CHAPTER - IV

SETTING THE CLOCK BACK TO ZERO PROPERTY OF A CLASS OF BIVARIARE DISTRIBUTIONS

1. INTRODUCTION:

In Chapter III we have discussed the setting the clock back to zero property for discrete distributions. characterizations of the SCBZ property in terms of the failure rate function r and the mean residual life function m were discussed.

In this Chapter we discuss two extensions of SCBZ property to bivariate case. One of these is proposed by Rao and Damaraju (1993) and the other one is our contribution. Each of these extensions is appropriate in a different reliability situation. Their physical interpretations are discussed and examples of bivariate reliability models satisfying these definition are presented.

In section 2, we define SCBZ property given by Rao and Damaraju (1993). We also propose an another variant of bivariate SCBZ property and discuss their physical interpretations. In section 3, we show that the mean residual life function and percentile life function takes a simpler form for distributions possessing SCBZ property. In section 4, we present some bivariate distributions satisfying these definitions.

In the next section we present the two definitions and their practical implications.

2. THE BIVARIATE SCBZ PROPERTY:

The following definition of SCBZ property is given by Rao and Damaraju (1993).

Definition 2.1.

A class of bivariate life distributions { $S(x,y,\beta)$, $x \geq 0$, $y \geq 0$, $\beta \in \Theta$ } is said to have the 'setting the clock back to zero' property if, for each $\beta \in \Theta$ and $x_o \geq 0$, the survival function satisfies the pair of equations,

$$\frac{S(x+x_o, x_o, \beta)}{S(x_o, x_o, \beta)} = S(x, x_o, \beta^*) \quad , \forall \quad x,y \geq 0 \tag{4.1}$$

$$\frac{S(x_o, y+x_o, \beta)}{S(x_o, x_o, \beta)} = S(x_o, y, \beta^{**}), \quad \forall \quad x,y \geq 0 \tag{4.2}$$

where $\beta^* = \beta^*(x_o) \in \bar{\Theta}$ and $\beta^{**}(x_o) \in \bar{\Theta}$, where $\bar{\Theta}$ denotes the closure of the parameter space Θ. In this definition β^* and β^{**} are allowed to take values on the boundary of the parameter space $\bar{\Theta}$ also, because this widens the class of distributions possessing SCBZ property. In section 4 we discuss examples of some distributions where β^* and β^{**} belong to the boundary of Θ.

Note that equations (4.1) and (4.2) can be written as

$$P(X > x+x_o \mid X > x_o, Y > x_o) = P(X^* > x, Y^* > x_o) \quad \forall x, x_o \geq 0.$$

and

$$P(Y > y+x_o \mid X > x_o, Y > x_o) = P(X^{**} > x_o, Y^{**} > y) \quad \forall y, x_o \geq 0.$$

where (X^*, Y^*) and (X^{**}, Y^{**}) have same distribution as that of (X,Y) except that the parameter $\underline{\beta}$ is changed to β^* and β^{**} respectively.

Interpreting X and Y as lifetimes of an organisms under risks 1 and 2 respectively, this means that the conditional distribution of the additional survival time of an individual due to risk 1 (2) (assuming that the risk 2 (1) has not killed the individual first) given that the individual has survived both the risks x_o time units remains in the same family, except for a change in the value of the parameters.

We propose an another variant of bivariate SCBZ property which is given below.

Definition 2.2.

A class of bivariate life distributions $\{S(x,y,\beta), x \geq 0, y \geq 0, \beta \in \Theta\}$ is said to have the 'setting the clock back to zero' property if, for each $\beta \in \Theta$ and $x_o, y_o \geq 0$, the survival function satisfies the pair of equations,

$$\frac{S(x_o+x, y_o, \beta)}{S(x_o, y_o, \beta)} = S(x, y_o, \beta^*) \quad \forall\ x, y > 0,\ x_o, y_o > 0. \tag{4.3}$$

$$\frac{S(x_o, y_o+y, \beta)}{S(x_o, y_o, \beta)} = S(x_o, y, \beta^{**}) \quad \forall\ x, y > 0,\ x_o, y_o > 0. \tag{4.4}$$

where $\beta^* = \beta^*(x_o, y_o) \in \bar{\Theta}$ and $\beta^{**}(x_o, y_o) \in \bar{\Theta}$, where $\bar{\Theta}$ denotes the closure of the parameter space Θ. In this definition also β^* and β^{**} are allowed to take values on the boundary of the parameter space Θ, because of the same reason as given for definition 2.1.

