Chapter IV

Nonparametric Quality Control Charts for Variability

4.1 Introduction :

To monitor both center of the distribution as well as variability $\overline{X}$ and R control charts will usually be used. When the chart for variability gives that the process is under control, then the chart for the center of the distribution such as mean, median are applied. The range R-chart is less efficient than the corresponding S^2 chart when the underlying distribution is normal. The S^2 chart for monitoring an increase in process variability gives an out of control signal if S^2 exceed the control limits $\hat{\sigma}^2\chi^2_\alpha$, where χ^2_α denotes the upper α percentage point of chi-square distribution with (n–1) d. f. and $\hat{\sigma}^2$ is an estimate of process variance. It is common practice to estimate σ from process data. This estimate is used for appropriate percentage point of the chi-square distribution in setting up the control limits. These limits can be used only when characteristic under study has normal distribution.

In case of non-normal data, effects of non-normality are more revere in control charts for variability than in case of charts for location. Thus here nonparametric charts for variability are needed.

Amin, Reynold and Bakir (1995) suggested NQCC for monitoring the increase in variance are based on signs computed within samples in the Shewhart charts given below.

4.2 NQCC based on Quartiles :

This control chart is based on first and third quartiles (Q_1 and Q_3), which are specified by process engineers or estimated from the process data when the process is in-control. Samples of n observations are taken at regular interval from the process. Let $X_i = (X_{i1}, X_{i2}, \ldots, X_{in})$ be sample taken at i^{th} sample point. At time t each observation be compared with Q_1 and Q_3.

Define,

$$U_{ij} = \begin{cases} 1 & \text{if } X_{ij} < Q_1 \text{ or } X_{ij} > Q_3 \\ 0 & \text{if } X_{ij} = Q_1 \text{ or } X_{ij} = Q_3 \\ -1 & \text{if } Q_1 < X_{ij} < Q_3 \end{cases} \qquad \ldots \qquad (4.1)$$

where X_{ij} is j^{th} observation in the i^{th} sample.

Further let $U_i = \sum_{j=1}^{n} U_{ij}$ be a statistic at i^{th} sample point, for i = 1, 2, ...

Then random variable $V_i = \frac{U_i + n}{2}$ follows a Binomial distribution with parameter n and p, where $p = P[\, X_{ij} \geq Q_3 \text{ or } X_{ij} \leq Q_1 \,]$.

Out-of-control signal is given if $V_i \geq c$, where V_i is the number of observed X_{ij}'s such that exceeds Q_3 plus the number of X_{ij}'s that fall below Q_1 in the i^{th} sample. This is relatively simple nonparametric control chart for which we need to determine Q_1 and Q_3, the in-control values of lower and upper quartiles.

The ARL values of Shewhart nonparametric charts for variability are compared to those of S^2 charts for normal, gamma, and double exponential distribution for n = 7. The ARL values of S^2 chart for non-normal distribution were obtained by simulation. The constant a_3 and ARL values are obtained by simulation based on 10000 runs, Amin, et. al. (1995).

The results are given in the Table 4.1 to Table 4.3 for various distributions.

Table 4.1 : *ARL values for Shewhart sign chart and S^2 charts for variability of Normal Distribution for n = 7 and c = 7.*

σ1/σ2	Sign Chart (Vi) : c = 7	S^2 chart a_3=2.905
1.0	128.0 (121.8)	128.0
1.1	74.9 (71.8)	39.4
1.2	48.6 (45.9)	16.8
1.3	34.2 (32.0)	8.9
1.4	25.4 (24.2)	5.6
1.5	19.8 (19.2)	3.9
1.6	16.0 (14.9)	3.0
1.7	13.2 (12.6)	2.4
1.8	11.2 (10.9)	2.0
1.9	9.7 (9.2)	1.8
2.0	8.6 (8.4)	1.6

Note : The figures in the bracket are simulated ARL values.

