CHAPTER 2

POINT AND INTERVAL ESTIMATION OF PROCESS CAPABILITY INDICES

2.1 INTRODUCTION

All the PCIs discussed in Chapter 1 depend on the unknown parameters viz, process standard deviation σ and/or the process mean μ. The true values of these indices can only be estimated. For estimation purpose we suppose that a random sample of size n giving values $X_1, X_2, \ldots, X_n$ is available on the quality characteristic which is normally distributed with mean μ and variance σ^2. The value μ is then estimated by

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

and the value σ is estimated by

$$S = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2}\quad ,$$

which is independent of $\bar{X}$.

The estimates of C_p, C_{pk} and C_{pm} obtained by replacing μ by $\bar{X}$ and σ by S have been discussed by Chan et al.(1988), Marcucci and Beazley (1988), Cheng and Spiring (1989), Bissell (1990), Boyles (1991) and Pearn et al.(1992) among others.

Though Montgomery (1996) and others have recommended minimum values of C_p, CPL, and CPU, we can not compare the estimated index with the minimum recommended value. Because the recommended minimum values are for the true indices and not for their estimates. Therefore, evenif the estimated index is larger than or equal to the minimum recommended value for the true index, one can not be 100% sure that the true index is larger than or equal to the minimum value. Rather, we can only claim that the true index is larger than or equal to the minimum value with a certain level of confidence. Chan et al.(1990), Boyles (1991), Kushler and Hurley (1992), Subbaiah and Taam (1993), Franklin and Wasserman (1992) consider the confidence bounds for the PCIs.

In this chapter we will discuss the point and interval estimates of the indices C_p, CPU, CPL, C_{pk}, and C_{pm} in details.

2.2 ESTIMATION OF THE INDEX C_p

2.2.1 Point Estimation of C_p

Since $C_p = \frac{d}{3\sigma}$, a natural estimate of C_p is

$$\hat{C}_p = \frac{d}{3S} \tag{2.2.1}$$

Lemma 2.2.1:

$$E(\hat{C}_p) = \frac{C_p}{b_n}$$

and
$$\mathrm{Var}(\hat{C}_p) = \left(\frac{n-1}{n-3} - \frac{1}{b_n^2}\right)\hat{C}_p^2,$$

where
$$b_n = \sqrt{\frac{2}{n-1}}\,\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n-2}{2}\right)}\,C_p.$$

Proof:

Since X_i's are independent and identically normal,

$$\frac{n-1}{\sigma^2}S^2 \sim Y,$$

where Y is a random variable having chi-square distribution with n - 1 degrees of freedom (d.f.).

$$\Rightarrow \qquad S^2 \sim \frac{\sigma^2}{n-1}Y$$

$$\Rightarrow \qquad S \sim \frac{\sigma}{\sqrt{n-1}}\sqrt{Y}$$

$$\Rightarrow \qquad \frac{3S}{d} \sim \frac{3\sigma}{d\sqrt{n-1}}\sqrt{Y}$$

$$\Rightarrow \qquad \frac{d}{3S} \sim \frac{d}{3\sigma}\frac{\sqrt{n-1}}{\sqrt{Y}}$$

That is, $\hat{C}_p \sim \dfrac{\sqrt{n-1}}{\sqrt{Y}}\, C_p$ (2.2.2)

Therefore, $E(\hat{C}_p) = \sqrt{n-1}\; C_p\; E(Y^{-1/2})$

and since $E(Y^{-1/2}) = \cancel{\sqrt{\dfrac{n-1}{2}}}\; \dfrac{\Gamma\left(\frac{n-2}{2}\right)}{\sqrt{2}\,\Gamma\left(\frac{n-1}{2}\right)}$,

we have $E(\hat{C}_p) = \dfrac{C_p}{b_n}$, (2.2.3)

where $b_n = \sqrt{\dfrac{2}{n-1}}\;\dfrac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n-2}{2}\right)}$ (2.2.4)

Similarly, variance of $\hat{C}_p$ is given by

$$\mathrm{Var}(\hat{C}_p) = \left(\frac{n-1}{n-3} - \frac{1}{b_n^2}\right)C_p^2 \quad ■ \qquad (2.2.5)$$

Thus, the estimate $\hat{C}_p$ is biased for C_p. An unbiased estimate can be obtained as $\tilde{C}_p = b_n\hat{C}_p$. A few values of b_n are presented in Table 2.1. An accurate approximation for b_n when $n \geq 15$ is

$$b_n \cong 1 - \frac{3}{4(n - 1)}$$

Table 2.1: *Values of b_n for some n.*

n	b_n	n	b_n
5	0.798	40	0.981
10	0.914	45	0.983
15	0.945	50	0.985
20	0.960	55	0.986
25	0.968	60	0.987

Since S is a consistent estimate of σ, it follows that $\hat{C}_p$ is consistent for C_p.

2.2.1 Interval Estimation of C_p

Lemma 2.2.2:

A 100(1 - α)% confidence interval for C_p, based on $\hat{C}_p$ is given by

$$\left(\sqrt{\frac{\chi^2_{n-1,\alpha/2}}{n - 1}}\ \hat{C}_p\ ,\ \sqrt{\frac{\chi^2_{n-1,1-\alpha/2}}{n - 1}}\ \hat{C}_p \right).$$

where $\chi^2_{n-1,\varepsilon}$ is ε quantile of χ^2_{n-1}.

Proof:

From (2.2.2) it follows that

$$(n-1)\left[\frac{C_p}{\hat{C}_p}\right]^2 \sim \chi^2_{n-1}$$

Hence,

$$1-\alpha = P\left[\chi^2_{n-1,\alpha/2} < (n-1)\left(\frac{C_p}{\hat{C}_p}\right)^2 < \chi^2_{n-1,1-\alpha/2}\right]$$

$$= P\left[\sqrt{\frac{\chi^2_{n-1,\alpha/2}}{n-1}}\,\hat{C}_p < C_p < \sqrt{\frac{\chi^2_{n-1,1-\alpha/2}}{n-1}}\,\hat{C}_p\right]$$

That is,

$$\left(\sqrt{\frac{\chi^2_{n-1,\alpha/2}}{n-1}}\,\hat{C}_p\ ,\ \sqrt{\frac{\chi^2_{n-1,1-\alpha/2}}{n-1}}\,\hat{C}_p\right) \qquad (2.2.6)$$

is a $100(1-\alpha)\%$ confidence interval for C_p.■

Lower and upper $100(1-\alpha)\%$ confidence bounds for C_p are, ofcourse,

$$\sqrt{\frac{\chi^2_{n-1,\alpha}}{n-1}}\,\hat{C}_p \quad \text{and} \quad \sqrt{\frac{\chi^2_{n-1,1-\alpha}}{n-1}}\,\hat{C}_p \quad \text{respectively.}$$

2.3 ESTIMATION OF THE INDEX C_{pk}

2.3.1 Point Estimation of C_{pk}

Since $C_{pk} = \dfrac{d - |\mu - T|}{3\sigma}$, the estimate of C_{pk} is given by

$$\hat{C}_{pk} = \frac{d - |\bar{X} - T|}{3S} \qquad (2.3.1)$$

$$= \frac{\frac{1}{3}\left(\frac{d}{\sigma} - \frac{|\bar{X} - T|\sqrt{n}}{\sqrt{n}\sigma} \right)}{S/\sigma}$$

$$= \frac{1}{3}\left(\frac{d}{\sigma} - \frac{1}{\sqrt{n}} \frac{\sqrt{n}|\bar{X} - T|}{\sigma} \right)\frac{\sigma}{S} \qquad (2.3.2)$$