Note that equations (4.3) and (4.4) can be written as

$$P(X > x_o+x \mid X > x_o,\ Y > y_o) = P(X^* > x,\ Y^* > y_o) \qquad \forall\ x, x_o, y_o \geq 0.$$

and

$$P(Y > y_o+y \mid X > x_o,\ Y > y_o) = P(X^{**} > x_o,\ Y^{**} > y) \qquad \forall\ y, x_o, y_o \geq 0.$$

where (X^*, Y^*) and (X^{**}, Y^{**}) have same distribution as that of (X,Y) except that the parameter $\underline{\beta}$ is changed to $\beta^*(x_o, y_o)$ and $\beta^{**}(x_o, y_o)$ respectively.

Interpreting X and Y as lifetimes of an organism under risks 1 and 2 respectively, these equations mean that the conditional probability of a component surviving additional x (y) time units under risk 1 (2) given that it has survived x_o time units under risk 1 and y_o time units under risk 2 is equal to the probability that it survives x (x_o) units under risk 1 and y_o (y) time units under risk 2, except the changed value of the parameter. Henceforth we refer to the SCBZ property defined in definition 2.1 and 2.2 as SCBZ(1) and SCBZ(2) property respectively.

In the next section we present some bivariate probability distributions which satisfy SCBZ(1) and SCBZ(2) properties.

3. **SOME EXAMPLES:**

Example 3.1.

Let us consider the class of bivariate pareto distributions having the survival function

$$S(x,y,\beta) = (1 + \sigma_1 x + \sigma_2 y)^{-\alpha}, \quad x \geq 0,\ y \geq 0.$$

where $\beta = \{\sigma_1, \sigma_2, \alpha\}$ and $\Theta = \{\{\sigma_1,\sigma_2,\alpha\} \mid \sigma_1,\sigma_2 > 0,\ \alpha > 1\}$.

Consider,

$$\frac{S(x+x_o, x_o, \beta)}{S(x_o, x_o, \beta)} = \frac{(1+\sigma_1 x_o + \sigma_2 x_o + \sigma_1 x)^{-\alpha}}{(1+\sigma_1 x_o + \sigma_2 x_o)^{-\alpha}}$$

$$= (1 + \sigma_1^* x + \sigma_2^* x_o)^{-\alpha}$$

$$= S(x, x_o, \beta^*), \quad \forall\ x, x_o \geq 0,$$

where $\beta^*(x_o) = (\sigma_1^*, \sigma_2^*, \alpha)$, $\sigma_1^*(x_o) = \sigma_1(1 + \sigma_1 x_o + \sigma_2 x_o)^{-1} > 0$, $\sigma_2^* = 0$.

Similarly

$$\frac{S(x_o, y+x_o, \beta)}{S(x_o, x_o, \beta)} = \frac{(1+\sigma_1 x_o + \sigma_2 x_o + \sigma_2 y)^{-\alpha}}{(1+\sigma_1 x_o + \sigma_2 x_o)^{-\alpha}}$$

$$= (1 + \sigma_1^* x_0 + \sigma_2^* y)^{-\alpha}$$

$$= S(x_0, y, \beta^{**}), \qquad \forall\ y, x_0 \geq 0,$$

where $\beta^{**}(x_0) = (\ \sigma_1^{**}, \sigma_2^{**}, \alpha\ ),\ \sigma_1^* = 0,\ \sigma_2^{**}(x_0) = \sigma_2(1 + \sigma_1 x_0 + \sigma_2 x_0)^{-1} > 0$. The vectors β^* and β^{**} belong to $\bar{\Theta}$, so that by definition 2.1, the class of bivariate Pareto distributions has the SCBZ(1) property.

