\underline{W}}(\underline{x} - u\underline{e})\ \theta\ e^{-\theta u} du \qquad (1.49a)$$

Note from condition (iii) that $F_{\underline{W}}$ puts the entire mass on the axes and is zero elsewhere.

$$S(\underline{x}) = \int_{\min(x_i)}^{\infty} S_{\underline{W}}(\underline{x} - u\underline{e})\ \theta\ e^{-\theta u} du \tag{1.49b}$$

Therefore, replacing $\underline{x}$ by $\underline{x} + t\underline{e}$ in (1.49a), we get

$$S(\underline{x} + t\underline{e}) = \int_0^{\infty} S_{\underline{W}}(\underline{x} + t\underline{e} - u\underline{e})\ \theta\ e^{-\theta u} du$$

$$S(\underline{x} + t\underline{e}) = \int_0^{\infty} S_{\underline{W}}(\underline{x} - (u-t)\underline{e})\ \theta\ e^{-\theta u} du$$

$$= \int_0^{t+\min(x_i)} S_{\underline{W}}(\underline{x} - (u-t)\underline{e})\ \theta\ e^{-\theta u} du + \int_{t+\min(x_i)}^{\infty} S_{\underline{W}}(\underline{x} - (u-t)\underline{e})\ \theta\ e^{-\theta u} du$$

Note from condition (iii) that $F_{\underline{W}}$ puts the entire mass on the axes and is zero elsewhere.

Since $(\underline{X} - (u-t)\underline{e}) > \underline{0}$ for $u \in (0,\ t + \min(x_i))$, the first integral vanishes in the above equation and we have,

$$S(\underline{x} + t\underline{e}) = \int_{t+\min(x_i)}^{\infty} S_{\underline{W}}(\underline{x} - (u-t)\underline{e})\ \theta\ e^{-\theta u} du$$

Putting $u-t = z$ we get $du = dz$ and range of z is $\min(x_i)$ to ∞. This gives,

$$S(\underline{x} + t\underline{e}) = \int_{\min(x_i)}^{\infty} S_{\underline{W}}(\underline{x} - z\underline{e})\ \theta\ e^{-\theta(t+z)} dz$$

$$= e^{-\theta t} \int_{\min(x_i)}^{\infty} S_{\underline{W}}(\underline{x} - z\underline{e})\, \theta\, e^{-\theta z}\, dz$$

$$= S(t\underline{e})\, S(\underline{x}) \qquad \text{(from (1.49b))}$$

Thus $\underline{X}$ has MLMP property ■

The second characterization of MLMP given by Ghurye and Marshall (1984). It follows from the following corollary to the Theorem 1.4.

Corollary 1.1. A random vector $\underline{X}$ has satisfies MLMP iff,

$$S(\underline{x}) = \int_0^{\infty} S_{\underline{W}}(\underline{x} - u\underline{e})\, \theta \exp(-\theta u)\, du, \qquad 0 < \theta < \infty,\ x \geq 0$$

where $\underline{X} = U\underline{e} + \underline{W}$, here $U = \min(X_1, X_2, \ldots, X_n)$.

Proof: In the proof of Theorem 1.4 we see that,

MLMP $\rightarrow$ Conditions (i) and (iv) $\rightarrow$ (1.49a) $\rightarrow$ MLMP, which shows that (1.49a) is equivalent to MLMP.

4.2. CHARACTERIZATION DUE TO KULKARNI (1998):

Kulkarni (1998) gave following characterization of MLMP which is an extension of the result given in section 2.3 for bivariate case. A detail proof of multivariate extension is provided because it is slightly different from the bivariate case.

Theorem 1.5: A necessary and sufficient condition for a random vector $\underline{S}$ having vector failure rate $\underline{r} = (R_1, R_2, R_3, \ldots. R_K)$ to have MLMP is that $\sum_{i=1}^{k} R_i(t) = c$ where c is constant.

Proof: Sufficiency: Suppose

$$\sum_{i=1}^{k} R_i = c \tag{1.50}$$

Here $R_i = \frac{\partial R}{\partial s_i}$, so that (1.50) is a Langrange's linear partial differential equation in k variables (cf. Miller (1960), pp. 95) having subsidiary equation,

$$\frac{ds_1}{1} = \frac{ds_2}{1} = \frac{ds_3}{1} = \ldots\ldots \frac{ds_k}{1} = \frac{dR}{c}, \tag{1.51}$$

which has k independent integrals $u_j(s_1,s_2,\ldots,s_k) = a_j$, $j = 1,2,\ldots,k$ where $u_j(s_1,s_2,\ldots,s_k) = s_1-s_j$, $j = 2,3,\ldots k$ and $u_1 = (cs_1 - R(\underline{s}))$. Therefore its solution is given by (cf. Miller (1960) pp. 95) $\phi(u_1,u_2,\ldots,u_k) = 0$, where ϕ is any function of $u_1,u_2,\ldots,u_k$. Assuming that the equation $\phi(u_1,u_2,\ldots,u_k) = 0$ can be solved for u_1 in terms of $u_2,u_3,\ldots,u_k$, we get

$u_1 = cs_1 - R(\underline{s}) = g(u_2,\ldots,u_k)$. This gives,

$$R(\underline{S}) = cs_1 - g(s_1-s_2,s_1-s_3,\ldots,s_1-s_k). \tag{1.52}$$

By putting $s_1 = 0$ in (1.52) and noting that $R(0,s_2,s_3,\ldots,s_k)$ is the cumulative hazard function of the marginal distribution of $X_2,X_3,\ldots,X_k$ we get,

