CHAPTER - II

SETTING THE CLOCK BACK TO ZERO PROPERTY OF A CLASS OF CONTINUOUS DISTRIBUTIONS

1. INTRODUCTION :

In Chapter I we have discussed the univariate lack of memory property, its extension to bivariate and multivariate case and some characterizations of bivarivate lack of memory property and multivariate lack of memory property. Some distributions possessing BLMP including those with non-exponential marginals were presented. It was shown that the only distribution having BLMP with exponential marginals is BVE given by Marshal and Olkin (1967).

In this chapter we discuss an another property of life time distributions, namely the 'setting the clock back to zero' (SCBZ) property, and its characterizations for univariate case.

Rao and Talwalkar (1989) have introduced the SCBZ property. This property can be viewed as generalization of the lack of memory property (LMP) of the exponential distribution, in the sense that for the exponential distribution, the conditional distribution of additional survival time of an unit of age t_0 say, is exactly the same as the original distribution, while for SCBZ property it belongs to the same family, but may be with a changed value of the parameter.

In section 2, we present the 'Setting the Clock Back to Zero' (SCBZ) property as a generalization of Lack of Memory property (LMP). In section 3, we present some distributions possessing the SCBZ property. It is observed that Weibull family dose not possesses the SCBZ property. In section 4, we describe some consequences of having SCBZ property and its application to some statistical problems, also we discuss the closure of SCBZ property in competing risk set-up. In section 5, we present some characterizations of SCBZ property in terms of failure rate function and mean residual life function.

In the next section we present the formal definition of SCBZ property.

2 THE SCBZ PROPERTY AS A GENERALIZATION OF LMP:

2.1 DEFINITION :

A family of lifetime distributions $\{ S(x,\beta),\ x \geq 0\ \beta \in \Theta \}$ is said to have the 'setting the clock back to zero' property, if for each $\beta \in \Theta$ and $x, x_o > 0$ the survival function satisfies the equation:

$$\frac{S(x + x_o, \beta)}{S(x_o, \beta)} = S(x, \beta^*), \text{ with } \beta^* = \beta^*(x_o) \in \Theta \qquad (2.1)$$

Here β may be single parameter or a vector of parameters and Θ is the appropriate parameter space. Also β^*, the new parameter or a vector parameter is a function of x_o.

An exponential distribution is the unique distribution possessing LMP while it can be shown that there are several distributions which possess SCBZ property.

Note that equation (2.1) can be written as

X?

$$P(X \geq x + x_o \mid X \geq x_o) = P(X^* \geq x) \quad \forall \quad x, x_o \geq 0,$$

where the random variable X^* has the same distribution as that of X, except that parameter β is replaced by $\beta^*(x_o) \in \Theta$.

This means that the conditional distribution of additional length of survival of an organism given that it has survived x_o time units remains in the same family except for a change in the parameter value.

Putting $\beta^*(x_o) = \beta \; \forall \; x_o$ in (2.1) we get the functional equation,

$$S(x + x_o, \beta) = S(x_o, \beta) \, S(x, \beta) \qquad \forall \; x, x_o \geq 0.$$

which is the usual lack of memory property (LMP) of exponential distribution. Thus lack of memory property is a special case of 'setting the clock back to zero' property, so that SCBZ property can be viewed as a generalization of LMP.

3. SOME DISTRIBUTIONS POSSESSING THE SCBZ PROPERTY.

In this section we present some distributions possessing the SCBZ property.

3.1. EXPONENTIAL DISTRIBUTION.

As noted in previous subsection, SCBZ property generalizes the lack of memory property of the exponential distribution. Therefore exponential distribution trivially possesses SCBZ property.