Table 4.2 : *ARL values for Shewhart sign chart and S^2 charts for variability of Gamma Distribution for n = 7 and c = 7.*

σ1/σ2	Sign Chart (Vi) : c = 7	S^2 chart a_3=2.905
1.0	128.0 (128.9)	125.5
1.1	72.6 (98.6)	50.0
1.2	44.4 (67.2)	24.5
1.3	29.0 (44.4)	14.5
1.4	21.1 (30.0)	9.6
1.5	14.7 (20.3)	6.8
1.6	11.1 (14.5)	5.3
1.7	9.1 (10.9)	4.4
1.8	7.7 (8.4)	3.7
1.9	6.7 (6.6)	3.2
2.0	6.0 (5.5)	2.8

Note : The figures in the bracket are simulated ARL values.

Table 4.3 : *ARL values for Shewhart sign chart and S^2 charts for variability of Double Exponential Distribution for $n = 7$ and $c = 7$.*

σ_1/σ_2	Sign Chart (Vi) : c = 7	S^2 chart a_3=9.0
1.0	128.0 (121.8)	126.6
1.1	82.4 (83.4)	60.3
1.2	57.0 (56.5)	33.4
1.3	41.8 (41.6)	20.6
1.4	32.0 (32.6)	13.5
1.5	25.4 (25.4)	9.5
1.6	20.8 (20.4)	7.0
1.7	17.4 (17.0)	5.6
1.8	14.8 (14.9)	4.5
1.9	12.9 (13.0)	3.8
2.0	11.3 (11.2)	3.2

Note : The figures in the bracket are simulated ARL values.

From Table 4.1 to Table 4.3 we observe that as far as detection of increase in variability is concerned S^2 based charts perform better than V_i based charts even in case of heavy tailed or asymmetric distribution.

4.3 Shewhart sign chart for process variability using quantiles

For Q_1 and Q_3 in-control ARL values of the S_{N_i} chart is fixed for given n. Therefore one cannot design control chart to give desired in-control ARL. Motivating from this fact we have proposed a general quantile based chart to detect increase in variability, which is given in the following.

Define,

$$U^*_{ij} = \begin{cases} 1 & \text{if } X_{ij} < \xi_p \text{ or } X_{ij} > \xi_{1-p} \\ 0 & \text{if } X_{ij} = \xi_p \text{ or } X_{ij} = \xi_{1-p} \\ -1 & \text{if } \xi_p < X_{ij} < \xi_{1-p} \end{cases} \quad \ldots \quad (4.2)$$

where ξ_p is lower p^{th} percentile of the process output in-control distribution. The chart is illustrated with p = 0.15 and is given below.

Using above definition of U^*_{ij} we define, U^*_i be the sign statistic as

$$U^*_i = \sum_{j=1}^{n} U^*_{ij}, \qquad \text{for } i = 1, 2, \ldots .$$

Then random variable $V^*_i = \frac{U^*_i + n}{2}$ follows a binomial distribution with parameters n and p, where $p = P[\,X_{ij} > \xi_{0.85} \text{ or } X_{ij} < \xi_{0.15}\,]$.

We define out of control criteria as a process gives a signal if $V_i^* \geq c^*$ where c* is to be chosen suitable. If we set n = 7 then out of control signal is given if 6 or 7 points fall outside $\xi_{0.15}$ or $\xi_{0.85}$ i.e. $V^*_i \geq 6$. This is also relatively simple nonparametric control chart for which we need to know, determine $\xi_{0.15}$ and $\xi_{0.85}$, the in-control values of lower and upper quantiles.

The ARL values of proposed Shewhart nonparametric chart for variability are simulated for normal and double exponential distribution for n = 7. The ARL values of S^2 chart for nonnormal distribution were obtained by simulation method.

Table 4.4 : *The ARL values for Shewhart chart for variability for* *Normal distribution* *when* $n = 7$ *using* $\xi_{0.15}$ *and* $\xi_{0.85}$, $c \geq 6$.