Next consider,

$$\frac{S(x_0+x, y_0, \beta)}{S(x_0, y_0, \beta)} = \frac{(1+\sigma_1 x_0 + \sigma_2 y_0 + \sigma_1 x)^{-\alpha}}{(1+\sigma_1 x_0 + \sigma_2 y_0)^{-\alpha}}$$

$$= (1 + \sigma_1^* x + \sigma_2^* y_0)^{-\alpha}$$

$$= S(x, y_0, \beta^*), \qquad \forall\ x, y_0 \geq 0.$$

where $\beta^*(x_0, y_0) = (\{\sigma_1^*, \sigma_2^*, \alpha\},\ \sigma_1^* = \sigma_1(1 + \sigma_1 x_0 + \sigma_2 y_0)^{-1} > 0,\ \sigma_2^* = 0.$

Similarly

$$\frac{S(x_0, y_0+y, \beta)}{S(x_0, y_0, \beta)} = \frac{(1+\sigma_1 x_0 + \sigma_2 y_0 + \sigma_2 y)^{-\alpha}}{(1+\sigma_1 x_0 + \sigma_2 y_0)^{-\alpha}}$$

$$= (1 + \sigma_1^* x_0 + \sigma_2^* y)^{-\alpha}$$

$$= S(x_0, y, \beta^{**}), \qquad \forall\ y, x_0 \geq 0.$$

where $\beta^{**}(x_0, y_0) = \{\sigma_1^{**}, \sigma_2^{**}, \alpha\},\ \sigma_1^* = 0,\ \sigma_2^{**} = \sigma_2(1 + \sigma_1 x_0 + \sigma_2 y_0)^{-1} > 0$. The vector β^* and β^{**} belong to $\bar{\Theta}$, so that by definition 2.2, the class of bivariate Pareto distributions has the SCBZ(2) property.

Example 3.2.

Let us consider bivariate exponential distribution due to Marshall and Olkin (1967) with survival function,

$$S(x,y,\beta) = \exp(-\lambda_1 x - \lambda_2 y - \lambda_{12}\max(x,y)), \quad x,y \geq 0.$$

The parameter vector is $\beta = (\lambda_1,\lambda_2,\lambda_{12})$ and the parameter space is $\Theta = \{(\lambda_1,\lambda_2,\lambda_{12}),\ \lambda_1,\lambda_2,\lambda_{12} \geq 0,\ \}$.

Consider,

$$\frac{S(x+x_o,x_o,\beta)}{S(x_o,x_o,\beta)} = \frac{\exp(-\lambda_1 x - \lambda_1 x_o - \lambda_2 x_o - \lambda_{12}\max(x+x_o,x_o))}{\exp(-\lambda_1 x_o - \lambda_2 x_o - \lambda_{12}\max(x_o,x_o))}$$

$$= \exp(-(\lambda_1 + \lambda_{12})x)$$

$$= S(x,x_o,\beta^*)$$

where $\beta^*(x_o) = (\lambda_1^*,\lambda_2^*,\lambda_{12}^*)$ and $\lambda_1^* = \lambda_1+\lambda_{12} > 0$, $\lambda_2^*=0$, $\lambda_{12}^*= 0$.

Similarly

$$\frac{S(x_o,y+x_o,\beta)}{S(x_o,x_o,\beta)} = \frac{\exp(-\lambda_1 x_o - \lambda_2 y - \lambda_2 x_o - \lambda_{12}\max(x_o,y+x_o))}{\exp(-\lambda_1 x_o - \lambda_2 x_o - \lambda_{12}\max(x_o,x_o))}$$

$$= \exp(-(\lambda_2 + \lambda_{12})y)$$

$$= S(x_o,y,\beta^{**})$$

where $\beta^{**}(x_o) = (\lambda_1^{**},\lambda_2^{**},\lambda_{12}^{**})$ and $\lambda_1^{**} = 0$ $\lambda_2^{**}=\lambda_2+ \lambda_{12} > 0$, $\lambda_{12}^{**}= 0$.

The vectors β^* and β^{**} belong to $\bar{\Theta}$. Thus by definition 2.1, Marshall and Olkin (1967) bivariate exponential distribution has the SCBZ(1) property.

Next we show that bivariate exponential distribution does not satisfy SCBZ(2) property. This is shown below.

For $x_0 < y_0 < x_0 + x$, consider

$$\frac{S(x_0+x,y_0,\beta)}{S(x_0,y_0,\beta)} = \frac{\exp(-\lambda_1 x - \lambda_1 x_0 - \lambda_2 y_0 - \lambda_{12}\max(x_0+x,y_0))}{\exp(-\lambda_1 x_0 - \lambda_2 y_0 - \lambda_{12}\max(x_0,y_0))}$$

$$= \exp(-\lambda_1 x + \lambda_{12}(x_0 + x - y_0))$$

This can not put in the form $\exp(-\lambda_1^* x - \lambda_2^* y_0 - \lambda_{12}^*\max(x,y_0)) = S(x,y_0,\beta^*)$ for any choice of λ_1^*, λ_2^*, λ_{12}^*.