Lemma 2.3.1:

$$E(\hat{C}_{pk}) = \frac{1}{3}\left[\frac{d}{\sigma} - \sqrt{\frac{2}{n\pi}} \exp\left\{ - \frac{n(\mu - T)^2}{2\sigma^2} \right\} - \frac{|\mu - T|}{\sigma}\left\{ 1 - 2\Phi\left(- \frac{|\mu - T|\sqrt{n}}{\sigma} \right) \right\} \right] \frac{1}{b_n}$$

and

$$\mathrm{Var}(\hat{C}_{pk}) = \frac{n - 1}{9(n - 3)}\left[\left(\frac{d}{\sigma} \right)^2 - 2\left(\frac{d}{\sigma} \right) \sqrt{\frac{2}{n\pi}} \exp\left\{- \frac{n(\mu - T)^2}{2\sigma^2}\right\} -2\left(\frac{d}{\sigma} \right) \frac{|\mu - T|}{\sigma} \left\{ 1 - 2\Phi\left(- \frac{\sqrt{n}|\mu - T|}{\sigma}\right) \right\} + \frac{1}{n} + \frac{(\mu - T)^2}{\sigma^2} \right] - [E(\hat{C}_{pk})]^2 .$$

Proof:

On assumption of normally, S and $\frac{\sqrt{n}|\bar{X} - T|}{\sigma}$ are independently distributed. The statistic $\frac{\sqrt{n}|\bar{X} - T|}{\sigma}$ has a folded normal distriution. Therefore we have

$$E\left(\frac{\sqrt{n}|\bar{X} - T|}{\sigma} \right) = \left(\frac{2}{\pi} \right)^{1/2} \exp\left\{ - \frac{n(\mu - T)^2}{2\sigma^2} \right\} + \frac{\sqrt{n}|\mu - T|}{\sigma} \left\{ 1 - 2\Phi\left(- \frac{\sqrt{n}|\mu - T|}{\sigma} \right) \right\} \tag{2.3.3}$$

and

$$E\left\{ \left(\frac{\sqrt{n}}{\sigma} |\bar{X} - T| \right)^2 \right\} = 1 + \frac{n(\mu - T)^2}{\sigma^2} \tag{2.3.4}$$

(Refer Leone et al. (1961) for details of (2.3.3) and (2.3.4).)

Also since

$(n - 1)\frac{S^2}{\sigma^2} \sim Y$, where Y is a chi-square random variable having n - 1 d.f., we have

$$\frac{S}{\sigma} \sim \frac{\sqrt{Y}}{\sqrt{n - 1}}$$

or $$\frac{\sigma}{S} \sim \sqrt{n - 1}\, Y^{-1/2}$$

Hence

$$E\left(\frac{\sigma}{S} \right) = \sqrt{n - 1}\, E(Y^{-1/2})$$

$$= \sqrt{\frac{n-1}{2}}\;\frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}$$

$$= \frac{1}{b_n} \qquad (2.3.5)$$

Hence, using (2.3.2) expectation of $\hat{C}_{pk}$ is given by

$$E(\hat{C}_{pk}) = \frac{1}{3}\left[\frac{d}{\sigma} - \frac{1}{\sqrt{n}}\,E\left(\frac{\sqrt{n}(\overline{X} - T)}{\sigma}\right)\right] E\left(\frac{\sigma}{S}\right)$$

$$= \frac{1}{3}\left[\frac{d}{\sigma} - \sqrt{\frac{2}{n\pi}}\exp\left\{-\frac{n(\mu - T)^2}{2\sigma^2}\right\} - \frac{|\mu - T|}{\sigma}\left\{1 - 2\Phi\left(-\frac{|\mu - T|\sqrt{n}}{\sigma}\right)\right\}\right]\frac{1}{b_n}$$

$$\ldots(2.3.6)$$

and

$$\mathrm{Var}(\hat{C}_{pk}) = E(\hat{C}_{pk}^2) - [E(\hat{C}_{pk})]^2$$

$$= \frac{n-1}{9(n-3)}\left[\left(\frac{d}{\sigma}\right)^2 - 2\left(\frac{d}{\sigma}\right)\sqrt{\frac{2}{n\pi}}\exp\left\{-\frac{n(\mu - T)^2}{2\sigma^2}\right\} - 2\left(\frac{d}{\sigma}\right)\frac{|\mu - T|}{\sigma}\left\{1 - 2\Phi\left(-\frac{\sqrt{n}|\mu - T|}{\sigma}\right)\right\} + \frac{1}{n} + \frac{(\mu - T)^2}{\sigma^2}\right]$$

$$- [E(\hat{C}_{pk})]^2 \;\blacksquare \qquad (2.3.7)$$

Some numerical values of $E(\hat{C}_{pk})$ and $Var(\hat{C}_{pk})$ are given in Table 2.2 and corresponding values of C_{pk} are given in Table 2.3.

Note that $\hat{C}_{pk}$ is a biased estimate of C_{pk}. The bias arises from two sources:

i) $E(S^{-1}) = b_n^{-1}\sigma^{-1} \neq \sigma^{-1}$. This bias is positive since $b_n < 1$.

ii) $E\left(\frac{\sqrt{n}|\bar{X} - T|}{\sigma}\right) > \frac{\sqrt{n}|\mu - T|}{\sigma}$. This leads to a negative bias, because $\frac{\sqrt{n}|\bar{X} - T|}{\sigma}$ has a negative sign in the numerator of C_{pk}.

The resultant bias is positive for all cases shown in Table 2.2 for which $\mu \neq T$. When $\mu = T$ the bias is positive for $n = 10$ but becomes negative for larger n. Ultimately, as $n \longrightarrow \infty$ the bias tends to zero.

Also $\hat{C}_{pk}$ is a consistent estimate of C_{pk}. This follows from the fact that $\bar{X}$ and S are consistent for μ and σ respectively and $d - |\mu - T|$ is a continuous function of μ.

2.3.2 Interval Estimation of C_{pk}

In this section first we shall find the lower confidence bounds for CPU and CPL and then the lower confidence bound for C_{pk}. Recall that when only a single specification limit is given,