3.2. LINEAR HAZARD EXPONENTIAL DISTRIBUTION.

The failure rate of the linear hazard exponential distribution is

$$r(x,\underline{\beta}) = a + bx, \quad a,b > 0, \qquad \forall\ x \geq 0.$$

Here $\underline{\beta} = (a,b)$ is the parameter and $\Theta = (0,\infty)^2$. The survival function is given by,

$$S(x,\beta) = \exp(-(ax + \frac{b}{2}x^2)), \qquad \forall\ x \geq 0,\ a,b > 0.$$

Therefore,

$$\frac{S(x + x_o,\ \beta)}{S(x_o,\beta)} = \frac{\exp(-ax-ax_o - \frac{b}{2}x^2 - bxx_o - \frac{b}{2}x_o^2)}{\exp(-ax_o - \frac{b}{2}x_o^2)}$$

$$= \exp[-[(a + bx_o)\ x + \frac{b}{2}x^2]]$$

$$= \exp[-[a^*x + \frac{b}{2}x^2]]$$

$$= S(x,\beta^*) \tag{2.2}$$

Here $\beta^* = (a^*,b^*)$, $a^* = a + bx_o$ and $b^* = b$. Note that $\beta^*(x_o) \in \Theta$, $\Theta = (0,\infty)^2$ for every $x_o \geq 0$. Hence linear hazard exponential distribution satisfies the SCBZ property.

3.3. GOMPERTZ DISTRIBUTION.

Gompertz distribution has failure rate

$r(x,\underline{\beta}) = k\, e^{\alpha x}$, $k > 0$, $x \geq 0$, $-\infty < \alpha < \infty$, where $\underline{\beta} = (k,\alpha)$ and $\Theta = (0,\infty)\times(-\infty,\infty)$.

The survival function of Gompertz distribution is

$$S(x,\beta) = \exp\left(- \int_0^x r(t,\beta)\, dt \right)$$

$$= \exp\left(- k \int_0^x e^{\alpha t} dt \right)$$

$$= \exp\left(- \frac{k}{\alpha} (e^{\alpha x} - 1)\right)$$

Therefore,

$$\frac{S(x + x_o, \beta)}{S(x_o,\beta)} = \frac{\exp\left(- \frac{k}{\alpha}(e^{\alpha(x + x_o)} - 1)\right)}{\exp\left(- \frac{k}{\alpha} (e^{\alpha x_o} - 1)\right)}$$

$$= \exp\left(- \frac{k^*}{\alpha^*} (e^{\alpha^* x} - 1)\right)$$

$$= S(x,\beta^*) \tag{2.3}$$

where $\beta^* = (k^*, \alpha^*)$, $k^* = k\, e^{\alpha x_o}$ and $\alpha^* = \alpha$. Note that, Since $k\, e^{\alpha x_o} \geq 0$, $k^* \in \Theta \ \forall \ x_o > 0$. Thus Gompertz distribution possesses the SCBZ property.

3.4 KRANE FAMILY:

Rao (1990) showed that the General Krane family of distributions possesses the SCBZ property.

The failure rate of General Krane family of distributions is,

$$r(x, \underset{-}{\beta}) = \beta_1 + 2\beta_2 x + 3\beta_3 x^2 + \dots\dots\dots + m\beta_m x^{m-1}, \quad x \geq 0.$$

where $\underset{-}{\beta} = (\beta_1, \beta_2, \dots\dots \beta_m) \in \Theta = (0,\infty)^m$.

The corresponding survival function is

$$S(x,\beta) = \exp\left(- \int_0^x r(t,\beta)\, dt\right)$$

$$= \exp(-(\beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \dots + \beta_m x^m))$$

Note that for any x and $x_o \geq 0$,

$$S(x+xo,\beta) = \exp\left[-\left\{\beta_1 (x+x_o) + \beta_2 (x+x_o)^2 + \beta_3 (x+x_o)^3 + \dots + \beta_m (x+x_o)^m\right\}\right]$$

$$= \exp\left[-\beta_1 x_o - \beta_2 x_o^2 - \beta_3 x_o^3 - \dots - \beta_m x_o^m\right]$$

$$\times \exp\left[-\beta'_1 x - \beta'_2 x^2 - \beta'_3 x^3 - \dots - \beta'_m x^m\right]$$

$$= S(x_o,\beta)\, \exp(-\beta'_1 x - \beta'_2 x^2 - \beta'_3 x^3 - \dots - \beta'_m x^m)$$

where $\beta'_1 = \beta_1 + 2\beta_2 x_o + 3\beta_3 x_o^2 + \ldots + m\beta_m x_o^{m-1}$

$$\beta'_2 = \beta_2 + 3\beta_3 x_o + \ldots\ldots + \frac{m}{2}(m-1)\beta_{m-1} x_o^{m-1}$$

$$\beta'_3 = \beta_3 + 4\beta_4 x_o + \ldots\ldots + \frac{m(m-1)(m-2)}{3!} x_o^{m-3}$$