σ_1/σ_2	Sign chart	S^2 chart
1.0	263.8 (246.3)	263.8
1.1	118.2 (111.1)	69.7
1.2	63.0 (60.9)	26.5
1.3	38.0 (38.2)	12.9
1.4	25.2 (23.7)	7.5
1.5	17.9 (17.1)	5.0
1.6	13.4 (13.2)	3.6
1.7	10.5 (10.1)	2.8
1.8	8.5 (8.4)	2.3
1.9	7.1 (7.3)	2.0
2.0	6.1 (6.0)	1.8

Note : The figures in the bracket are simulated ARL values.

Table 4.5 : *The ARL values for Shewhart chart for variability for* ***Double Exponential distribution*** *when $n = 7$ using $\xi_{0.15}$ and $\xi_{0.85}, c \geq 6$.*

σ_1/σ_2	Sign chart	S^2 chart
1.0	263.8 (261.8)	263.8
1.1	142.5 (142.0)	113.1
1.2	85.7 (85.7)	59.4
1.3	56.0 (56.8)	36.0
1.4	39.0 (39.2)	24.0
1.5	28.7 (28.0)	17.5
1.6	21.9 (22.1)	13.5
1.7	17.3 (17.2)	10.8
1.8	14.1 (13.8)	9.0
1.9	11.7 (11.9)	7.7
2.0	9.9 (9.9)	6.8

Note : The figures in the bracket are simulated ARL values.