$- g(-s_2,-s_3,\ldots,-s_k)$

$$= R_{X_2,X_3,\ldots,X_k}(s_2,s_3,\ldots,s_k),\ s_i \geq 0,\ i = 2,\ldots,k, \qquad (1.53)$$

where $R_{X_2,X_3,\ldots,X_k}$ is the marginal cumulative hazard function of $X_2,X_3,\ldots,X_k$. Therefore from (1.52) we get,

$$R(\underline{s}) = cs_1 + R_{X_2,X_3,\ldots,X_k}((s_2-s_1),\ldots,(s_k-s_1)),\ s_i \geq s_1,\ i = 2,\ldots,3.$$

Similarly, if $s_{i_o} = \min(s_1,s_2,\ldots,s_k)$, then by taking

$u_j^* = s_{i_o} - s_j$, $j = 1,2,3,\ldots,k$; $j \neq i_o$ and $u_{i_o}^* = cs_{i_o} - R(\underline{s})$, it can be seen that $u_i^* = a_j$, $j = 1,2,\ldots k$; are also integrals of the subsidiary equation (1.51). Then by similar argument, we get

$$R(\underline{s}) = cs_{i_o} + R_o((s_1-s_{i_o}),\ldots,(s_{i_o-1}-s_{i_o}),(s_{i_o+1}-s_{i_o}),\ldots(s_k-s_{i_o})),$$

$$s_i \geq s_{i_o},\ i = 1,2,\ldots k. \qquad (1.54)$$

where R_o denotes the cumulative hazard function of the marginal of distribution of $(X_1,\ldots,X_{i_o-1}, X_{i_o+1},\ldots,X_k)$. Putting $s_i = t$, for $i = 1,2,3,\ldots,k$, in equation (1.54), we get $R(t,t,\ldots,t) = ct$, so that again from (1.54) we have,

$$R(s_1+t,s_2+t,\ldots,s_k+t) = c(s_{i_o}+t)+R_o((s_1-s_{i_o}),\ldots,(s_{i_o-1}-s_{i_o}),(s_{i_o+1}-s_{i_o})$$

$$,\ldots\ldots,(s_k - s_{i_o}),$$

$$= cs_{i_0} + ct + R_0((s_1 - s_{i_0}),\ldots,(s_{i_0-1} - s_{i_0}),(s_{i_0+1} - s_{i_0}),\ldots,(s_k - s_{i_0}))$$

$$= R(\underline{s}) + R(t,t,\ldots,t)$$

But $S(\underline{t}) = e^{-R(\underline{t})}$. Therefore we get,

$$S(s_1+t, s_2+t, \ldots, s_k+t) = S(\underline{s})\, S(t,t,\ldots,t), \quad \forall\ \underline{s}.t \geq 0.$$

Thus $\underline{S}$ has MLMP.

Necessity : Conversely, suppose $\underline{S}$ has MLMP. Then,

$$S(s_1+t, s_2+t, \ldots, s_k+t) = S(s_1,\ldots,s_k)\, S(t\,\underline{1}) \quad \forall\ \underline{s} \geq 0,\ t > 0. \quad (1.55)$$

Putting $s_i = s$, $i = 1,2,3,\ldots,k$ and noting that $S(s,s,\ldots,s)$ is same as survival function of $\min(S_1,\ldots,S_k)$. and by characterization of univariate LMP, we must have $S(s,s,\ldots,s) = e^{-cs}$ for some $c > 0$. Therefore, putting $s = 0$, in equation (1.55) we get,

$$S(t, s_2+t, \ldots, s_k+t) = S(0, s_2, \ldots, s_k)\, S(t,t,t,\ldots t)$$

$$= e^{-ct}\, S_{X_2,X_3,\ldots,X_k}(s_2,\ldots s_k), \quad \underline{s} \geq 0,\ t \geq 0.$$

or equivalently,

$$S(s_1,\ldots,s_k) = e^{-cs_1}\, S_{X_2,X_3,\ldots,X_k}(s_2 - s_1, \ldots s_k - s_1), \quad s_i \geq s_1, \quad i = 1,2,\ldots,k.$$

Taking logarithm on both sides and multiplying by -1 on both sides, we get

$$-\log(S(s_1,\ldots,s_k)) = cs_1 - \log(S_{X_2,X_3,\ldots,X_k}(s_2 - s_1, \ldots, s_k - s_1))$$

$$R(s_1,\ldots,s_k) = cs_1 + R_{X_2,X_3,\ldots,X_k}(s_2-s_1,\ldots,s_k-u_1),\ s_i \geq s_1,$$

$$i = 2,\ldots,k.$$

By similar arguments if $s_i = \min(s_1,s_2,\ldots,s_k)$ we have

$$R(s_1,\ldots,s_k) = cs_i + R_0(s_1-s_i,\ldots,s_{i-1}-s_i,s_{i+1}-s_i,\ldots,s_k-s_i), \quad (1.56)$$

$$s_j \geq s_i,\ j= 1,\ldots,i-1,i+1,\ldots k$$

where R_0 denotes the cumulative hazard function of the marginal distribution of $(X_1,\ldots,X_{i_0-1},X_{i_0+1},\ldots,X_k)$.