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$$\beta'_{m-1} = \beta_{m-1} + m\,\beta_m\, x_o$$

Therefore,

$$\frac{S(x+x_o,\beta)}{S(x_o,\beta)} = \frac{S(x_o,\beta)\ \exp(-\beta_1' x - \beta_2' x^2 - \beta_3' x^3 + \ldots .- \beta_m' x^m)}{S(x_o,\beta)}$$

$$= \exp(-(\beta_1' x + \beta_2' x^2 + \beta'_3 x^3 + \ldots\ldots\ldots + \beta_m' x^m\ ))$$

$$= S(x,\beta^*) \tag{2.4}$$

where $\underline{\beta}^*(x_o) = (\beta_1', \beta_2', \ldots\ldots \beta_m') \in \Theta\ \forall\ \ x_o \geq 0$. Hence the General Krane family of distributions possesses the SCBZ property.

We mention here the following points worth noting about the Krane family:

i) The choice $m = 1$, $\beta_i = 0$, $i \geq 2$ provides exponential distribution.

ii) The choice $m = 2$, $\beta_i = 0$, $i \geq 3$ provides the linear hazard exponential distribution.

iii) The choice $\beta_i = 0, i = 1,2,3,\ldots\ldots m-1$ and $\beta_m = 1$, corresponds to a subfamily of Weibull distributions where the parameter is an integer. However it is worth noting that the Weibull family as a whole dose not possess SCBZ property, as shown below. The Weibull family has the failure rate,

$$r(x,p) = p\, x^{p-1}, \; p > 0, \; x \geq 0. \qquad \text{Here, } \beta = p \text{ and } \Theta = (0,\infty).$$

The survival function of the Weibull distribution is

$$S(x,\beta) = \exp\left(-\int_0^x r(t,p)\, dt\right)$$

$$= \exp(-x^p)$$

Therefore,

$$\frac{S(x+x_o,\beta)}{S(x_o,\beta)} = \frac{\exp(-(x+x_o)^p)}{\exp(-x_o^{\,p})}$$

$$= \exp\left\{-\left[(x+x_o)^p - x_o^p\right]\right\}$$

$$\neq S(x,P^*) \qquad \text{for any } P^* > 0,$$

Since the term $\exp(-[(x+x_o)^p - x_o^{\,p}])$ can not be put in the form of $e^{-x^{p^*}}$, for some $P^* > 0$. Hence Weibull distribution does not possess SCBZ property.

3.5. THE FAMILY INTRODUCED BY RAO (1990).

Chiang and Conforti (1989) introduced a model in connection with time to tumor data, in which failure rate is given by,

$$h(x,\beta,\nu) = \frac{\beta}{\nu}(1 - \alpha e^{-\nu x}), \quad x \geq 0,\ \theta = (\alpha,\beta,\nu) \in \Theta,\ \Theta = (0,\infty)^3.$$

Rao (1990) introduced a more general form of this distribution, where the failure rate function is given by,

$$h(x,\beta,\nu) = \frac{\beta}{\nu}(1 - \alpha e^{-\nu x}),\ x \geq 0,\ \theta = (\alpha,\beta,\nu) \in \Theta. \qquad (2.5)$$

Here $\alpha = 1$ gives the Chiang and Conforti (1989) model. In (2.5), it is understood that the failure rate $h(x,\alpha,\beta,\nu) > 0$. Since $\alpha > 0$, $\beta > 0$ and $\nu > 0$, this is equivalent to $(1 - \alpha e^{-\nu x}) \geq 0$ or $x \geq x_1$, where $x_1 = \frac{1}{\nu}\log\alpha$. Thus $h(x,\alpha,\beta,\nu) > 0$ for $x \geq x_1$. Note that if $0 < \alpha < 1$ then $x_1 < 0$ hence failure rate $h(x,\alpha,\beta,\nu)$ is non-negative for $x \geq 0$. The vector parameter is $\underline{\theta} = \{\alpha, \beta, \nu\}$ and $\Theta = (0,\infty)^3$. the survival function corresponding to the general form given in (2.5) is,