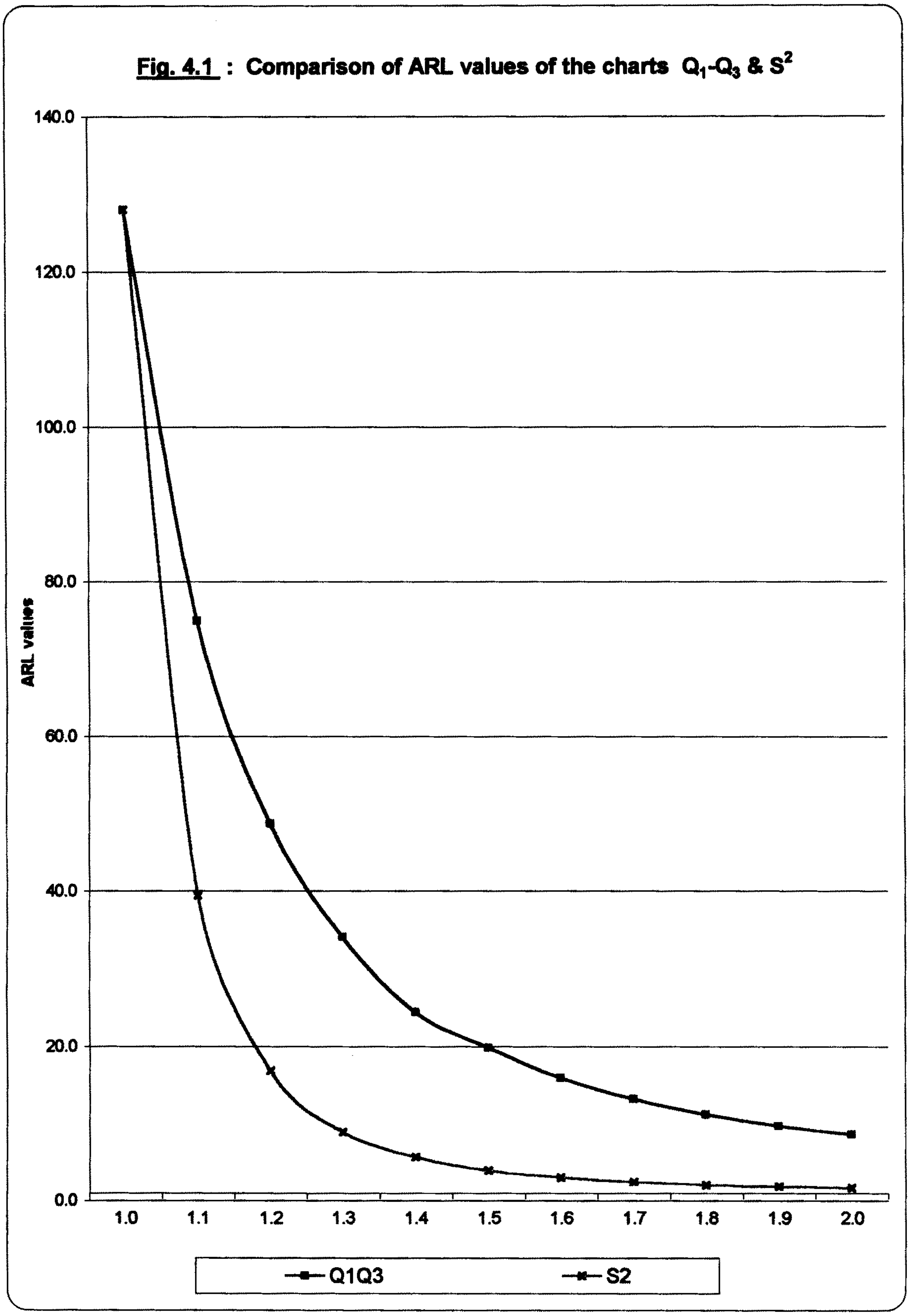
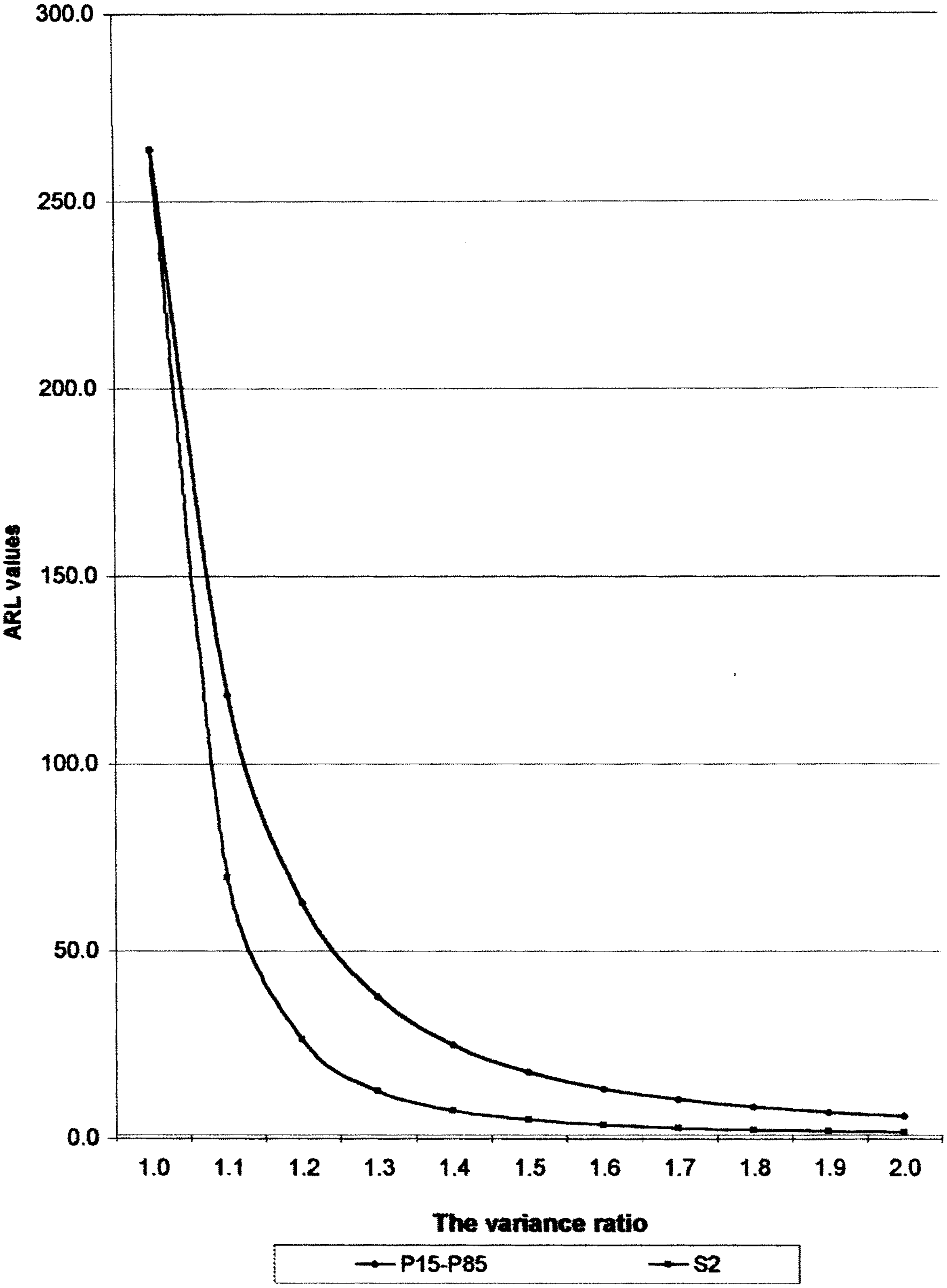