Differentiating both sides of (1.56) partially with respect to s_i, $i = 1,2,\ldots,k$ we get,

$$R_j = \partial R_0(s_1-s_i,\ldots,s_{i-1}-s_i,s_{i+1}-s_i,\ldots,s_k-s_i)/\partial w_j,\ j\neq i,\ j = 1,2,\ldots k.$$

and $R_i = c - \sum_{j\neq i}^{k} \partial R_0(s_1-s_i,\ldots,s_{i-1}-s_i,s_{i+1}-s_i,\ldots,s_k-s_i)/\partial w_j$, where $w_j=s_j-s_i$, $j = 1,2,\ldots,k$. So that $R_1 + R_2 + \quad +R_k = c$. (cf. CALCULUS by Apostol (1969) Theorem 8.8 pp. 264)

Thus $\underline{S}$ has MLMP. ∎

5. SOME SUPPLEMENTARY RESULTS.

5.1. THE DISTRIBUTION HAVING BLMP AND EXPONENTIAL MARGINALS:

Marshall and Olkin (1967) studied bivariate distributions having BLMP along with exponential marginals. They observed that the only distribution having BLMP and having exponential marginals is the bivariate distribution with survival function,

$$S(x,y) = \exp(-\lambda_1 x - \lambda_2 y - \lambda_{12}\max(x,y)),\ x,y \geq 0,\ \lambda_1,\lambda_2,\lambda_{12}>0 \quad (1.57)$$

This is presented in the following theorem. Henceforth we refer to (1.57) as survival function of the bivariate exponential distribution (BVE).

Theorem 1.6: The BVE is the only bivariate distribution with exponential marginals satisfying (1.10).

Proof: Suppose (1.10) holds. Since we demand exponential marginals, let $S_1(x) = e^{-\delta_1 x}$ and $S_2(x) = e^{-\delta_2 x}$, $\delta_1 > 0$, $\delta_2 > 0$ be the marginal survival functions.

Then from (1.10) we have,

$$S(x,y) = \begin{cases} e^{-\theta y-\delta_1(x-y)} & \text{for } x \geq y, \\ e^{-\theta x-\delta_2(y-x)} & \text{for } y \geq x, \end{cases} \qquad (1.58)$$

for some $\theta > 0$.

Since $S(x,y)$ is decreasing in y, from (1.58), we must have $\theta > \delta_1$, so that $\lambda_2 = \theta - \delta_1 > 0$. Similarly, since $S(x,y)$ is decreasing in x, we must have $\theta > \delta_2$, hence $\lambda_1 = \theta - \delta_2 > 0$. Let $\lambda_{12} = \delta_1 + \delta_2 - \theta$. To insure that $\lambda_{12} > 0$, we must show that $\delta_1 + \delta_2 > \theta$. To show this, consider the univariate distribution corresponding to $Z = \min(X,Y)$. That is, $G_Z(x) = F(x,x)$

$$G_Z(x) = 1 - e^{-\delta_1 x} - e^{-\delta_2 x} + e^{-\theta x},\ x \geq 0.$$

Differentiating G w.r.t. x, we get, the density of $Z = \min(X,Y)$,

$$f_Z(x) = \delta_1 e^{-\delta_1 x} + \delta_2 e^{-\delta_2 x} - \theta e^{-\theta x}.$$ Since $f_Z(x) \geq 0 \ \forall\ x \geq 0$,

letting x tend to zero, we get, $\lambda_{12} = \delta_1 + \delta_2 - \theta > 0$.

From the choice $\lambda_1 = \theta - \delta_2$, $\lambda_2 = \theta - \delta_1$ and $\lambda_{12} = \delta_1 + \delta_2 - \theta$ we get, $\theta = \lambda_1 + \lambda_2 + \lambda_{12}$, $\delta_1 = \lambda_1 + \lambda_{12}$ and $\delta_2 = \lambda_2 + \lambda_{12}$.

Substituting these values in (1.58), we get,

$$S(x,y) = \begin{cases} e^{-(\lambda_1+\lambda_2+\lambda_{12})y - (\lambda_1+\lambda_{12})x + (\lambda_1+\lambda_{12})y} & \text{for } x \geq y \\ e^{-(\lambda_1+\lambda_2+\lambda_{12})x - (\lambda_2+\lambda_{12})y + (\lambda_2+\lambda_{12})x} & \text{for } y \geq x \end{cases}$$

$$= \begin{cases} e^{-\lambda_1 x - \lambda_2 y - \lambda_{12} x} & \text{for } x \geq y \\ e^{-\lambda_1 x - \lambda_2 y - \lambda_{12} y} & \text{for } y \geq x \end{cases}$$

$$= e^{-\lambda_1 x - \lambda_2 y - \lambda_{12}\max(x,y)}, \text{ for } x,y \geq 0,$$

which is the survival function of BVE of Marshall and Olkin (1967). Thus the theorem is proved. ■

5.2. CONDITIONS ON MARGINALS OF A BIVARIATE DISTRIBUTION HAVING BLMP:

In this subsection, we show that the function given in equation (1.10) need not be a bivariate survival function for any arbitrary choice of the marginal survival functions S_1 and S_2. This is shown below in Example 5.2.1, taking S_1 and S_2 to be survival functions of the univariate Weibull distribution. Further, conditions on the marginal survival functions S_1 and S_2 are given under which the function S in (1.10) is a bivariate survival function.

5.2.1. **AN EXAMPLE.**

Take the survival functions S_1 and S_2 as

$S_1(x) = e^{-x^{p_1}}$ if $x \geq 0$, $p_1 > 0$ and

$S_2(y) = e^{-x^{p_2}}$ if $y \geq 0$, $p_2 > 0$.

Note that S_1 and S_2 are survival functions of Weibull distributions. Substituting these in equation (1.10) we get,

$$S(x,y) = \begin{cases} e^{-\theta y} \; e^{-(x-y)^{p_1}} & x \geq y \geq 0, \\ e^{-\theta x} \; e^{-(y-x)^{p_2}} & y \geq x \geq 0, \end{cases} \tag{1.59}$$

We examine below whether S in (1.59) is a bivariate survival function. Note that one of the necessary conditions for S to be a survival functionis $S(x,y) \downarrow$ in $x,y \geq 0$.