Table 2.2: *Moments of* $\hat{C}_{pk}$

d/σ	$\tau = (\mu - T)/\sigma$ 0.0		0.5		1.0		1.5	
	*EV**	*Var**	*EV*	*Var*	*EV*	*Var*	*EV*	*Var*
n = 10								
2	0.637	0.035	0.542	0.034	0.365	0.024	0.182	0.017
3	1.002	0.079	0.906	0.073	0.729	0.054	0.547	0.036
4	1.367	0.143	1.271	0.131	1.094	0.103	0.912	0.076
5	1.732	0.226	1.636	0.209	1.459	0.171	1.277	0.135
6	2.096	0.329	2.001	0.307	1.824	0.260	1.641	0.213
n = 20								
2	0.633	0.014	0.520	0.014	0.347	0.010	0.174	0.007
3	0.980	0.031	0.867	0.028	0.695	0.021	0.521	0.014
4	1.327	0.055	1.215	0.050	1.042	0.039	0.868	0.029
5	1.674	0.086	1.562	0.079	1.389	0.064	1.215	0.050
6	2.022	0.124	1.909	0.115	1.736	0.096	1.563	0.079
n = 30								
2	0.635	0.009	0.513	0.009	0.342	0.006	0.171	0.005
3	0.977	0.019	0.856	0.018	0.685	0.013	0.513	0.009
4	1.319	0.034	1.198	0.031	1.027	0.024	0.856	0.018
5	1.662	0.053	1.540	0.048	1.369	0.039	1.198	0.031
6	2.004	0.076	1.882	0.070	1.711	0.059	1.540	0.048
n = 40								
2	0.637	0.007	0.510	0.006	0.340	0.004	0.170	0.003
3	0.977	0.014	0.850	0.013	0.680	0.009	0.510	0.006
4	1.317	0.025	1.190	0.022	1.020	0.017	0.850	0.013
5	1.657	0.038	1.530	0.035	1.360	0.028	1.190	0.022
6	1.997	0.055	1.870	0.050	1.700	0.042	1.530	0.035

* *EV* = $E(\hat{C}_{pk})$, *Var* = $Var(\hat{C}_{pk})$.

Table 2.3: *Values of C_{pk}.*

d/σ	$\tau = (\mu - T)/\sigma$			
	0.0	*0.5*	*1.0*	*1.5*
2	2/3	1/2	1/3	1/6
3	1	5/6	2/3	1/2
4	4/3	7/6	1	5/6
5	5/3	3/2	4/3	7/6
6	2	11/6	5/3	3/2

one uses either CPU or CPL to measure the process capaility. We define them as

$$CPU = \frac{USL - \mu}{3\sigma} \quad \text{and} \quad CPL = \frac{\mu - LSL}{3\sigma} .$$

Their estimates are

$$\hat{CPU} = \frac{USL - \bar{X}}{3S} \quad \text{and} \quad \hat{CPL} = \frac{\bar{X} - LSL}{3S} \quad \text{respectively.}$$

Let $LSL = \bar{X} - k_1 S$ and $USL = \bar{X} + k_2 S$, where k_1 and k_2 are such that a $100(1 - \alpha)\%$ lower confidence bound c_u for CPU satisfies

$$P[CPU > c_u] = 1 - \alpha \qquad (2.3.8)$$

From above equation we have

$$1 - \alpha = P\left[\frac{USL - \mu}{3\sigma} > c_u \right]$$

$$= P[\ \bar{X} + k_2 S - \mu > 3c_u\sigma\ ]$$

$$= P[\ \bar{X} - \mu > 3c_u\sigma - k_2 S\ ]$$

$$= P[\ \bar{X} - \mu \ < \ k_2 S - 3c_u\sigma\ ]$$

$$= P\left[\frac{\bar{X} - \mu}{S/\sqrt{n}} < \sqrt{n}k_2 + \frac{3\sqrt{n}c_u\sigma}{S} \right]$$

$$= P\left[\frac{\bar{X} - \mu}{S/\sqrt{n}} + \frac{3\sqrt{n}c_u}{S/\sigma} < \sqrt{n}k_2 \right]$$

$$= \quad P[T_{n-1}(\delta) \ < \ \sqrt{n}k_2\ ], \qquad (2.3.9)$$

where $\delta = 3\sqrt{n}c_u$, $k_2 = 3\hat{CPU}$ and $T_k(x)$ is a non-central t-variable with k d.f. and non-centrality parameter x.

Similarly a $100(1 - \alpha)\%$ lower confidence bound c_l for CPL satisfies

$$P[T_{n-1}(\delta_o) \ < \ \sqrt{n}k_1\ ] \ = \ 1 - \alpha \qquad (2.3.10)$$

where $k_1 = 3\hat{CPL}$ and $\delta_o = 3\sqrt{n}c_l$.

Since (2.3.9) and (2.3.10) are of the same form, it follows that the relationship between c_u and $\hat{CPU}$ is same as that between c_l and $\hat{CPL}$. Given $\hat{CPU}$ we can solve for c_u from (2.3.9). Similarly given $\hat{CPL}$ we can solve for c_l from (2 3 10). Chou et al. (1990) have provided a table which gives the 95% lower confidence bounds c_u (or c_l) for CPU (or CPL), for various values of n and $\hat{CPU}$ (or $\hat{CPL}$).

Obtaining a confidence bound for C_{pk} is not straight forward since C_{pk} is minimum of the two functions that depend jointly on $\bar{X}$ and S. A 100(1 - α)% lower confidence bound c_k for C_{pk} satisfies

$$P[C_{pk} > c_k] = 1 - \alpha \qquad (2.3.11)$$

Above equation can be written as

$$P[\ CPL > c_k,\ CPU > c_k\ ] = 1 - \alpha$$

since LSL = $\bar{X} - k_1 S$ and USL = $\bar{X} + k_2 S$, we have

$$P\left[\frac{\bar{X} - k_1 S - \mu}{3\sigma} < -c_k\ ,\ \frac{\bar{X} + k_2 S - \mu}{3\sigma} > c_k\right] = 1 - \alpha$$

This equation is similer to the one given in Chou and Owen (1984). Therefore, it can be written as

$$P[\ T_{n-1}(\delta_1) \leq t_1 \text{ and } T_{n-1}(\delta_2) \geq t_2\ ] = 1 - \alpha$$

$$(2.3.12)$$

where $t_1 = k_1\sqrt{n}$, $t_2 = -k_2\sqrt{n}$, $\delta_1 = 3c_k\sqrt{n}$ and $\delta_2 = -3c_k\sqrt{n}$.

Given $\widehat{CPU}$ and $\widehat{CPL}$, we can solve for c_k from (2.3.12). Assuming $\widehat{CPU} = \widehat{CPL}$, Chou et.al.(1990) has provided lower 95% confidence bounds for C_{pk}. However, this assumption in general, leads to conservative lower bounds. Simulation study by Franklin and Wasserman (1992a) have shown that the actual coverage due to these bounds is about 96 to 97%.

Alternatively, since (2.3.9) and (2.3.10) provide exact lower bounds for CPU and CPL respectively and both are of the same form, one can use either of them to find the lower bound for C_{pk}, because $C_{pk} = \min(CPU, CPL)$. Comparison of Table 4, in Chou et al. (1990), which is based on (2.3.9) or (2.3.10) and Table 5, in Chou et al. (1990), which gives lower 95% lower confidence bounds for C_{pk} assuming $\widehat{CPU} = \widehat{CPL}$ and using (2.3.12), shows that every entry in Table 4 is larger than the corresponding entry in Table 5. Thus, Table 4 is less conservative than Table 5. Although this method has the disadvantage of requiring an algorithm for the non-central t-distribution or the use of tables.

Alternatively since $\widehat{CPL} = \dfrac{\bar{X} - LSL}{3S}$, we have

$$3\sqrt{n}\widehat{CPL} = \frac{\sqrt{n}(\bar{X} - LSL)}{S} = \frac{\sqrt{n}(\bar{X} - \mu)}{S} + \frac{\sqrt{n}(\mu - LSL)}{S}$$

and it follows that $3\sqrt{n}\widehat{CPL}$ has a non-central t-distribution with $n - 1$ d.f. and non-centrality parameter $\sqrt{n}(\mu - LSL)/\sigma$. Hence,

$$E(3\sqrt{n}\widehat{CPL}) = \frac{\sqrt{n}(\mu - LSL)}{\sigma}$$

and

$$E(\widehat{CPL}) = \frac{(\mu - LSL)}{3\sigma} = CPL.$$

Johnson and Kotz (1970b) gave a simple asymptotic approximation to the variance of non-central t-distribution.