$$\begin{aligned} S(x,\beta) &= \exp\left(-\int_0^x h(t,\beta)\, dt\right) \\ &= \exp\left(-\frac{\beta}{\nu}\int_0^x (1 - \alpha e^{-\nu t})\, dt\right) \\ &= \exp\left[-\frac{\beta}{\nu}\left\{x - \frac{\alpha}{\nu}(1 - e^{-\nu x})\right\}\right],\ \alpha,\beta,\nu > 0,\ x > x_1. \qquad (2.6)\end{aligned}$$

Therefore,

$$\frac{S(x+x_o,\beta)}{S(xo,\beta)} = \frac{\exp\left[-\frac{\beta}{\nu}\left[(x + x_o) - \frac{\alpha}{\nu}(1 - e^{-\nu(x + x_o)})\right]\right]}{\exp(-\frac{\beta}{\nu}(x_o - \frac{\alpha}{\nu}(1 - e^{-\nu x_o}))]}$$

$$= \exp\left[-\frac{\beta}{\nu}\left[x - \frac{\alpha}{\nu}\, e^{-\nu x_o}(1-e^{-\nu x_o})\right]\right]$$

$$= \exp\left[-\frac{\beta}{\nu}\left[x - \frac{\alpha'}{\nu}(1-e^{-\nu x})\right]\right]$$

$$= S(x,\beta^*) \qquad \forall\, x \geq x_1,\ x_o \geq 0, \tag{2.7}$$

where $\alpha' = \alpha\, e^{-\nu x_o} > 0\ \forall\, x_o \geq 0$ and $\beta^* = (\alpha', \beta, \nu) \in \Theta = (0,\infty)^3$. Thus the family of survival distributions described in (2.5) and hence that described by Chiang and Conforti has SCBZ property.

In the next section we describe some consequences of having SCBZ property and its application to some statistical problems.

4. SOME CONSEQUENCES OF SCBZ PROPERTY:

4.1. SIMPLICITY INDUCED IN THE FUNCTIONAL FORM OF MEAN RESIDUAL LIFE FUNCTION:

The life expectancy or mean residual life function (MRLF) of a living organism is denoted e_{x_o} and is defined as the expected remaining life of the organism given that it has survived x_o time units. We show here that the MRLF of a distribution having SCBZ property takes a particularly simpler form and equals the expected value of a random variable having probability distribution from the same family but with changed value of the parameter.

Consider,

$$e_{x_o} = E(\, X \mid X \geq x_o) - x_o$$

$$= \frac{1}{S(x_o,\beta)} \int_{x_o}^{\infty} x \ f(x,\beta) \ dx - x_o$$

$$= \int_0^{\infty} \frac{S(x+x_o,\beta) \ dx}{S(x_o,\beta)}$$

$$= \int_o^{\infty} S(x,\beta^*) \ dx \qquad \text{(From (2.1))}$$

$$= E_{\beta^*}(X), \quad \text{where } \beta^* = \beta^*(x_o) \in \Theta. \qquad (2.8)$$

In the next section we discuss the life expectancy of some distributions possessing SCBZ property.

4.2. LIFE EXPECTANCY OF SOME DISTRIBUTIONS POSSESSING SCBZ PROPERTY:

4.2.1. The distribution introduced by Rao (1990):

In the above section we have shown that the Rao's distribution has SCBZ property. Then from (2.7) and (2.8), the life expectancy for Rao's distribution is given by,

$$e_{x_o} = \int_o^{\infty} S(x,\theta^*) \ dx$$

$$= \int_o^{\infty} \exp[- \frac{\beta}{\nu}(x - \frac{\alpha'}{\nu} (1-e^{-\nu x}))] \ dx, \ (\text{here } \alpha' = \alpha \ e^{-\nu x_o}).$$

$$= \int_0^{\infty} \exp[-\frac{\beta}{\nu}x + \beta\frac{\alpha'}{\nu^2}(1-e^{-\nu x})]\,dx$$

$$= \int_0^{\infty} \exp[-\frac{\beta}{\nu}x]\exp(\alpha' c - \alpha' c e^{-\nu x})\,dx, \quad \text{where } c = \beta/\nu^2$$

$$= \exp(\alpha' c)\int_0^{\infty} \exp[-\frac{\beta}{\nu}x]\exp(-\alpha' c\, e^{-\nu x})\,dx \qquad (2.9)$$

Putting $y = \alpha' c\, e^{-\nu x}$ we get $e^{-\nu x} = y/(\alpha' c)$.