Thus Marshall and Olkin (1967) bivariate exponential distribution does not satisfy SCBZ(2) property.

Example 3.3.

Let us consider the Gumbel bivariate exponential distribution with survival function,

$$S(x,y,\beta) = \exp(-\lambda_1 x - \lambda_2 y - \delta xy), \quad x,y \geq 0.$$

The parameter vector is $\beta = (\lambda_1, \lambda_2, \delta)$ and the parameter space $\Theta = \{(\lambda_1,\lambda_2,\delta), \lambda_1, \lambda_2, 0 < \delta < \lambda_1\lambda_2\}$.

Consider,

$$\frac{S(x+x_0,x_0,\beta)}{S(x_0,x_0,\beta)} = \frac{\exp[-\lambda_1 x - \lambda_1 x_0 - \lambda_2 x_0 - \delta(x+x_0)x_0]}{\exp[-\lambda_1 x_0 - \lambda_2 x_0 - \delta x_0 x_0]}$$

$$= \exp[-(\lambda_1 + \delta x_o)x]$$

$$= S(x, x_o, \beta^*) \qquad \forall\ x, x_o \geq 0,$$

where $\beta^*(x_o) = (\lambda_1^*, \lambda_2^*, \delta^*)$ and $\lambda_1^* = \lambda_1 + \delta x_o > 0$, $\lambda_2^* = 0$, $\delta^* = 0$.

Similarly the ratio

$$\frac{S(x_o, y+x_o, \beta)}{S(x_o, x_o, \beta)} = \frac{\exp[-\lambda_1 x_o - \lambda_2 y - \lambda_2 x_o - \delta(y+x_o)x_o]}{\exp[-\lambda_1 x_o - \lambda_2 x_o - \delta x_o x_o]}$$

$$= \exp[-(\lambda_2 + \delta x_o)y]$$

$$= S(x_o, y, \beta^{**}) \qquad \forall\ x_o, y \geq 0,$$

where $\beta^{**}(x_o) = (\lambda_1^{**}, \lambda_2^{**}, \delta^{**})$ and $\lambda_1^{**} = 0$ $\lambda_2^{**} = \lambda_2 + \delta x_o > 0$, $\delta^{**} = 0$.

The vectors β^* and β^{**} belong to $\bar{\Theta}$. Thus by definition 2.1, Gumbel bivariate exponential distribution has the SCBZ(1) property.

Next consider,

$$\frac{S(x_o+x, y_o, \beta)}{S(x_o, y_o, \beta)} = \frac{\exp(-\lambda_1 x - \lambda_1 x_o - \lambda_2 y_o - \delta(x+x_o)y_o)}{\exp(-\lambda_1 x_o - \lambda_2 y_o - \delta x_o y_o)}$$

$$= \exp(-(\lambda_1 + \delta y_o)x)$$

$$= \exp(-\lambda_1^* x - \lambda_2^* y_o - \delta^* x y_o)$$

$$= S(x, y_o, \beta^*) \qquad \forall\ x, y_o \geq 0.$$

where $\beta^*(x_o, y_o) = (\lambda_1^*, \lambda_2^*, \delta^*)$ and $\lambda_1^* = \lambda_1 + \delta y_o > 0$, $\lambda_2^* = 0$, $\delta^* = 0$.

Similarly

$$\frac{S(x_o, y_o+y, \beta)}{S(x_o, y_o, \beta)} = \frac{\exp(-\lambda_1 x_o - \lambda_2 y_o - \lambda_2 y - \delta(x_o(y_o+y))}{\exp(-\lambda_1 x_o - \lambda_2 y_o - \delta x_o y_o)}$$

$$= \exp(-(\lambda_2 + \delta x_o)y)$$

$$= \exp(-\lambda_1^{**} x_o - \lambda_2^{**} y - \delta^{**} x_o y)$$

$$= S(x_o, y, \beta^{**}) \quad \forall\ x_o, y \geq 0.$$

where $\beta^{**}(x_o, y_o) = (\lambda_1^{**}, \lambda_2^{**}, \delta^{**})$ and $\lambda_2^{**} = \lambda_2 + \delta x_o > 0$, $\lambda_1^{**} = 0$, $\delta^{**} = 0$.