Using the same (3, pp 204), we have

$$\mathrm{Var}(3\sqrt{n}\hat{CPL}) \cong 1 + \frac{1}{2}\,\frac{n(\mu - LSL)^2}{\sigma^2(n - 1)}$$

Hence $\quad \mathrm{Var}(\hat{CPL}) \cong \dfrac{1}{9n} + \dfrac{(\mu - LSL)^2}{18\sigma^2(n - 1)}$

$$= \frac{1}{9n} + \frac{(\ CPL\ )^2}{2(n - 1)} \quad .$$

Hence, use of normal approximation to the sampling distribution of $\hat{CPL}$ yields an approximate $100(1 - \alpha)\%$ lower confidence bound for CPL as

$$\hat{CPL} - z_{1-\alpha}\sqrt{\frac{1}{9n} + \frac{(\ \hat{CPL}\ )^2}{2(n - 1)}} \qquad (2.3.13)$$

where $z_{1-\alpha}$ is the $(1 - \alpha)$th quantile of the standard normal distriution. Similarly

$$\hat{CPU} - z_{1-\alpha}\sqrt{\frac{1}{9n} + \frac{(\ \hat{CPU}\ )^2}{2(n - 1)}} \qquad (2.3.14)$$

gives an approximate $100(1 -\alpha)\%$ lower connfidence bound for CPU.

Again, since C_{pk} = min(CPU, CPL), we can use (2.3.13) or (2.3.14) to find lower confidence bound for C_{pk}. That is, $100(1 -\alpha)\%$ lower confidence bound for C_{pk} is given by

$$\hat{C}_{pk} - z_{1-\alpha}\sqrt{\frac{1}{9n} + \frac{\hat{C}_{pk}^{2}}{2(n-1)}} \qquad (2.3.15)$$

This approach was also proposed by Bissell (1990), who derived formula (2.3.15) using a Taylor series argument. Also using this approach, an approximate $100(1 - \alpha)\%$ two sided confidence interval for C_{pk} can be given as

$$\hat{C}_{pk} \pm z_{1-\alpha/2}\sqrt{\frac{1}{9n} + \frac{\hat{C}_{pk}^{2}}{2(n-1)}} \qquad (2.3.16)$$

While ease of use is an important consideration in choosing a method for determining confidence bounds, the performance of the method is also an important issue. For confidence bounds, the natural performance characteristic is the 'miss rate' of the method. If the derived confidence level is $100(1 - \alpha)\%$, then the nominal miss rate is α. The actual miss rate is the probability a random sample results in a computed lower bound that is greater than the true value of the index. For an exact method the actual miss rate equals the nominal miss rate, but when the method involves an approximation, the actual miss rate can differ from the nominal miss rate. An approximate method for which the difference is small can be said to perform well.

Simulation methods are often used to estimate the actual miss rate when it can not be determined analytically. We have conducted a simulation study with various parameter settings to investigate the performance of the bounds of C_{pk} given by (2.3.15) and (2.3.16). 4000 simulated samples were generated for each parameter setting using MINITAB. For each sample (2.3.15) and (2.3.16) were computed. The proportions of the intervals given by (2.3.15) and (2.3.16), covering the true value of C_{pk} are as given in Table 2.4 and Table 2.5 respectively. It is observed from Table 2.4 and Table 2.5 that bounds using (2.3.15) and (2.3.16) are quite acceptable.

While doing this simulation study we could have also reported estimates of $E(\hat{C}_{pk})$ and $Var(\hat{C}_{pk})$, however, for brevity we have omitted these values.

2.4 ESTIMATION OF THE INDEX C_{pm}

2.4.1 Point Estimation of C_{pm}

The original estimate of C_{pm} in Chan et al (1988) is defined as

$$\hat{C}_{pm} = \frac{USL - LSL}{6S_{T_{n-1}}} \qquad (2.4.1)$$

Table 2.4: *Coverage proportions of Bissell's 100(1 $-\alpha$)% lower confidence bounds for C_{pk}.*

C_{pk}	α 0.10	0.05	0.01
n = 30			
0.2222	0.893	0.944	0.989
0.3333	0.892	0.946	0.989
0.4444	0.890	0.944	0.989
0.6667	0.891	0.945	0.991
1.0000	0.942	0.976	0.996
1.3333	0.882	0.946	0.990
2.0000	0.917	0.962	0.993
2.6667	0.888	0.950	0.994
4.0000	0.902	0.957	0.994
n = 100			
0.2222	0.896	0.946	0.986
0.3333	0.908	0.948	0.988
0.4444	0.897	0.954	0.991
0.6667	0.902	0.948	0.989
1.0000	0.950	0.979	0.998
1.3333	0.899	0.950	0.992
2.0000	0.923	0.963	0.993
2.6667	0.893	0.944	0.995
4.0000	0.912	0.954	0.990

Table 2.5: *Coverage proportions of Bissell's 100(1 −α)% confidence intervals for C_{pk}.*

C_{pk}	α 0.10	0.05	0.01
n = 30			
0.2222	0.896	0.951	0.990
0.3333	0.894	0.956	0.990
0.4444	0.894	0.953	0.989
0.6667	0.899	0.954	0.990
1.0000	0.896	0.942	0.989
1.3333	0.902	0.943	0.993
2.0000	0.903	0.942	0.991
2.6667	0.901	0.957	0.989
4.0000	0.896	0.953	0.989
n = 100			
0.2222	0.905	0.955	0.990
0.3333	0.903	0.947	0.991
0.4444	0.906	0.955	0.988
0.6667	0.898	0.953	0.989
1.0000	0.889	0.954	0.988
1.3333	0.887	0.949	0.993
2.0000	0.900	0.955	0.992
2.6667	0.897	0.954	0.992
4.0000	0.902	0.952	0.989

where $S^2_{T_{n-1}} = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - T)^2$

Boyles (1991) has given an alternative estimate of Cpm as

$$\tilde{C}_{pm} = \frac{USL - LSL}{6S_{T_n}} \qquad (2.4.2)$$

where $S^2_{T_n} = \frac{1}{n}\sum_{i=1}^{n}(X_i - T)^2$

When sample size is large there is a little difference between $\hat{C}_{pm}$ and $\tilde{C}_{pm}$. However, the inferential aspects of $\hat{C}_{pm}$ and $\tilde{C}_{pm}$ are different for small or moderate sample size.

Let $B(\cdot)$ and $MSE(\cdot)$ denote the bias and the mean square error of an estimate respectively. The biases of $\hat{C}_{pm}$ and $\tilde{C}_{pm}$ are given by

$$B(\hat{C}_{pm}) = E(\hat{C}_{pm}) - C_{pm}$$

and $B(\tilde{C}_{pm}) = E(\tilde{C}_{pm}) - C_{pm}$, respectively.