Raising the power on both sides by, $c = \beta/\nu^2$ we get,

$$e^{-\frac{\beta}{\nu}x} = (y/(\alpha' c))^{\beta/\nu^2}$$

Differentiating both sides w.r.t x we have,

$$-(\beta/\nu)\, e^{-\frac{\beta}{\nu}x} = (\beta/\nu^2)\,\frac{y^{(\beta/\nu^2)-1}}{(\alpha' c)^c}\,dy.$$

Further, at $x = 0$, $y = \alpha' c$ and at $x = \infty$, $y = 0$.

Therefore equation (2.9) becomes,

$$e_{x_o} = -\frac{e(\alpha' c)}{\nu(\alpha' c)^c}\int_{\alpha' c}^{0} y^{c-1}\, e^{-y}\,dy$$

$$= \frac{e(\alpha' c)}{\nu(\alpha' c)^c}\int_0^{\alpha' c} y^{c-1}\, e^{-y}\,dy$$

$$= \frac{\exp(\alpha' c)}{\nu}(\alpha' c)^{-c}\,\Gamma_{\alpha' c, c}$$

where $\overline{|a,b}$ is incomplete gamma function defined by,

$$\overline{|a,b} = \int_0^a y^{b-1} e^{-y} dy.$$

This is the expectation of the remaining life of the organism given that it has survived x_o time units. If we put $x_o = 0$ the above mean residual lifetime function (MRLF) reduces to the ordinary expectation of life.

4.2.2. Life expectancy of linear hazard exponential distribution.

In subsection 3.2 it is shown that the linear hazard exponential distribution possesses the SCBZ property. Therefore from (2.8) we have,

$$e_{x_o} = \int_o^{\infty} S(x,\beta^*) \, dx$$

$$= \int_o^{\infty} \exp[-[a^* x + \frac{b}{2} x^2]] \, dx \qquad \text{(from(2.2))}$$

$$= \int_o^{\infty} \exp\, [-(b/2)[[x + \frac{a^*}{b}]^2 - a^{*2}/b^2]]dx$$

$$= \exp\,(a^{*2}/2b) \int_o^{\infty} \exp\,[-\alpha(x+\beta)^2]dx$$

where $\alpha = b/2$, $\beta = a^*/b$. Now, suppose $\alpha = 1/2\sigma^2$ and $\mu = -\beta$ i.e. $\sigma^2 = 1/2\alpha$ and $\mu = -\beta$. Then we get,

$$e_{x_o} = \exp\,(a^{*2}/2b) \int_o^{\infty} \exp\,[-1/2\sigma^2 \,(x-\mu)^2] \, dx$$

$$= \sigma \sqrt{2\Pi} \quad (1-\phi((0-\mu)/\sigma) \exp\,(a^{*2}/2b)$$

$$= \sqrt{2\Pi}\ \frac{1}{\sqrt{2\alpha}}\ (1 - \phi(\beta\sqrt{2\alpha}))\ \exp\ (a^{*2}/2b) \quad \text{(Substituting for } \mu \text{ and } \sigma)$$

$$= \sqrt{2\Pi/b}\quad (1 - \phi((a^{*}/b)\sqrt{2b/2}))\ \exp\ (a^{*2}/2b) \quad \text{(Substituting the values of } \alpha \text{ and } \beta\text{)}$$

$$= \exp\ (a^{*2}/2b)\ \sqrt{2\Pi/b}\quad (1 - \phi(a^{*}/\sqrt{b}\ ))$$

where from equation (2.2), we have $a^{*} = a + bx_o$.