Further $S(x,y) \downarrow x$ $\forall\ y > 0$ and $S(x,y) \downarrow y$ $\forall\ x > 0$, if $\frac{\partial S(x,y)}{\partial x} < 0$ $\forall\ x,y > 0$, and $\frac{\partial S(x,y)}{\partial y} < 0$ $\forall\ x,y > 0$, respectively.

Now, Differentiating equation (1.59) w.r.t. x we get,

$$\frac{\partial S(X,Y)}{\partial x} = \begin{cases} -p_1 \; e^{-\theta y} \; e^{-(x-y)^{p_1}} \; (x-y)^{p_1-1} & x \geq y \geq 0 \quad (1.60a) \\ e^{-\theta x} \; e^{-(y-x)^{p_2}} \left[p_2 (y-x)^{p_2-1} - \theta\right] & y \geq x \geq 0 \quad (1.60b) \end{cases}$$

Check! — [illegible] depend!

and Differentiating equation (1.59) w.r.t y, we get

$$\frac{\partial S(X,Y)}{\partial y} = \begin{cases} e^{-\theta y}\, e^{-(x-y)^{p_1}}\left[p_1 (x-y)^{p_1-1} - \theta\right] & x \ge y \ge 0 \quad (1.61a) \\ -p_2\, e^{-\theta x}\, e^{-(y-x)^{p_2}} (y-x)^{p_2-1} & y \ge x \ge 0 \quad (1.61b) \end{cases}$$

Note that the r.h.s. of (1.60b) and (1.61a) is positive for some y and x respectively which implies that S is not decreasing in x for every $y \ge 0$ and y for every $x \ge 0$. Therefore condition (a) is not satisfied, so that S is not a survival function. This shows that for any arbitrary choice of S_1 and S_2 in (1.10) need not yield a bivariate survival function S. Marshall and Olkin (1967) have obtained conditions on the marginal densities f_1 and f_2 (equivalently on the survival functions S_1 and S_2) under which the function S(x,y) in (1.10) is a bivariate survival function. We present these in the next Theorem.

Theorem 1.7: Let $F_j(x)$ be distribution functions with absolutely continuous densities $f_j(x)$ for which $\lim_{z\to\infty} f_j(z) = 0$, j=1,2. In order that S(x,y) given in (1.10) be a bivariate survival function, it is necessary and sufficient that,

$$\text{i) } \theta \le f_1(0) + f_2(0) \le 2\theta. \qquad (1.62)$$

$$\text{ii) } \frac{d\log(f_j(z))}{dz} \ge -\theta, \text{ for all } z \ge 0,\ j=1,2. \qquad (1.63)$$

Proof: Let us define a bivariate function

$$S(x,y) = \begin{cases} e^{-\theta y}\; S_1(x-y), & \text{for } x \ge y \ge 0 \\ e^{-\theta x}\; S_2(y-x), & \text{for } y \ge x \ge 0 \end{cases} \tag{1.64}$$

where $S_j = 1 - F_j$, $j = 1,2$.

It is required to obtain conditions on S_1 and S_2 under which S given in (1.64) is a survival function.

Suppose S is a survival function. By Lebesque decomposition theorem, every distribution function has a unique decomposition given by

$$\overline{F}(x,y) = \alpha\, \overline{F}_a(x,y) + (1-\alpha)\, \overline{F}_s(x,y), \qquad 0 \le \alpha \le 1.$$

where F_a is absolutely continuous distribution function and F_s is singular distribution function. Equivalently,

$$S(x,y) = \alpha\, S_\alpha(x,y) + (1-\alpha)\, S_s(x,y), \qquad 0 \le \alpha \le 1. \tag{1.65}$$

where S, S_α and S_s are corresponding survival functions.

First we determine $S_a(x,y)$ and $S_s(x,y)$ for the function S given in (1.64).

In order to find the absolutely continuous part of S, we equate the mixed derivatives of S obtained from (1.64) and (1.65). This gives,

$$\frac{\partial^2 S(x,y)}{\partial x\, \partial y} = \alpha\, f_a(x,y)$$

$$= \begin{cases} e^{-\theta y} \; [f_1'(x-y) + \theta f_1(x-y)], & \text{for } x \geq y \geq 0 \\ e^{-\theta x} \; [f_2'(y-x) + \theta f_2(y-x)], & \text{for } y \geq x \geq 0 \end{cases} \tag{1.66}$$

where $f_a(x,y)$ is absolutely continuous probability density function of x and y.

Consider

$$\int_{x\geq y} \alpha \; f_a(x,y)dx\; dy = \int_0^{\infty}\int_0^{x} e^{-\theta y}\left[f_1'(x-y)+\theta \; f_1(x-y)\right]dy\; dx$$

$$= \int_0^{\infty}\left\{\left[-e^{-\theta y}f_1(x-y)\right]_0^x - \theta \int_0^x e^{-\theta y}f_1(x-y)dy + \theta\left[-e^{-\theta y}F_1(x-y)\right]_0^x \right.$$

$$\left. + \; \theta^2 \int_0^x e^{-\theta y} \; F_1(x-y) \; dy \right\} dx$$

$$= \int_0^{\infty}\left\{f_1(x)-e^{-\theta x}f_1(0)-\theta \; \left[-e^{-\theta y}F_1(x-y)\right]_0^x - \theta^2 \int_0^x e^{-\theta y}F_1(x-y)\; dy \right.$$