The vector β^{*} and β^{**} belong to $\bar{\Theta}$. Thus by definition 2.2, Gumbel bivariate exponential distribution has the SCBZ(2) property.

We note here that the parameter β^{*} is independent of x_o while β^{**} is independent of y_o.

Example 3.4.

Consider a bivariate life distribution, whose joint survival function is

$$S(x, y, \theta) = \exp\left(\gamma\ [1 - e^{\alpha x + \beta y}]\right), \quad \forall\ x, y \geq 0$$

where $\theta = \{\alpha, \beta, \gamma\}$ and $\Theta = (\ \{\alpha, \beta, \gamma\},\ \alpha, \beta > 0,\ \gamma \geq 1)$.

Consider,

$$\frac{S(x+x_o, x_o, \theta)}{S(x_o, x_o, \theta)} = \frac{\exp\left(\gamma[1 - e^{\alpha(x+x_o)+\beta x_o}]\right)}{\exp\left(\gamma[1 - e^{\alpha x_o + \beta x_o}]\right)}$$

$$= \exp\left(\gamma\ e^{\alpha x_o}\ e^{\beta x_o}[1 - e^{\alpha x}]\right)$$

$$= \exp\left\{\gamma^* \, [1 - e^{\alpha x}]\right\}, \text{ where } \gamma^* = \gamma \, e^{(\alpha + \beta) x_o}$$

$$= S(x, x_o, \theta^*)$$

where $\theta^* = \{\alpha^*, \beta^*, \gamma^*\}$; $\alpha^* = \alpha$, $\beta^* = 0$, $\gamma^* = \gamma \, e^{(\alpha+\beta)x_o} > 1$ for $x_o > 0$.

Similarly the ratio

$$\frac{S(x_o, y+x_o, \theta)}{S(x_o, x_o, \theta)} = \frac{\exp\left\{\gamma[1 - e^{\alpha x_o + \beta y + \beta x_o}]\right\}}{\exp\left\{\gamma[1 - e^{\alpha x_o + \beta x_o}]\right\}}$$

$$= \exp\left\{\gamma \, e^{\alpha x_o} \, e^{\beta x_o}[1 - e^{\beta y}]\right\}$$

$$= \exp\left\{\gamma^{**}[1 - e^{\beta y}]\right\}, \text{ where } \gamma^{**} = \gamma \, e^{(\alpha + \beta) x_o}$$

$$= S(x_o, y, \theta^{**})$$

where $\theta^{**} = \{\alpha^{**}, \beta^{**}, \gamma^{**}\}$, $\alpha^{**} = 0$, $\beta^{**} = \beta$, $\gamma^{**} = \gamma \, e^{(\alpha+\beta)x_o} > 0$.

The vectors θ^* and θ^{**} belong to $\bar{\Theta}$. So that by definition 2.1 bivariate life distribution has the SCBZ(1) property.

Further

$$\frac{S(x_o+x, y_o, \theta)}{S(x_o, y_o, \theta)} = \frac{\exp\left\{\gamma[1 - e^{\alpha(x+x_o)+\beta y_o}]\right\}}{\exp\left\{\gamma[1 - e^{\alpha x_o + \beta y_o}]\right\}}$$

$$= \exp\left\{\gamma \, e^{\alpha x_o} \, e^{\beta y_o} \, [1 - e^{\alpha x}]\right\}$$

$$= \exp\left\{\gamma^* \, [1 - e^{\alpha x}]\right\}, \text{ where } \gamma^* = \gamma \, e^{(\alpha x_o + \beta y_o)}$$

$$= S(x, y_o, \theta^*) \qquad \forall \; x_o, y_o \geq 0,$$

where $\theta^* = \{\alpha^*, \beta^*, \gamma^*\}$, $\alpha^* = \alpha$, $\beta^* = 0$, $\gamma^* = \gamma\, e^{(\alpha x_0 + \beta y_0)} > 0$.