Fig. 4.1 : Comparison of ARL values of the charts Q_1-Q_3 & S^2
140.0
120.0
100.0
80.0
60.0
40.0
20.0
0.0
ARL values
1.0
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
2.0
Q1Q3
S2

Fig 4.2 : Comparision of ARL values of the charts $\xi_{0.15} - \xi_{0.85}$ & S^2



4.3.1 Conclusion :

From Table 4.4 and Table 4.5 and Fig. 4.1 and Fig. 4.2, we observe that ($\xi_{0.15}$, $\xi_{0.85}$) based chart improves over (Q_1, Q_3) based chart in the sense that, the difference in ARL of S^2 and ARL of ($\xi_{0.15}$, $\xi_{0.85}$) based charts is relatively smaller than that of (Q_1, Q_3) based chart.

To conclude the dissertation we have the following

(1) As far as detection in a shift is concerned SN_i charts perform better than the Shewhart's chart for heavy tailed and asymmetric distribution. Therefore use of the same is recommended in these situations. However for normal distribution or normal like light-tailed distributions the SN_i based chart is not encouraged to use.

(2) CUSUM charts based on SN_i-statistic can be used even in normal case as well as heavy-tailed distributions. However EWMA charts based on SN_i are not encouraged to use.

(3) As far as detection of increase in variation is concerned SN_i based charts are poor in their performance and there is no nonparametric competitor for S^2 based charts.

4.4 Future Study :

There is good scope to construct various distribution-free statistic and study the corresponding non-parametric control charts. One of the ways is to modify S_{N_i} as follows.

$$U_{ij} = \begin{cases} 1 & \text{if } X_{ij} > \mu_0 + k \\ 0 & \text{if } \mu_0 - k < X_{ij} < \mu_0 + k \\ -1 & \text{if } X_{ij} < \mu_0 - k \end{cases}$$

and denote modified S_{N_i} as

$$S_{N_i}^{k} = \sum_{j=1}^{k} U_{ij}$$

Motivation to the modification is to ignore closer X_{ij}'s and only include outside X_{ij}. Choice of 'k' will be an important aspect. S_{N_i}-chart discussed in chapter II is the case for k = 0. A study of optimal value of 'k' will be undertaken for various situations in future.