$$\left. + \; \theta \; F_1(x) - \theta \; e^{-\theta x} \; F_1(0) + \theta^2 \int_0^x e^{-\theta y} \; F_1(x-y) \; dy \right\} dx$$

$$= \int_0^{\infty}\left\{f_1(x) - e^{-\theta x}f_1(0) - \theta F_1(x) + \theta e^{-\theta x}F_1(0) + \theta F_1(x) \right\} dx$$

$$= \int_0^{\infty} f_1(x) \; dx - f_1(0)\int_0^{\infty} e^{-\theta x}dx$$

$$= 1 - \frac{f_1(0)}{\theta} \tag{1.67}$$

Similarly, it can be shown that

$$\int_{x \le y} \alpha\, f_a(x,y)\, dx dy = \int_0^\infty \int_x^\infty e^{-\theta x}[f_2'(y-x) + \theta f_2(y-x)]dx\ dy$$

$$= 1 - \frac{f_2(0)}{\theta} \tag{1.68}$$

From (1.67) and (1.68), we get,

$$\int_0^\infty \int_0^\infty \alpha\, f_a(x,y)\, dx\ dy = 1 - \frac{f_1(0)}{\theta} + 1 - \frac{f_2(0)}{\theta}$$

$$= 2 - [f_1(0) + f_2(0)]/\theta$$

Since $f_\alpha(x,y)$ is density, we have

$$\int_0^\infty \int_0^\infty f_a(x,y)\ dx\ dy = 1$$

$$\therefore \alpha = 2 - [f_1(0) + f_2(0)]/\theta \tag{1.69}$$

From (1.65) and (1.69), it follows that the absolute continuous part $S_\alpha(x,y)$ of $S(x,y)$ has density $f_\alpha(x,y)$ given by

$$f_\alpha(x,y) = \begin{cases} \frac{1}{\alpha} & e^{-\theta y}[f_1'(x-y) + \theta f_1(x-y)] \quad \text{for } x \ge y \ge 0 \\ \frac{1}{\alpha} & e^{-\theta x}[f_2'(y-x) + \theta f_2(y-x)] \quad \text{for } y \ge x \ge 0 \end{cases} \tag{1.70}$$

where $\alpha = 2 - [f_1(0)+f_2(0)]/\theta$

Next, from (1.70), we have

$$S_\alpha(x,x) = \int_x^\infty \int_x^\infty f_a(u,v)\ du\ dv$$

$$= \frac{1}{\alpha} \int_x^\infty \int_u^\infty e^{-\theta u} \left[f_2'(v-u) + \theta\ f_2(v-u)\right] dv\ du$$

$$+ \frac{1}{\alpha} \int_x^\infty \int_v^\infty e^{-\theta v} \left[f_1'(u-v) + \theta\ f_1(u-v)\right] du\ dv$$

$$= \frac{1}{\alpha}\int_x^\infty e^{-\theta u}\Big[f_2(v-u)+\theta F_2(v-u)\Big]_u^\infty du + \frac{1}{\alpha}\int_x^\infty e^{-\theta v}\Big[f_1(u-v)+\theta F_1(u-v)\Big]_v^\infty dv$$

$$= \frac{1}{\alpha}\left\{\int_x^\infty e^{-\theta u}\Big[\theta-f_2(0)\Big]du + \int_x^\infty e^{-\theta v}\Big[\theta-f_1(0)\Big]dv\right\}$$

$$= \frac{1}{\alpha\,\theta}\, e^{-\theta x}\left\{2\theta - [f_1(0) + f_2(0)]\right\}$$

$$= \frac{1}{\alpha\,\theta}\, e^{-\theta x}\,\theta\,\alpha \qquad \text{(from (1.69))}$$

$$= e^{-\theta x} \qquad (1.71)$$

But, from (1.64) we have $S(x,x) = e^{-\theta x}$, $\quad x \ge 0 \qquad (1.72)$

and from (1.65), we have

$$S(x,x) = \alpha\, S_a(x,x) + (1-\alpha)\, S_s(x,x)$$

$$\Rightarrow S_s(x,x) = \frac{S(x,x) - \alpha\, S\ (x,x)}{(1-\alpha)}$$

$$= \frac{e^{-\theta x} - \alpha\, e^{-\theta x}}{(1-\alpha)}$$

$$= e^{-\theta x}, \qquad x \ge 0$$

From (1.64), it is clear that the singular part of S is concentrated on the line x = y.

Therefore, we have,

$$S_s(x,y) = \begin{cases} S_s(x,x) & \text{if } x > y, \\ S_s(y,y) & \text{if } y > x, \end{cases}$$

$$= \begin{cases} e^{-\theta x} & \text{if } x > y, \\ e^{-\theta y} & \text{if } y > x, \end{cases}$$

Hence,

$$S_s(x,y) = e^{-\theta \max(x,y)}, \quad x,y \geq 0$$

Further we note that S given in (1.65) is survival function only if $0 \leq \alpha \leq 1$ and S_a and S_s are both survival functions. Now $0 \leq \alpha \leq 1$ together with (1.69) gives,

$$\theta \leq f_1(0) + f_2(0) \leq 2\theta \tag{1.73}$$

Further S_α is a survival function if and only if f_α is density function. Therefore, we must have,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_\alpha(x,y)\, dx\, dy = 1 \tag{1.74}$$

and $$f_\alpha(x,y) \geq 0 \quad \forall\, x,y. \tag{1.75}$$

Here (1.74) follows from (1.69) and from (1.70), it follows that, (1.75) holds if,

$$f'_j(z) + \theta\, f_j(z) \geq 0, \; j = 1,2 \;\forall\, z \geq 0.$$

$$\Rightarrow \frac{f'_j(z)}{f_j(z)} \geq -\theta \;\Rightarrow\; \frac{\partial \log(f_j)}{\partial z} \geq -\theta \tag{1.76}$$

From (1.73) and (1.76) the theorem follows. ∎

As Corollaries of Theorem 1.7 Block and Basu (1974) proved the following results.