Similarly

$$
\begin{aligned}
\frac{S(x_0, y_0 + y, \theta)}{S(x_0, y_0, \theta)} &= \frac{\exp\left\{\gamma[1 - e^{\alpha x_0 + \beta y_0 + \beta y}]\right\}}{\exp\left\{\gamma[1 - e^{\alpha x_0 + \beta x_0}]\right\}} \\
&= \exp\left\{\gamma\, e^{\alpha x_0}\, e^{\beta y_0}[1 - e^{\beta y}]\right\} \\
&= \exp\left\{\gamma^{**}\,[1 - e^{\beta y}]\right\}, \quad \text{where } \gamma^{**} = \gamma\, e^{(\alpha x_0 + \beta y_0)} \\
&= S(x_0, y, \theta^{**}) \quad \forall\, x_0, y_0 \geq 0,
\end{aligned}
$$

where $\theta^{**} = \{\alpha^{**}, \beta^{**}, \gamma^{**}\}$, $\alpha^{**} = 0$, $\beta^{**} = \beta$, $\gamma^{**} = \gamma\, e^{(\alpha x_0 + \beta y_0)} > 0$. The vectors θ^* and θ^{**} belong to $\bar{\Theta}$. So that by definition 2.2 bivariate life distribution has the SCBZ(2) property.

It can be observed that the expressions for the mean residual life function and percentile residual life function of a distribution are greatly simplified as a consequence of having SCBZ property of either kind. This is discussed in the next section.

4. SOME CONSEQUENCES OF SCBZ PROPERTY:

4.1 CONSEQUENCES OF SCBZ(1) PROPERTY:

4.1.1. Simplified form of mean residual life function:

Let (X,Y) be the lifetimes of an organism under risks 1 and 2. The mean residual life function (MRLF) of an individual who has survived x_0 units under risks 1 and 2 is represented by the vector

$\underline{e}_{x_o} = \{e_{x_o}^{(1)}, e_{x_o}^{(2)}\}$. It is given by

$$e_{x_o}^{(1)} = E(X - x_o | X \geq x_o,\ Y \geq x_o)$$

$$= \frac{1}{S(x_o, x_o, \beta)} \int_{x_o}^{\infty} \int_{x_o}^{\infty} x\ f(x, y, \beta)\ dx\ dy - x_o$$

$$= \int_0^{\infty} \frac{S(x+x_o, x_o, \beta)}{S(x_o, x_o, \beta)}\ dx \tag{4.5}$$

Similarly,

$$e_{x_o}^{(2)} = E(Y - x_o | X \geq x_o,\ Y \geq x_o)$$

$$= \frac{1}{S(x_o, x_o, \beta)} \int_{x_o}^{\infty} \int_{x_o}^{\infty} y\ f(x, y, \beta)\ dx\ dy - x_o$$

$$= \int_0^{\infty} \frac{S(x_o, y+x_o, \beta)}{S(x_o, x_o, \beta)}\ dy \tag{4.6}$$

Now suppose X has SCBZ(1) property.

i.e. $\dfrac{S(x+x_o, x_o, \beta)}{S(x_o, x_o, \beta)} = S(x, x_o, \beta^*)$, where $\beta^*(x_o) \in \bar{\Theta}$.

and $\dfrac{S(x_o, y+x_o, \beta)}{S(x_o, x_o, \beta)} = S(x_o, y, \beta^{**})$, where $\beta^{**}(x_o) \in \bar{\Theta}$.

Therefore equations (4.5) and (4.6) reduce to

$$e_{x_o}^{(1)} = \int_0^{\infty} S(x, x_o, \beta^*)\ dy$$

$$= E(X^{*}) \qquad \forall \quad x_o \geq 0. \tag{4.7}$$

and

$$e^{(2)}_{x_o} = \int_0^{\infty} S(x_o, y, \beta^{**})\, dy$$

$$= E(X^{**}) \qquad \forall \quad x_o \geq 0. \tag{4.8}$$

This means that if the expected value of the random variablehas known closed form expressions, then $e^{(1)}_{x_o}$ and $e^{(2)}_{x_o}$ can be very easily computed. Thus using equations (4.7) and (4.8) and noting that the marginals of BVE are $\exp(\lambda_1 + \lambda_{12})$ and $\exp(\lambda_2 + \lambda_{12})$ respectively, The MRLF of BVE is given by

$$e^{1}_{x_o} = (\lambda_1 + \lambda_{12})^{-1} \quad \text{and} \quad e^{2}_{x_o} = (\lambda_2 + \lambda_{12})^{-1}$$

and noting that marginals of Gumbels distribution are $\exp(\lambda_1 + \delta x_o)$ and $\exp(\lambda_2 + \delta x_o)$ respectively, the MRLF of Gumbels distribution is given by

$$e^{1}_{x_o} = (\lambda_1 + \delta x_o)^{-1} \quad \text{and} \quad e^{2}_{x_o} = (\lambda_2 + \delta x_o)^{-1}.$$