The mean square errors of $\hat{C}_{pm}$ and $\tilde{C}_{pm}$ are given by

$$MSE(\hat{C}_{pm}) = Var(\hat{C}_{pm}) + B(\hat{C}_{pm})$$

and $MSE(\tilde{C}_{pm}) = Var(\tilde{C}_{pm}) + B(\tilde{C}_{pm})$, respectively.

Lemma 2.4.1:

(i) $B(\hat{C}pm) < B(\tilde{C}pm)$

(ii) $B(\tilde{C}pm) \geq 0$

and (iii) $B(\hat{C}pm) \geq -Cpm\left(1 - \sqrt{\frac{n-1}{n}}\right)$.

Proof:

(i) follows from the fact that

$$\hat{C}pm = \tilde{C}pm\sqrt{\frac{n-1}{n}} \leq \tilde{C}pm.$$

(ii) follows from Jenson's inequality, namely,

$$E(\tilde{C}pm) = E\left[g(S_{T_n}^2)\right] \geq g\left[E(S_{T_n}^2)\right] = Cpm,$$

where $g(S_{T_n}^2) = \dfrac{USL - LSL}{6S_{T_n}}$ is a convex function.

Now from (i) and (ii) above, we have

$$E(\hat{C}pm) = \sqrt{\frac{n-1}{n}}\,E(\tilde{C}pm) \geq \sqrt{\frac{n-1}{n}}\,Cpm.$$

Hence,

$$B(\hat{C}pm) = E(\hat{C}pm) - Cpm$$

$$\geq \sqrt{\frac{n-1}{n}}\,Cpm - Cpm$$

$$= -C_{pm}\left(1 - \sqrt{\frac{n-1}{n}}\right).$$

Hence the proof of (iii). ∎

Theorem 2.4 1.

If $|B(\hat{C}_{pm})| < B(\tilde{C}_{pm})$, then $MSE(\hat{C}_{pm}) < MSE(\tilde{C}_{pm})$.

Proof:

Since $\hat{C}_{pm} = \sqrt{\frac{n-1}{n}}\,\tilde{C}_{pm}$,

$$Var(\hat{C}_{pm}) = \frac{n-1}{n} Var(\tilde{C}_{pm}) < Var(\tilde{C}_{pm})$$

Therefore, $MSE(\hat{C}_{pm}) = Var(\hat{C}_{pm}) + \left[B(\hat{C}_{pm})\right]^2$

$$< Var(\tilde{C}_{pm}) + \left[B(\tilde{C}_{pm})\right]^2$$

$$= MSE(\tilde{C}_{pm}),$$

because $\left[B(\hat{C}_{pm})\right]^2 < \left[B(\tilde{C}_{pm})\right]^2$, when $|B(\hat{C}_{pm})| < B(\tilde{C}_{pm})$. ∎

Now since $S^2_{T_n} = \frac{1}{n}\sum_{i=1}^{n}(X_i - T)^2$,

$$\frac{nS^2_{T_n}}{\sigma^2} = \sum_{i=1}^{n}\left(\frac{X_i - T}{\sigma}\right)^2$$

$$= \sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma} + \frac{\mu - T}{\sigma} \right)^2 .$$

And since X_i's are iid $N(\mu, \sigma^2)$, it follows that $nS^2_{T_n}/\sigma^2$ has a noncentral chi-square distribution with n d.f. and non-centrality parameter $n(\mu - T)^2/\sigma^2 = n\delta^2 = \lambda$. We denote this distribution by $\chi^2_n(\lambda)$. In order to compute $E(\tilde{C}_{pm})$ we require following lemma.

Lemma 2.4.2:

Let $U \sim \chi^2_n(\lambda)$, then

$$E(U^r) = 2^r \sum_{j=0}^{\infty} A_j(\lambda) \frac{\Gamma(n/2 + j + r)}{\Gamma(n/2 + j)}$$

where $A_j(\lambda) = (\lambda/2)^j \dfrac{e^{-\lambda/2}}{j!}$.

Proof: See Appendix A.

We shall use this result to prove the following lemma.

Lemma 2.4.3:

$$E(\tilde{C}_{pm}) = \sqrt{\frac{n}{2}}\, C_p \sum_{j=0}^{\infty} A_j(\lambda) \frac{\Gamma((n-1)/2 + j)}{\Gamma(n/2 + j)} \qquad (2.4.3)$$

$$E(\tilde{C}^2_{pm}) = \frac{n}{2} C_p^2 \sum_{j=0}^{\infty} A_j(\lambda) \frac{\Gamma((n-1)/2 + j)}{\Gamma(n/2 + j)} \qquad (2.4.4)$$

Proof:

$$\text{Since} \quad \frac{nS^2_{T_n}}{\sigma^2} \sim \chi^2_n(\lambda),$$

$$S^2_{T_n} \sim \frac{\sigma^2}{n} Y,$$

where Y is a non-central chi-square random variable with n [illegible] d.f. and non-centrality parameter λ.

$$\Rightarrow \quad S_{T_n} \sim \frac{\sigma}{\sqrt{n}} \sqrt{Y}$$

$$\Rightarrow \quad \frac{6S_{T_n}}{USL - LSL} \sim \frac{6\sigma}{USL - LSL} \frac{1}{\sqrt{n}} \sqrt{Y}$$

$$\Rightarrow \quad \frac{USL - LSL}{6S_{T_n}} \sim \frac{(USL - LSL)\sqrt{n}}{6\sigma} Y^{-1/2}$$

That is $\quad \tilde{C}_{pm} \sim \sqrt{n}\, C_p\, Y^{-1/2}$

Therefore,

$$E(\tilde{C}_{pm}) = \sqrt{n}\, C_p\, E(Y^{-1/2})$$

$$= \sqrt{n}\, C_p\, 2^{-1/2} \sum_{j=0}^{\infty} A_j(\lambda) \frac{\Gamma((n-1)/2 + j)}{\Gamma(n/2 + j)}$$

and

$$E(\tilde{C}^2_{pm}) = nC^2_p\, E(Y^{-1})$$

$$= \frac{n}{2} C_p^2 \sum_{j=0}^{\infty} A_j(\lambda) \frac{\Gamma((n-1)/2 + j)}{\Gamma(n/2 + j)} \quad \blacksquare$$

From the relation $\hat{C}_{pm} = \sqrt{\frac{n-1}{n}}\, \tilde{C}_{pm}$, we have following corollary.

Corollary 2.4.3:

$$E(\hat{C}_{pm}) = \sqrt{\frac{n-1}{2}}\, C_p \sum_{j=0}^{\infty} A_j(\lambda) \frac{\Gamma((n-1)/2 + j)}{\Gamma(n/2 + j)} \qquad (2.4.5)$$

$$E(\hat{C}_{pm}^2) = \frac{n-1}{2} C_p^2 \sum_{j=0}^{\infty} A_j(\lambda) \frac{\Gamma((n-1)/2 + j)}{\Gamma(n/2 + j)} \qquad (2.4.6)$$

Since both S_{T_n} and $S_{T_{n-1}}$ are consistent for $\sigma' = \sqrt{E(X - T)^2}$, it follows that both $\hat{C}_{pm}$ and $\tilde{C}_{pm}$ are concistent for C_{pm}.

2.4.2. Interval Estimation of C_{pm}

A number of approximate confidence intervals for C_{pm} are considered in Subbaiah and Taam (1993). The approximations developed are based on the large sample properties of the two estimates of C_{pm}. Using the fact that $nS_{T_n}^2/\sigma^2$ has a noncentral Chi square distriution following confidence intervals for C_{pm} are considered, which are based on the estimate $\tilde{C}_{pm}$.