Corollary 1.2: Let $F_1(x)$ and $F_2(y)$ have absolutely continuous densities f_1 and f_2. Then $S(x,y)$ given by (1.10) is an absolutely continuous bivariate survival function if and only if,

i) $f_1(0) + f_2(0) = \theta$ for some $\theta > 0$.

$$\text{ii)} \quad \frac{d\log(f_j(z))}{dz} \geq -\theta, \text{ for all } z \geq 0,\ j = 1,2. \tag{1.77}$$

Proof : From Theorem 1.7, it follows that S in (1.10) is a survival function if and only if

i) $\theta \leq f_1(0) + f_2(0) \leq 2\theta$.

$$\text{ii)} \quad \frac{d\log(f_j(z))}{dz} \geq -\theta, \quad \text{for all } z \geq 0,\ j=1,2.$$

Further from (1.65) it is clear that S is an absolutely continuous survival function iff $\alpha = 1$. Therefore, Putting $\alpha = 1$ in (1.69) gives,

$$2 - [f_1(0) + f_2(0)] / \theta = 1$$

$$\Rightarrow \quad f_1(0) + f_2(0) = \theta \tag{1.78}$$

Thus from (1.76) and (1.78) the Corollary follows

Corollary 1.3: Suppose (f_1, f_2) and (g_1, g_2) are marginal densities satisfying conditions (1.73) and (1.75).

Define,

$$h_1 = \gamma f_1 + (1-\gamma)\, g_1, \quad 0 \leq \gamma \leq 1$$

$$h_2 = \gamma f_2 + (1-\gamma)\, g_2$$

Then h_1 and h_2 also satisfy the conditions (1.73) and (1.75), and

hence the function $S = \begin{cases} e^{-\theta y}\,\bar{H}_1(x-y) & \text{if } x \ge y \ge 0 \\ e^{-\theta x}\,\bar{H}_2(y-x) & \text{if } y \ge x \ge 0 \end{cases}$

is also a bivarivate survival function and has BLMP.

Proof: Suppose that (f_1, f_2) and (g_1, g_2) are marginal densities satisfying the conditions,

i) $\theta \le f_1(0) + f_2(0) \le 2\theta$ and $\theta \le g_1(0) + g_2(0) \le 2\theta$.

ii) $\dfrac{d\log(f_i(z))}{dz} \ge -\theta$ and $\dfrac{d\log(g_i(z))}{d\,z} \ge -\theta$, $i = 1,2$

Now define

$$h_1(z) = \gamma f_1(z) + (1-\gamma)g_1(z)$$

$$h_2(z) = \gamma f_2(z) + (1-\gamma)g_2(z)$$

putting $z = 0$ we get,

$$h_1(0) = \gamma f_1(0) + (1-\gamma)g_1(0)$$

$$h_2(0) = \gamma f_2(0) + (1-\gamma)g_2(0)$$

Now,

$$h_1(0) + h_2(0) = \gamma[f_1(0) + f_2(0)] + (1-\gamma)[g_1(0) + g_2(0)]$$

but $f_1(0) + f_2(0) \ge \theta$ and $g_1(0) + g_2(0) \ge \theta$

$$\Rightarrow \quad h_1(0) + h_2(0) \ge \gamma\theta + (1-\gamma)\theta$$

$$\Rightarrow \quad h_1(0) + h_2(0) \ge \theta \qquad (1.79)$$

also $f_1(0) + f_2(0) \le 2\theta$ and $g_1(0) + g_2(0) \le 2\theta$

$$\Rightarrow \quad h_1(0) + h_2(0) \leq \gamma\, 2\theta + (1-\gamma)\, 2\theta$$

$$\Rightarrow \quad h_1(0) + h_2(0) \leq 2\theta \qquad (1.80)$$

From (1.79) and (1.80) it follows that

$$\theta \leq h_1(0) + h_2(0) \leq 2\theta$$

Also, $h_i = \gamma f_i + (1-\gamma) g_i, \qquad i = 1,2$

$$\text{If} \quad \frac{d\log f_i(z)}{dz} \geq -\theta$$

$$\text{i.e.} \quad \frac{1}{f_i}\frac{df_i}{dz} \geq -\theta$$

$$\Rightarrow \quad \frac{df_i}{dz} \geq -\theta f_i, \quad i = 1,2$$

Similarly

$$\frac{dg_i}{dz} \geq -\theta g_i, \quad i = 1,2$$

Now

$$\frac{d\log(h_i)}{dz} = \frac{1}{h_i}\,\frac{dh_i}{dz}$$

$$= \frac{1}{h_i}\left[\gamma \frac{df_i}{dz} + (1-\gamma)\frac{dg_i}{dz}\right]$$

but $\frac{df_i}{dz} \geq -\theta f_i$, and $\frac{dg_i}{dz} \geq -\theta g_i$, $i = 1,2$

$$\frac{d\log(h_i)}{dz} \geq \frac{1}{h_i}\left[\gamma(-\theta f_i) + (1-\gamma)(-\theta g_i)\right]$$

$$\geq -\frac{1}{h_i}\,\theta\,[\gamma f_i + (1-\gamma)g_i]$$

$$\geq -\theta\,\frac{1}{h_i}\,h_i$$

$$\geq -\theta \qquad\qquad i = 1,2.$$

Hence h_1 and h_2 also satisfy the conditions (1.73) and (1.75).