4.3. LIFE EXPECTANCY OF GOMPERTZ DISTRIBUTION:

In subsection 3.3 it is shown that the Gompertz distribution possesses the SCBZ property. Therefore from (2.8) we have,

$$e_{x_o} = \int_o^{\infty} S(x,\beta^{*})\ dx$$

$$= \int_o^{\infty} \exp(-\ \frac{k^{*}}{\alpha^{*}}\ (e^{\alpha^{*}x} - 1))\ dx$$

$$= \exp(-(k^{*}/\alpha^{*}))\int_o^{\infty} e^{-(k^{*}/\alpha)e^{\alpha x}} dx \qquad (\text{Since } \alpha^{*} = \alpha\ ) \qquad (2.10)$$

Putting $(k^{*}/\alpha)e^{\alpha x} = y$ we get, $(k^{*}/\alpha)\alpha\ e^{\alpha x}dx = dy$

$$\Rightarrow dx = (1/\alpha)\ y^{-1}\ dy.$$

at $x = 0$, $y = k^{*}/\alpha$ and at $x = \infty$, $y = \infty$.

Therefore (2.10) becomes

$$e_{x_o} = \exp(-(k^*/\alpha^*)) \int_{k^*/\alpha}^{\infty} e^{-y} y^{-1} dy$$

The integral on the R.H.S. is convergent and its values are tabulated in Rao (1990).

4.4. CLOSURE UNDER COMPETING RISK SET-UP:

In this section we discuss the closure of SCBZ property in competing risk set-up. The result is proved in the following Theorem.

Theorem 2.1: Let X and Y denote the lifetimes of an organism under risk 1 and risk 2 respectively. Let Z = min(X,Y) denote the observed lifetime of the organism. If the survival distribution of X and Y have the SCBZ property, then the survival distribution of Z under the assumption of independent risk also has SCBZ property.

Proof : Let failure rate under the two risks be denoted by $r_1(x,\beta)$ and $r_2(x,\nu)$, so that the corresponding survival functions are

$$S_1(x,\beta) = \exp(- \int_0^x r_1(u,\beta)\, du)$$

$$= \exp(-H_2(x,\beta)\, du)$$

and

$$S_2(x,\nu) = \exp(- \int_0^x r_2(u,\nu)\, du)$$

$$= \exp(-H_2(x,\nu)\, du)$$

where $H_1(x,\beta)$ and $H_2(x,\nu)$ are the cumulative hazard functions for the two risks of deaths, where $\beta \in \Theta_1$ and $\nu \in \Theta_2$. Here β, ν are single parameter or vector parameters.

Suppose survival functions of these families have the SCBZ property.

$$\text{i.e.} \quad \frac{S_1(x+x_0,\beta)}{S_1(x_0,\beta)} = S_1(x,\beta^*) \tag{2.11}$$

$$\text{and} \quad \frac{S_2(x+x_0,\nu)}{S_2(x_0,\nu)} = S_2(x,\nu^*) \tag{2.12}$$

where $\beta^* = \beta^*(x_0) \in \Theta_1$ and $\nu^* = \nu^*(x_0) \in \Theta_2$.

The observed lifetime of the organism is $Z = \min(X,Y)$. Under the assumption of independent risk of death, the survival function of Z is

$$\begin{aligned}
S_Z(x,\theta) &= P(\ Z > x\ ) \\
&= P(\min(X,Y) > x) \\
&= P(\ X > x,\ Y > x) \\
&= P(\ X > x)\ P(\ Y > x)\ \text{(Since X and Y are independent)} \\
&= S_1(x,\beta)\ S_2(x,\nu)
\end{aligned}$$

Here $\theta = (\beta,\nu)$ is the vector of parameters for the observed life length Z. Now consider

$$\frac{S_Z(x+x_o,\theta)}{S_Z(x_o,\theta)} = \frac{S_1(x+x_o,\beta)}{S_1(x_o,\beta)}\ \frac{S_2(x+x_o,\nu)}{S_2(x_o,\nu)}$$

$$= S_1(x,\beta^*)\ S_2(x,\nu^*) \qquad \text{(from (2.11) and (2.12))}$$

$$= S_Z(x,\theta^*)$$

where $\theta^* = (\beta^*,\ \nu^*) \in \Theta$, ($\Theta = \Theta_1 \times \Theta_2$).