Theorem 2.4.2:

A 100(1 - α)%, approximate confidence intervals for Cpm are given by

$$\text{(i)} \quad \left(\tilde{C}pm \sqrt{\chi^2_{\tilde{v},\alpha/2} / \tilde{v}} \quad , \quad \tilde{C}pm \sqrt{\chi^2_{\tilde{v},1-\alpha/2} / \tilde{v}} \right) , \qquad (2.4.7)$$

where $\tilde{v} = \dfrac{n(1 + \tilde{\delta}^2)^2}{(1 + 2\tilde{\delta}^2)}$, $\tilde{\delta} = \dfrac{\bar{x} - T}{S}$,and $\chi_{v,\alpha}$ is the α quantile of central chi-square variable with $\tilde{v}$ d.f.

$$\text{and (ii)} \quad \tilde{C}pm \left(1 \pm z_{1-\alpha/2} \sqrt{\frac{1}{2\tilde{v}}} \right) , \qquad (2.4.8)$$

where $z_{1-\alpha/2}$ is the (1 - α/2) quantile of standard normal variable and $\tilde{v}$ is large.

Proof:

Patnaik (1949) provided an approximation to non-central chi-square quantile with a constant multiple of a central chi-square quantile, namely, $\chi^2_n(\lambda) \approx c\chi^2_v$, where

$$c = \frac{(1 + 2\delta^2)}{1 + \delta^2} , \quad v = \frac{n(1 + \delta^2)^2}{1 + 2\delta^2} \text{ and } \delta = \frac{\mu - T}{\sigma}$$

Therefore, we have

$$1 - \alpha \approx P\left[c\chi^2_{v,\alpha/2} < \frac{nS^2_{T_n}}{\sigma^2} < c\chi^2_{v,1-\alpha/2} \right]$$

$$= P\left[\chi^2_{v,\alpha/2} < \frac{nS^2_{T_n}}{\sigma^2 c} < \chi^2_{v,1-\alpha/2} \right]$$

$$= P\left[\chi^2_{v,\alpha/2} < \frac{n}{c}\left(\frac{\tilde{C}pm}{Cpm} \right)^2 (1 + \delta^2) < \chi^2_{v,1-\alpha/2} \right]$$

$$\left\{\text{since } \left(\frac{\tilde{C}pm}{Cpm} \right)^2 = \frac{S^2_{T_n}}{\sigma^2 (1 + \delta^2)} \right\}$$

$$= P\left[\chi^2_{v,\alpha/2} < \left(\frac{\tilde{C}pm}{Cpm} \right)^2 v < \chi^2_{v,1-\alpha/2} \right]$$

$$\left\{ \text{since } \frac{n}{c}(1 + \delta^2) = \frac{n(1 + \delta^2)^2}{1 + 2\delta^2} = v \right\}$$

$$= P\left[\sqrt{\chi^2_{v,\alpha/2} / v}\ \tilde{C}pm < Cpm < \sqrt{\chi^2_{v,1-\alpha/2} / v}\ \tilde{C}pm \right]$$

The first result follows by estimating v and δ by $\tilde{v}$ and $\tilde{\delta}$ respectively.

The second result is based on the normal approximation of χ^2_v for large v. That is, $\chi^2_v \approx N(v, 2v)$, then $\sqrt{\chi^2_v / v} \approx N(1, 1/2v)$, because square-root is a monotone transformation and a chi-square variate is nonnegative.∎

Simulation study done by Subbaiah and Taam (1993) shows that coverage probabilities of these intervals are very close to the nominal value and the miss probabilities on both the low and high sides are nearly equal. Also performance of these intervals is better than any other interval discussed by them.

The 100(1 - α)% lower confidence bounds can be given by

$$\tilde{C}pm\sqrt{\chi^2_{\tilde{v},\alpha} / \tilde{v}} \quad \text{and} \quad \tilde{C}pm\left(1 - z_{1-\alpha}\sqrt{\frac{1}{2\tilde{v}}}\right).$$

2.5 ESTIMATION OF THE INDEX Cpv

Spiring (1997) has suggested an estimate of Cpv as

$$Cpv = \frac{USL - LSL}{6\sqrt{\hat{\sigma}^2 + \omega(\overline{X} - T)^2}}, \tag{2.5.1}$$

where $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2$.

It follows that

(i) $\frac{n}{\sigma^2}\hat{\sigma}^2 \sim \chi^2_{n-1}$

(ii) $\overline{X} \sim N(\mu, \frac{\sigma^2}{n})$ and

(iii) $\overline{X}$ and $\hat{\sigma}^2$ are independent.

Since T is nonstochaic, it follows that $\frac{n}{\sigma^2}(\overline{X} - T)^2 \sim \chi^2_1(\lambda)$

with noncentrality parameter $\lambda = n\left(\frac{\mu - T}{\sigma}\right)^2$ and if ω is nonstochastic, it follows that $\frac{\omega n}{\sigma^2}(\bar{X} - T)^2 \sim \omega\,\chi_1^2(\lambda)$.

Defining $Q^2_{n,\lambda} = \frac{n}{\sigma^2}\left[\hat{\sigma}^2 + \omega(\bar{X} - T)^2\right]$, $Q^2_{n,\lambda}$ becomes a linear comination of two independent chi-squared distributions

$$\chi^2_{n-1} + \omega\,\chi_1^2(\lambda)$$

Let $Q^2_{n,\lambda}(x)$ be a c.d.f. associated with $Q^2_{n,\lambda}$, Press (1966) showed that the $Q^2_{n,\lambda}(x)$ can be expressed as a mixture of central chi-squared distributions with general form

$$Q^2_{n,\lambda}(x) = \sum_{i=1}^{\infty} d_i\,\chi^2_{n+2j}(x)$$

with the d_i's being weights such that $\sum_{i=1}^{\infty} d_i = 1$, where the d_i's are the functions of the d.f., the non-centrality parameter λ and the weight function ω. The functional forms of the d_i's are given in Press (1966), which for $Q^2_{n,\lambda}$ are as follows:

$$d_o = \omega^{-1/2} e^{-\lambda/2}$$

$$d_i = \sum_{j=o}^{i} \sum_{k=o}^{j} e^{-\lambda/2}\left(\frac{\lambda}{2}\right)^{j-k} \omega^{-1/2-j+k}(1 - \omega^{-1})^{k+i-j}$$

$$\times \frac{\Gamma(i - j + 0.5)}{\Gamma(0.5)\,\Gamma(i - j + 1)} \binom{j - 1}{k}$$

for $i = 1,2,3,\ldots$

Lemma 2.5.1:

A $100(1-\alpha)\%$ confidence interval for C_{pv} is given by

$$\left(\sqrt{\frac{Q^2_{n,\lambda,\alpha/2}}{n(1+\frac{\omega\lambda}{n})}}\,\hat{C}_{pv} \quad , \quad \sqrt{\frac{Q^2_{n,\lambda,1-\alpha/2}}{n(1+\frac{\omega\lambda}{n})}}\,\hat{C}_{pv} \right),$$

where $Q_{n,\lambda,\varepsilon}$ is the ε quantile of $Q_{n,\lambda}$.