Therefore from Theorem 1.7, it is clear that the function,

$$S(x,y) = \begin{cases} e^{-\theta y}\,\bar{H}_1 & \text{for } x \geq y \geq 0 \\ e^{-\theta x}\,\bar{H}_2 & \text{for } y \geq x \geq 0 \end{cases}$$

where $\bar{H}_1 = \gamma\bar{F}_1 + (1-\gamma)\bar{G}_1$ and $\bar{H}_2 = \gamma\bar{F}_2 + (1-\gamma)\bar{G}_2$ is also bivariate survival function and has BLMP. ∎

5.3. To check the conditions of the Theorem 1.4 for the Weibull and Gamma distributions.

Here we show that Weibull and Gamma distributions can not be marginal distribution of a bivarite distribution having BLMP.

5.3.1. WEIBULL DISTRIBUTION:

The p.d.f. of weibull distribution is

$$f(z) = f(z,\beta,\delta_1) = \beta\delta_1 z^{\beta-1}\exp(-\delta_1 z^{\beta}),\ z \geq 0, \beta > 0, \delta_1 > 0. \quad (1.81)$$

Taking logarithm on both sides, we get

$$\log(f(z)) = \log(\beta\delta_1) + (\beta-1)\log(z) - \delta_1 z^{\beta}$$

Differentiating with respect to z, we get

$$\frac{d\log(f(z))}{dz} = (\beta-1)/z - \delta_1 \beta z^{\beta-1}$$

$$\therefore \lim_{z\to\infty} \frac{d\log(f(z))}{dz} = \lim_{z\to\infty} (\beta-1)/z - \delta_1\beta \lim_{z\to\infty} z^{\beta-1}$$

$$= -\infty, \qquad \text{if } \beta > 1$$

and

$$\therefore \lim_{z\to 0} \frac{d\log(f(z))}{dz} = \lim_{z\to 0} (\beta-1)/z - \delta_1\beta \lim_{z\to 0} z^{\beta-1}$$

$$= -\infty, \qquad \text{if } \beta < 1$$

Hence condition (ii) of Theorem 1.7 is not satisfied. Therefore the function given by (1.81) cannot be a survival function when either of the marginals is Weibull. The two conditions of Theorem 1.7 are satisfied when $\beta = 1$, which corresponds to the exponential distribution.

5.3.2. THE GAMMA DISTRIBUTION.

The p.d.f. of gamma distribution is

$$g(z) = g(z,\beta,\delta z) = \frac{1}{\Gamma\beta} \delta_2^{\beta} z^{\beta-1} \exp(-\delta_2 z), \; z \geq 0, \beta > 0, \delta > 0. \quad (1.82)$$

Taking logarithm on both sides, we get

$$\log(g(z)) = \log(\Gamma\beta) + \beta\log(\delta_2) + (\beta-1)\log(z) - \delta_2 z$$

Differentiating with respect to z, we get

$$\frac{d\log(g(z))}{dz} = (\beta-1)/z - \delta_2$$

$$\lim_{z\to\infty} \frac{d\log(g(z))}{dz} = -\delta_2, \qquad \text{if } \beta > 1$$

and

$$\lim_{z \to 0} \frac{d\log(g(z))}{dz} = (\beta-1) \lim_{z \to 0} (1/z) - \lim_{z \to \infty} \delta_2$$

$$= -\infty \qquad \text{if } \beta < 1$$

Hence condition (ii) of Theorem 1.7 is not satisfied.

Therefore the function given by (1.82) cannot be a survival function when either of the marginals is gamma distribution. The two conditions of Theorem 1.7 are satisfied when $\beta = 1$, which corresponds to the exponential distribution.

REMARK 1.2: A result similar to the one given in Theorem 1.7 for bivariate case is obtained for multivariate case by Ghurey and Marshall (1984) we quote the result below.

RESULT 1.1: Suppose that (1.9b) holds for some $\theta > 0$. Then S is survival function if and only if

i) S_1 and S_2 are both degenerate at zero.

or

ii) S_1 and S_2 are both absolute continuous with right hand derivatives

$$g(u) = \lim_{\delta \to \infty} \frac{S_1(u) - S_1(u+\delta)}{\delta}, \quad h(u) = \lim_{\delta \to \infty} \frac{S_2(u) - S_2(u+\delta)}{\delta}$$

which are right continuous, are of bounded variation and have at most a countable number of discontinuities; further

a) $\exp(-\theta u)\, g(u)$ is non-decreasing in $u \geq 0$.

b) $\exp(-\theta u)\, h(u)$ is non-decreasing in $u \geq 0$.

c) $S_1(u) + S_2(u) \geq (1 - \exp(-\theta u))$ for all $u \geq 0$.

The proof of the above result is very lengthy and is omitted.

5.4. SOME DISTRIBUTIONS HAVING BLMP:

5.4.1. Let $S_X(x)$ and $S_Y(y)$ be the marginal survival function of X and Y given by,

$$S_X(x) = \frac{\lambda}{\lambda_1+\lambda_2} \exp(-(\lambda_1+\lambda_{12})x) - \frac{\lambda_{12}}{\lambda_1+\lambda_2} \exp(-\lambda x), \qquad \text{for } x > 0$$

and

$$S_Y(y) = \frac{\lambda}{\lambda_1+\lambda_2} \exp(-(\lambda_2+\lambda_{12})y) - \frac{\lambda_{12}}{\lambda_1+\lambda_2} \exp(-\lambda y), \qquad \text{for } y > 0$$

$\lambda_1, \lambda_2, \lambda_{12} \geq 0$ and $\lambda = \lambda_1+\lambda_2+\lambda_{12}$.

we verify the conditions of Theorem 1.7 for S_X and S_Y.