Thus the family of life distributions of the observed life length of an organism also possesses the SCBZ property. ■

Remark 2.1: Let $X_1, X_2, X_3, \ldots, X_k$ be independent random variables each possessing SCBZ property and $Z = \min(X_1, X_2, X_3, \ldots, X_k)$. Then using inductive argument it is easy to observe that the distribution of Z also has SCBZ property. We note in this case that the hazard function of Z is given by $r_Z(t) = \sum_{i=1}^{k} r_{X_i}(t)$. The closure result proved in Theorem 2.1 in fact is true for more general case provided in the following theorem.

Theorem 2.2: Suppose that the failure rate function of random variable Z has form

$$r_Z(t,\beta) = \sum_{i=1}^{k_1} r_{X_i}(t,\beta_i) - \sum_{i=k_1+1}^{k} r_{X_i}(t,\beta_i), \quad k_1 < k \tag{2.13}$$

where the probability distribution of the k random variables $X_1, \ldots, X_k$ have SCBZ property. Then the survival function of the distribution of Z also has SCBZ property.

Proof : The survival function of the distribution of Z is

$$S_Z(x,\beta) = \exp(-\int_0^t r_Z(U,\beta)\, du)$$

$$= \exp(-\int_0^t \sum_{i=1}^{k_1} r_{Xi}(u,\beta_i)\, du + \int_0^t \sum_{i=k_1+1}^{k} r_{Xi}(u,\beta_i)\, du$$

$$= \frac{\exp(-\sum_{i=1}^{k_1} \int_0^t r_i(u,\beta_i)\, du)}{\exp(-\sum_{i=k_1+1}^{k} \int_0^t r_i(u,\beta_i)\, du)}$$

$$= \frac{\prod_{i=1}^{k_1} \exp(-\int_0^t r_i(u,\beta_i)\, du)}{\prod_{i=k_1+1}^{k} \exp(-\int_0^t r_i(u,\beta_i)\, du)}$$

$$= \frac{\prod_{i=1}^{k_1} S_i(t,\beta_i)}{\prod_{i=k_1+1}^{k} S_i(t,\beta_i)}$$

Therefore

$$\Rightarrow \frac{S_Z(t+t_o,\beta)}{S_Z(t_o,\beta)} = \frac{\prod_{i=1}^{k_1} S_i(t+t_o,\beta_i)}{\prod_{i=k_1+1}^{k} S_i(t+t_o,\beta_i)} \; \frac{\prod_{i=k_1+1}^{k} S_i(t_o,\beta_i)}{\prod_{i=1}^{k} S_i(t_o,\beta_i)}$$

$$= \frac{\prod_{i=1}^{k_1} \frac{S_i(t+t_o,\beta_i)}{S_i(t_o,\beta_i)}}{\prod_{i=k_1+1}^{k} \frac{S_i(t+t_o,\beta_i)}{S_i(t_o,\beta_i)}}$$

$$= \frac{\prod_{i=k}^{k_1} S_i(t,\beta_i^*)}{\prod_{i=k_1+1}^{k} S_i(t,\beta_i^*)} \qquad \text{(Since each } X_i \text{ has SCBZ property)}$$

$$= S_Z(t,\beta^*)$$

Hence the distribution Z also possesses SCBZ property.

Remark 2.2: We note here that for any arbitrary choice of univariate failure rate functions r_i $i = 1,2,\ldots,k$, the R.H.S. of (2.13) need not correspond to failure rate function. For example if failure rate function of two random variables X_1 and X_2 are $r_{X_1} \equiv 1$ and $r_{X_2} \equiv 2$ with $k_1 = 1$ and $k_2 = 2$. Then $r_{X_1}(x) - r_{X_2}(x) \equiv -1 < 0 \ \forall\ x$ is not a failure rate function. The choice of r_i $i = 1,2,\ldots,k$ should be such that the in the R.H.S. of (2.13) $r_Z(t) \geq 0 \ \forall\ t$ and $\int_o^{\infty} r_Z(t)\, dt = \infty$.

In the next section we discuss some characterizations of SCBZ property.

5. CHARACTERIZATIONS OF SCBZ PROPERTY:

The SCBZ property (2.1) can also be characterized in terms of the failure rate function r and the mean residual life function m.