4.1.2. Simplified form of Percentile Life Function (PRLF):

Let (X,Y) be the lifetimes of an organism under risks 1 and 2. The Percentile residual life function (PRLF) of an individual who has survived x_o units under risks 1 and 2 is represented by the vector $\underline{p}(x_o, \underline{\alpha}) = \{ p_1(x_o, \alpha_1), p_2(x_o, \alpha_2) \}$. It is given by

$$P(X > p_1(x_0,\alpha_1)+x_0 \mid X > x_0,\ Y > x_0) = 1 - \alpha_1$$

i.e. $$\frac{P(X > p_1(x_0,\alpha_1)+x_0, x_0)}{P(X > x_0, Y > x_0)} = 1 - \alpha_1$$

i.e. $$\frac{S(p_1(x_0,\alpha_1)+x_0, x_0, \beta)}{S(x_0, x_0, \beta)} = 1 - \alpha_1 \qquad (4.9)$$

Similarly

$$P(X > p_2(x_0,\alpha_2)+x_0 \mid X > x_0,\ Y > x_0) = 1 - \alpha_2$$

i.e. $$\frac{P(X > p_2(x_0,\alpha_2)+x_0, x_0)}{P(X > x_0, Y > x_0)} = 1 - \alpha_2$$

i.e. $$\frac{S(x_0, p_2(x_0,\alpha_2)+x_0, \beta)}{S(x_0, x_0, \beta)} = 1 - \alpha_2 \qquad (4.10)$$

Now suppose X has SCBZ(1) property. Then equation (4.9) and (4.10) becomes,

$$S(p_1(x_0,\alpha_1)+x_0, x_0, \beta^*) = 1 - \alpha_1. \quad \text{where } \beta^* \in \bar{\Theta}. \qquad (4.11)$$

$$S(x_0, p_2(x_0,\alpha_2)+x_0, \beta^{**}) = 1 - \alpha_2. \quad \text{where } \beta^{**} \in \bar{\Theta}. \qquad (4.12)$$

Thus $p_1(x_0,\alpha_1)$ and $p_2(x_0,\alpha_2)$ are roots of the equations (4.11) and (4.12) respectively. Thus using equations (4.11) and (4.12) and noting that the marginals of BVE are $\exp(\lambda_1 + \lambda_{12})$ and $\exp(\lambda_2 + \lambda_{12})$ respectively, The PRLF of BVE is given by

$$p_1((x_0,\alpha_1)+x_0,\beta^*) = -(\lambda_1 + \lambda_{12})^{-1}\ \text{In}(1 - \alpha) \quad \text{and}$$

$$p_2((x_0,\alpha_2)+x_0,\beta^{**}) = -(\lambda_2 + \lambda_{12})^{-1}\ \text{In}(1 - \alpha).$$

and noting that marginals of Gumbels distribution are $\exp(\lambda_1+\delta x_0)$ and $\exp(\lambda_2+\delta x_0)$ respectively, the MRLF of Gumbels distribution is given by

$$p_1((x_0,\alpha_1)+x_0,\beta^*) = -(\lambda_1 + \delta x_0)^{-1}\text{In}(1 - \alpha) \quad \text{and}$$

$$p_2((x_0,\alpha_2)+x_0,\beta^{**}) = -(\lambda_2 + \delta x_0)^{-1}\ \text{In}(1 - \alpha).$$

Next we present similar consequences due to SCBZ(2) property.

4.2 CONSEQUENCES OF SCBZ(2) PROPERTY:

4.2.1. Simplified form of mean residual life function:

Let (X,Y) be the lifetimes of an organism under risks 1 and 2. The mean residual life function (MRLF) of an individual who has survived x_0 units under risk 1 and y_0 units under risk 2 is represented by the vector $\underline{e}_{(x_0,y_0)} = \{e^{(1)}_{(x_0,y_0)}, e^{(2)}_{(x_0,y_0)}\}$.