First we find the p.d.f. of x and y.

$$f_1(x) = -S'_X(x)$$

$$= \frac{\lambda(\lambda_1+\lambda_{12})}{\lambda_1+\lambda_2} \exp(-(\lambda_1+\lambda_{12})x) - \frac{\lambda_{12}}{\lambda_1+\lambda_2} \lambda \exp(-\lambda x), \quad x \geq 0$$

Similarly,

$$f_2(y) = -S'_Y(y)$$

$$= \frac{\lambda(\lambda_2+\lambda_{12})}{\lambda_1+\lambda_2} \exp(-(\lambda_2+\lambda_{12})y) - \frac{\lambda_{12}}{\lambda_1+\lambda_2} \lambda \exp(-\lambda y), \quad y \geq 0$$

Now

$$f_1(0) = \frac{\lambda(\lambda_1+\lambda_{12})}{\lambda_1+\lambda_2} - \frac{\lambda_{12}}{\lambda_1+\lambda_2}\lambda$$

$$= \frac{\lambda\lambda_1}{(\lambda_1+\lambda_2)}$$

and $$f_2(0) = \frac{\lambda(\lambda_2+\lambda_{12})}{\lambda_1+\lambda_2} - \frac{\lambda_{12}}{\lambda_1+\lambda_2}\lambda$$

$$= \frac{\lambda\lambda_2}{(\lambda_1+\lambda_2)}$$

$$\therefore\ f_1(0) + f_2(0) = \lambda$$

Therefore S_X and S_Y satisfy condition (i) of Theorem 1.7.

Also, $$\frac{d\log(f_1(x))}{dx} \geq -\lambda$$

$$\Rightarrow f_1'(x) + \lambda f_1(x) \geq 0$$

$$\therefore\ f_1'(x) = \frac{\lambda(\lambda_1+\lambda_{12})^2}{\lambda_1+\lambda_2}\exp(-(\lambda_1+\lambda_{12})x) + \frac{\lambda_{12}}{\lambda_1+\lambda_2}\lambda^2\exp(-\lambda x), \quad x \geq 0$$

Now

$$f_1'(x) + \lambda f_1(x) = \frac{\lambda(\lambda_1+\lambda_{12})^2}{\lambda_1+\lambda_2}\exp(-(\lambda_1+\lambda_{12})x) + \frac{\lambda_{12}}{\lambda_1+\lambda_2}\lambda^2\exp(-\lambda x)$$

$$+ \frac{\lambda^2(\lambda_1+\lambda_{12})}{\lambda_1+\lambda_2}\exp(-(\lambda_1+\lambda_{12})x) - \frac{\lambda_{12}}{\lambda_1+\lambda_2}\lambda^2\exp(-\lambda x)$$

$$=[-(\lambda_1+\lambda_{12}) + \lambda]\,\frac{\lambda(\lambda_1+\lambda_{12})}{\lambda_1+\lambda_2}\,\exp(-(\lambda_1+\lambda_{12})x)$$

$$f'_1(x) + \lambda f_1(x) \geq 0 \;\Rightarrow -(\lambda_1+\lambda_{12})+\lambda \geq 0$$

$$\Rightarrow\; -\lambda_1 - \lambda_{12} + \lambda_1 + \lambda_2 + \lambda_{12} \geq 0$$

$\therefore\ \lambda_2 > 0 \qquad \forall\ x \geq 0$, which obviously holds.

Hence condition (ii) of Theorem 1.7 holds.

Therefore the function S given by (1.10) i.e.

$$S(x,y) = \begin{cases} e^{-\lambda y}\, S_X(x-y) & \text{if } x \geq y, \\ e^{-\lambda x}\, S_Y(y-x) & \text{if } y \geq x. \end{cases}$$

has BLMP. ■

Note that S given above is the survival function of the distribution proposed by Block and Basu (1974).

5.4.2. Let (X,Y) have bivariate exponential distribution with parameter $(\lambda_1,\lambda_2,\lambda_{12})$. i.e. $(X,Y) \sim BVE(\lambda_1,\lambda_2,\lambda_{12})$.

The joint bivariate survival function is

$$S(x,y) = \exp(-\lambda_1 x - \lambda_2 y - \lambda_{12}\max(x,y)) \quad \forall\ x,y \geq 0.$$

Therefore

$$S(x+t,y+t) = \exp(-\lambda_1 x - \lambda_1 t - \lambda_2 y - \lambda_2 t - \lambda_{12}\max(x+t,y+t))$$

$$= \exp(-\lambda_1 x - \lambda_1 t - \lambda_2 y - \lambda_2 t - \lambda_{12}(\max(x,y)+t))$$

$$= \exp(-[\lambda_1 + \lambda_1 + \lambda_{12}]t)\,\exp(-\lambda_1 x - \lambda_2 y - \lambda_{12}\max(x,y))$$

$$= \exp(-\lambda t)\ \exp(-\lambda_1 x - \lambda_2 y - \lambda_{12} \max(x,y))$$

where $\lambda = \lambda_1 + \lambda_1 + \lambda_{12}$.

$$S(x+t, y+t) = S(t,t)\ S(x,y) \quad \forall\ x,y,t \geq 0$$

Thus the equation (1.9a) holds.

Hence the BVE has BLMP.

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