A characterization of SCBZ property in terms of the failure rate function is given in the following Theorem.

Theorem 2.3: A non-negative continuous random variable X has SCBZ property if and only if

$$r(x+t,\beta) = r(x,\beta^*), \qquad \text{with } \beta^* = \beta^*(x_o) \tag{2.14}$$

where $r(.,\beta)$ is the failure rate function of X.

Proof : Assume that X has SCBZ property. Now from equation (2.1), we get

$$\frac{S(x + x_o, \beta)}{S(x_o,\beta)} = S(x,\beta^*)$$

$$\Rightarrow \qquad S(x + x_o,\beta) = S(x_o,\beta)\, S(x,\beta^*) \qquad \forall\ x \geq 0$$

Taking logarithm sides and multiplying by -1 on both sides we get,

$$-\log(S(x + x_o,\beta)) = -\log(S(x_o,\beta)) - \log(S(x,\beta^*))$$

Differentiating both sides with respect to x, we get

$$\frac{f(x+x_o,\beta)}{S(x+x_o,\beta)} = \frac{f(x,\beta^*)}{S(x,\beta^*)}$$

$$\Rightarrow \qquad r(x+x_o,\beta) = r(x,\beta^*) \qquad \forall\ x \geq 0,\ x_o > 0$$

Thus X has SCBZ property.

Conversely, Assume that (2.14) holds.

i.e. $r(x+x_o,\beta) = r(x,\beta^*)$ $\forall\ x \geq 0,\ x_o > 0$

Integrating both sides over the range 0 to x we get,

$$\int_0^x r(u+x_o,\beta)du = \int_0^x r(u,\beta^*)du \tag{2.15}$$

put $u + x_o = t \Rightarrow du = dt$ and range of t is x_o to $x + x_o$

Therefore the equation (2.15) becomes,

$$\int_{x_o}^{x+x_o} r(t,\beta)dt = \int_{x_o}^{x+x_o} r(t-x_o,\beta^*)dt$$

$$\Rightarrow \quad H(x + x_o,\beta) - H(x_o,\beta) = H(x,\beta^*)$$

$$\Rightarrow \quad \frac{\exp(-H(x+x_o,\beta))}{\exp(-H(x_o,\beta))} = \exp(-H(x,\beta^*)$$

$$\frac{S(x+x_o,\beta)}{S(x_o,\beta)} = S(x,\beta^*) \qquad \forall\ x,x_o \geq 0.$$

Thus X has SCBZ property.

In next Theorem gives a characterization of SCBZ property in terms of the mean residual life function.

Theorem 2.4: A non-negative continuous random variable X has SCBZ property if and only if

$$m(x+x_o,\beta) = m(x,\beta^*) \qquad \forall\ x,x_o \geq 0. \tag{2.16}$$

where $m(.,\beta)$ is the mean residual life function.

Proof : Suppose X has SCBZ property. Let

$$m(x+x_o,\beta) = \int_0^\infty \frac{S(u+x+x_o,\beta)}{S(x+x_o,\beta)}\,du$$

$$= \int_0^\infty \frac{S(u+x,\beta)\;S(x_o,\beta)}{S(x,\beta)\;S(x_o,\beta)}\,du$$

$$= \int_0^\infty \frac{S(u+x,\beta)}{S(x,\beta)}\,du$$

$$= m(x,\beta^*) \qquad \forall\; x,x_o \geq 0.$$

Thus X has SCBZ property.

Conversely, assume that the relation (2.16) holds.

Let $$m(x+x_o,\beta) = m(x,\beta^*) \qquad \forall\; x,x_o \geq 0.$$

$$\Rightarrow \qquad \frac{d(m(x+x_o,\beta))}{d(x+x_o)\beta)}\;\frac{d(x+x_0)}{dx} = \frac{d(m(x,\beta^*))}{dx}$$

$$\Rightarrow \qquad \frac{1\;+m'(x+x_o,\beta))}{m(x+x_o,\beta)} = \frac{1\;+\;m'(x,\beta^*)}{m(x,\beta^*)}$$

By the Theorem 2.3 $\underline{X}$ has SCBZ property.

Hence Theorem 2.4 is proved.

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