Proof:

We have

$$1-\alpha = P\left[Q^2_{n,\lambda,\alpha/2} < Q^2_{n,\lambda} < Q^2_{n,\lambda,1-\alpha/2} \right]$$

$$= P\left[Q^2_{n,\lambda,\alpha/2} < \frac{n}{\sigma^2}\,[\,\hat{\sigma}^2 + \omega(\overline{X}-T)^2\,] < Q^2_{n,\lambda,1-\alpha/2} \right]$$

$$= P\left[\sqrt{\frac{\sigma^2}{n} Q^2_{n,\lambda,\alpha/2}} < \sqrt{\hat{\sigma}^2 + \omega(\overline{X}-T)^2} < \sqrt{\frac{\sigma^2}{n} Q^2_{n,\lambda,1-\alpha/2}} \right]$$

$$= P\left[\frac{USL - LSL}{6\sqrt{\frac{\sigma^2}{n} Q^2_{n,\lambda,1-\alpha/2}}} < \frac{USL - LSL}{6\sqrt{\hat{\sigma}^2 + \omega(\overline{X}-T)^2}} < \frac{USL - LSL}{6\sqrt{\frac{\sigma^2}{n} Q^2_{n,\lambda,\alpha/2}}} \right]$$

$$= P\left[\sqrt{\frac{n(1+\frac{\omega\lambda}{n})}{Q^2_{n,\lambda,1-\alpha/2}}}\,C_{pv} < \hat{C}_{pv} < \sqrt{\frac{n(1+\frac{\omega\lambda}{n})}{Q^2_{n,\lambda,\alpha/2}}}\,C_{pv} \right]$$

$$\text{Since } C_{pv} = \frac{USL - LSL}{6\sigma\sqrt{1 + \frac{\omega\lambda}{n}}}$$

$$= P\left[\sqrt{\frac{Q^2_{n,\lambda,\alpha/2}}{n(1 + \frac{\omega\lambda}{n})}}\, \hat{C}_{pv} < C_{pv} < \sqrt{\frac{Q^2_{n,\lambda,1-\alpha/2}}{n(1 + \frac{\omega\lambda}{n})}}\, \hat{C}_{pv} \right].$$

Thus

$$\left(\sqrt{\frac{Q^2_{n,\lambda,\alpha/2}}{n(1 + \frac{\omega\lambda}{n})}}\, \hat{C}_{pv} \;,\; \sqrt{\frac{Q^2_{n,\lambda,1-\alpha/2}}{n(1 + \frac{\omega\lambda}{n})}}\, \hat{C}_{pv} \right) \tag{2.5.2}$$

is a $100(1 - \alpha)\%$ confidence interval for C_{pv}.∎

For $\omega = 0$, $C_{pv} = C_p$ with confidence interval

$$\left(\sqrt{\frac{\chi^2_{n-1,\alpha/2}}{n}}\, \hat{C}_p \;,\; \sqrt{\frac{\chi^2_{n-1,1-\alpha/2}}{n}}\, \hat{C}_p \right), \tag{2.5.3}$$

which is same as (2.2.6), with replacing n - 1 by n.

Similarly, for $\omega = 1$, $C_{pv} = C_{pm}$ with confidence interval

$$\left(\sqrt{\frac{\chi^2_n(\lambda)_{,\alpha/2}}{n(1 + \frac{\lambda}{n})}}\, \hat{C}_{pm} \;,\; \sqrt{\frac{\chi^2_n(\lambda)_{,1-\alpha/2}}{n(1 + \frac{\lambda}{n})}}\, \hat{C}_{pm} \right) \tag{2.5.4}$$

Applying approximation due to Patnaik (1949), it follows that this is same as the confidence interval given in (2.4.7), with replacing $\tilde{C}_{pm}$ by $\hat{C}_{pm}$.

The confidence intervals of various indices discussed above have been constructed using the results related to the normal distribution. However, it is possible to construct the approximate confidence intervals with required level of significance without using the normality of the process measurements. In the following we discuss such intervals.

2.6 BOOTSTRAP CONFIDENCE INTERVALS FOR PCIs

Franklin and Wasserman (1992) have proposed three nonparametric bootstrap confidence intervals for each of the three indices, viz. C_p, C_{pk} and C_{pm}. Although the practical interpretations of the indices are questionable when normality does not hold, it has been found that in the normal process environment, one of the bootstrap confidence intervals performs comparable to the confidence intervals based on normality.

Let $X_1, \ldots, X_n$ be a sample of size n taken from a process. A bootstrap sample denoted by $X_1^*, \ldots, X_n^*$ is a sample of size n drawn with replacemnet from the original sample. There are a total of n^n such resamples possible. These resamples would then be used to calculate n^n values of $\hat{C}_p^*$, $\hat{C}_{pk}^*$ and $\hat{C}_{pm}^*$. Each of these would be an estimate of C_p, C_{pk} and C_{pm} respectively and the entire collectin would constitute the (complete) bootstrap distrubution for $\hat{C}_p$, $\hat{C}_{pk}$ and $\hat{C}_{pm}$.

Bootstrap sampling is equivalent to sampling with replacement from the empirical probability distrubution function. Thus, the bootstrap distrubution of $\hat{C}_p$, $\hat{C}_{pk}$ and $\hat{C}_{pm}$ are estimates of the distriutions of $\hat{C}_p$, $\hat{C}_{pk}$ and $\hat{C}_{pm}$. In practice, usually only a random sample of the n^n possible resamples is drawn, the statistic is calculated for each of these, and the resulting empirical distrubution is referred to as the bootstrap distribution of the statistic. A rough minimum of 1000 bootstrap resamples are usually sufficient to compute reasonably accurate confidence interval estimates.

We assume that 1000 bootstrap resamples are taken and 1000 bootstap estimates of C_p, C_{pk} and C_{pm} are calculated and ordered from smallest to largest. Let the generic notations $\hat{C}$ and $\hat{C}^*(i)$ deonte the estiamte of a capability index and the associated ordered bootstrap estimates respectively. The three proposed confidence intervals are as follows:

1. The Standard Bootstrap (SB)

From the 1000 boostrap estimates $\hat{C}^*(i)$ Calculate the sample average

$$\hat{C}^* = \frac{1}{1000}\sum_{i=1}^{1000} \hat{C}^*(i)$$

and the sample standard deviation

$$S_c^* = \sqrt{\frac{1}{999}\sum_{i=1}^{1000} [\hat{C}^*(i) - \hat{C}^*]^2}$$

The quantity S_c^* is actually an estimate of the standard deviation of $\hat{C}$ and thus, if the distribution of $\hat{C}$ is approximately normal, the $100(1 - \alpha)\%$ SB confidence interval for C is

$$\hat{C} \pm z_{1-\alpha/2} S_c^* \tag{2.6.1}$$

Note that the interval is centered at the value of $\hat{C}$ derived from the original data and the bootstrap method is only used to estimate its standard deviation.