It is given by

$$e^{(1)}_{(x_0,y_0)} = E(X - x_0 \mid X \geq x_0,\ Y \geq y_0)$$

$$= \frac{1}{S(x_0,y_0,\beta)} \int_{x_0}^{\infty}\int_{y_0}^{\infty} x\ f(x,y,\beta)\ dx\ dy - x_0$$

$$= \int_0^\infty \frac{S(x_o+x, y_o, \beta)}{S(x_o, y_o, \beta)}\, dx \qquad (4.13)$$

Similarly

$$e^{(2)}_{(x_o,y_o)} = E(Y - y_o \mid X \geq x_o,\ Y \geq y_o)$$

$$= \frac{1}{S(x_o, y_o, \beta)} \int_{x_o}^{\infty}\int_{y_o}^{\infty} y\ f(x, y, \beta)\ dx\ dy - y_o$$

$$= \int_0^\infty \frac{S(x_o, y_o+y, \beta)}{S(x_o, y_o, \beta)}\, dy \qquad (4.14)$$

Now suppose X has SCBZ(2) property.

$$\text{i.e. } \frac{S(x_o+x, y_o, \beta)}{S(x_o, y_o, \beta)} = S(x, y_o, \beta^*)$$

and

$$\frac{S(x_o, y_o+y, \beta)}{S(x_o, y_o, \beta)} = S(x_o, y, \beta^{**})$$

Therefore equations (4.13) and (4.15) become,

$$e^{(1)}_{(x_o,y_o)} = \int_0^\infty S(x, y_o, \beta^*)\ dx,$$

$$= E(X^*) \qquad \forall\ x_o, y_o \geq 0, \text{ where } \beta^*(x_o, y_o) \in \bar{\Theta}. \qquad (4.15)$$

and

$$e^{(2)}_{(x_o,y_o)} = \int_0^{\infty} S(x_o,y,\beta^{**})\,dy,$$

$$= E(X^{**}) \qquad \forall\ x_o,y_o \geq 0,\ \text{where } \beta^{**}(x_o,y_o) \in \bar{\Theta}. \qquad (4.16)$$

4.2.2. Simplified form of the Percentile Life Function (PRLF):

Let (X,Y) be the lifetimes of an organism under risks 1 and 2. The Percentile residual life function (PRLF) of an individual who has survived x_o units under risk 1 and y_o units under risk 2 is represented by the vector $\underline{p}(x_o,\underline{\alpha}) = \{\ p_1(x_o,\alpha_1),\ p_2(x_o,\alpha_2)\ \}$. It is given by

$$P(X > p_1(x_o,\alpha_1)+x_o \mid X > x_o,\ Y > y_o) = 1 - \alpha_1$$

i.e. $$\frac{P(X > p_1(x_o,\alpha_1,)+x_o, Y > y_o)}{P(X>x_o,\ Y>y_o)} = 1 - \alpha_1$$

i.e. $$\frac{S(p_1(x_o,\alpha_1)+x_o,y_o,\beta)}{S(x_o,y_o,\beta)} = 1 - \alpha_1 \qquad (4.17)$$

Similarly

$$P(Y > p_2(y_o,\alpha_2)+y \mid X > x_o,\ Y > y_o) = 1 - \alpha_2$$

i.e. $$\frac{P(Y > p_2(y_o,\alpha_2,)+y_o, X > x_o)}{P(X>x_o,\ Y>y_o)} = 1 - \alpha_2$$

i.e. $$\frac{S(x_o,p_2(y_o,\alpha_2)+y_o,\beta)}{S(x_o,y_o,\beta)} = 1 - \alpha_2 \qquad (4.18)$$

Now suppose X has SCBZ(2) property. Then equations (4.17) and (4.18) become,

$$S(p_1(x_0,\alpha_1)+x,y_0,\beta^*) = 1 - \alpha_1 \quad \text{where } \beta^* \in \bar{\Theta}. \tag{4.19}$$

$$S(x_0,p_2(y_0,\alpha_2)+y,\beta^{**}) = 1 - \alpha_2. \quad \text{where } \beta^{**} \in \bar{\Theta}. \tag{4.20}$$

where $p_1(x_0,\alpha_1)$ and $p_2(y_0,\alpha_2)$ are roots of the equations (4.19). and (4.20) respectively.

—— 0 X 0 ——