2. The Percentile Bootstrap (PB)

Choose the $\alpha/2$ percent and $1-\alpha/2$ percent points of the distribution of $\hat{C}^*(i)$ as the end points of the confidence interval That is, choose.

$$[\ \hat{C}^*(1000\alpha),\ \hat{C}^*(1000(1-\alpha)\ ]$$

as the $100(1 - \alpha)\%$ approximate confidence interval for C. For a

90% confidence interval this would be

$$[\ \hat{C}^*(50),\ \hat{C}^*(950)\ ]$$

3. The Biased-corrected Percentile Bootstrap (BCPB)

It is possible that bootstrap distributins obtained using only a sample of the complete bootstrap distrubution may be shifted higher or lower than would be expected (that is, a biased distribution). Thus, a third method has been developed to correct for this potential bias. First using the ordered distribution of $\hat{C}^*$, calculate the probability

$$P_o = P\ [\hat{C}^* \le \hat{C}]$$

second, calculate

$$Z_o = \Phi^{-1}(P_o)$$

$$P_L = \Phi(2Z_o - z_{1-\alpha/2})$$

$$P_U = \Phi(2Z_o + z_{1-\alpha/2})$$

Finally, the BCPB confidence interval is

$$[\ \hat{C}^*(1000P_L),\ \hat{C}^*(1000P_U)\ ]$$

To compare the performance of the three proposed confidence intervals Franklin and Wasserman (1992) carried out a simulation when the underlying process is either normal, skewed or heavy tailed. Their simulation study had shown that when the underlying process is normal the SB method is superior. The other two methods are conservative in the sense that their actual coverage probabilities are less than the nominal value ($1 - \alpha$). If the underlying proecess is non-normal, it has been found that all the three methods are conservative. But since the practical interpretations of the indices are questionable when the process is non-normal, we needs not pay attention to this fact.

Our interest is now to compare the SB confidence interval to those based on normality discussed in the previous sections, in normal process environment. The performance of a confidence interval can be assessed using the two norms, namely, the extent to which the actual coverage probability meets the nominal value ($1 - \alpha$) and its average width. To compare the SB confidence intervals to the ones based on normality, we carried out a series of simulation with different parameter settings. For each parameter setting a sample of size $n = 30$ or 60 was drawn and 1000 bootstrap resamples (each of sige n) were drawn from that single sample. A 90% SB confidence interval was constructed and its width was measured. This single simulation was then repeated 400 times. Thus, we were able to calculate the proportion of times the SB confidence interval traps the true index value.

This proportion then provided an estimate of actual coverage probability of the SB confidence interval.

The procedure was repeated for each of three indices, viz, C_p, C_{pk} and C_{pm}. The simulation results for C_p, C_{pk} and C_{pm} are given in Table 2.6, Table 2.7 and Table 2.8 respectively.

Recall that the exact $100(1 - \alpha)\%$ confidence interval based on normality for the index Cp is given by (2.2.6). The expected width of this interval is

$$\left(\sqrt{\frac{\chi^2_{n-1,1-\alpha/2}}{n-1}} - \sqrt{\frac{\chi^2_{n-1,\,\alpha/2}}{n-1}} \right) E(\hat{C_p})$$

$$= \left(\sqrt{\frac{\chi^2_{n-1,1-\alpha/2}}{n-1}} - \sqrt{\frac{\chi^2_{n-1,\,\alpha/2}}{n-1}} \right) \frac{C_p}{b_n} \qquad (2.6.2)$$

where bn is as defined in (2.2.4). Using (2.6.2) we calculated the expected width of the 90% confidence interval of C_p given by (2.2.6), for each parameter setting and the values are given in Table 2.6. Observing Table 2.6 it follows that the average width of the SB confidence interval is negligibly greter than the expected width of the exact confidence interval. based on normality. Also the coverage proportions of the SB confidence intervals is very close to the nominal value 0.90.

In Section 2.3.2, we have seen that the most favourable approximate confidence interval based on normality for the index C_{pk} is the confidence interval given by (2.3.16) due to Bissell (1990). To compare the SB confidence interval for C_{pk} with this we again carried out a simulation with the same parameter settings to find the average width and coverage proportions of the Bissell's confidence intrval. For each parameter setting 400 random samples, each of size n = 30 or 60 were drawn. For each sample the Bissell's confidence interval was found out and its width was measured. Also the actual coverage proportion was estimated. The results are given in table 2.7 Observeing Table 2.7 it follows that the SB confidence interval for C_{pk} is as good as the confidence interval due to Bissell (1990).

The best available approximate confidence intervals based on normality, for the index C_{pm} are given by (2.4.7) and (2.4.8). To compare the SB confidence interval for C_{pm} with these intervals we again carried our a simulation with the same parameter settings to find the averge width and the actual coverage proportions of the confindence intervals (2.4.7) and (2.4.8). The result are given in Table 2.8. Observing Table 2.8, it follows that the performance of all the three intervals is almost identical.

Table 2.6: *Coverage Proportions and Average Widths of 90% Confidence Intervals for C_p.*

Confidence Interval	C_p	n = 30 coverage	n = 30 avg. width	n = 60 coverage	n = 60 avg. width
SB*	1.000	0.885	0.463	0.915	0.310
exact		0.900	0.442	0.900	0.306
SB	1.333	0.905	0.619	0.905	0.420
exact		0.900	0.589	0.900	0.408
SB	1.667	0.900	0.772	0.893	0.516
exact		0.900	0.736	0.900	0.510
SB	2.000	0.905	0.922	0.905	0.625
exact		0.900	0.883	0.900	0.612

* SB - *Standard Bootstrap.*

Table 2.7: *Coverage Proportions and Average Widths of 90% Confidence Intervals for C_{pk}.*

Confidence Interval	C_{pk}	n = 30 coverage	n = 30 avg.width	n =60 coverage	n =60 avg.width
SB	1.000	0.878	0 .455	0.880	0.314
Bissell	(μ=0, σ=1)	0.896	0.468	0.891	0.328
SB	1.333	0.918	0.660	0.815	0.441
Bissell	(μ=2, σ=0.25)	0.902	0.626	0.902	0.433
SB	1.667	0.905	0.799	0.888	0.543
Bissell	(μ=0.5, σ=0.5)	0.905	0.766	0.901	0.531
SB	2.000	0.888	0.888	0.880	0.616
Bissell	(μ=0, σ=0.5)	0.898	0.884	0.896	0.620

Table 2.8: *Coverage Proportions and Average Widths of 90% Confidence Intervals for C_{pm}.*

Confidence Interval	C_{pm}	n = 30 coverage	n = 30 avg.width	n = 60 coverage	n = 60 avg.width
	1.000 (μ=0, σ=1)				
SB		0.888	0.437	0.898	0.304
Normal 1*		0.894	0.434	0.896	0.303
Normal 2*		0.897	0.436	0.896	0.303
	0.496 (μ=2, σ=0.25)				
SB		0.898	0.036	0.928	0.026
Normal 1		0.880	0.036	0.878	0.026
Normal 2		0.880	0.036	0.879	0.026
	1.414 (μ=0.5, σ=0.5)				
SB		0.908	0.543	0.910	0.374
Normal 1		0.899	0.514	0.895	0.368
Normal 2		0.897	0.516	0.894	0.368
	2.000 (μ=0, σ=0.5)				
SB		0.895	0.871	0.913	0.614
Normal 1		0.900	0.869	0.904	0.607
Normal 2		0.900	0.873	0.903	0.608

* Normal 1 *and* Normal 2 *are the confidence intervals based on normality, given by (2.4.7) and (2.4.8) respectively.*

The simulation is carried out using MINITAB. The MINITAB macros are given in Appendix B.