CHAPTER 5

AN ALTERNATIVE VIEW OF MINIMUM ABERRATION CRITERION

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5.1 INTRODUCTION

In the previous chapter, we have discussed the minimum aberration criterion for distinguishing between designs of the same maximum resolution. When there are two or more designs of the maximum resolution, the minimum aberration criterion picks a design with fewer words of the minimum length. If there are two or more designs having this property, then among these designs it picks up a design with the similar property for the next non - zero wordlength.

When there is no idea about the magnitude of any of the two - factor interactions, in other words, when there is no prior knowledge about the relative sizes of the two - factor interactions, the experimenter will be interested in estimating as many two - factor interactions as possible giving equal importance to all the two - factor interactions. Note that, as seen in the Chapter 4, if d_1 has less aberration than d_2, then under d_1 more two - factor interactions are expected to be estimated, because in d_1 fewer two - factor interactions are expected to be aliased with each other. Thus in the situation described above, the experimenter will prefer design d_1.

Cheng, Steinberg and Sun (1999) revealed an another interesting aspect of minimum aberration design that a minimum aberration design maximizes the number of two - factor interactions which are not aliased with main effects, consequently making a maximum number of main effects estimable. Also they observed that minimum aberration designs tend to make the values of the number of two factor interactions in various alias sets very uniform in some sense.

Details about these points are discussed in section 5.2. The two different criteria : (i) Estimation capacity (ii) the expected number of suspect two-factor interactions for assessment of model robustness are discussed in section 5.3. It is shown that minimum aberration designs are optimal for both of these criteria when the number of significant two-factor interactions is not large.

At the end, in Section 5.4, we present survey of current literatures.

5.2 AN ALTERNATE VIEW OF MINIMUM ABERRATION CRITERION

5.2.1 NOTATIONS AND ILLUSTRATION

In a 2^{k-p} design 'd' of resolution III or higher, there are $2^p - 1$ of the total $2^k - 1$ factorial effects in the defining relation. The remaining $2^k - 2^p$ effects are partitioned into $g = 2^{k-p} - 1$ alias sets each of size 2^p and k of these g alias sets contain main effects. Let $f = g - k$ be the number of alias sets which do not contain any main effects. Denote these alias sets by, $M_1, M_2, ..., M_f$, and the k alias sets containing main effects by $M_{f+1}, ..., M_g$. For $1 \leq i \leq g$, let $m_i(d)$ denote the number of two - factor interactions in M_i.

For each design 'd' of resolution III or higher, there are $3A_3(d)$ two - factor interactions which are aliased with main effects. This is because from each word of length 3 in the defining relation, there are $\binom{3}{2} = 3$ two - factor interactions which are aliases of main effects. (For example, if the word ABC is there in the defining relation, then the interactions AB, BC, AC respectively are aliased with the main effects C, A and B) Therefore, the total number of two - factor interactions which are aliased with main effects is given by,

$$\sum_{i=f+1}^{g} m_i(d) = 3A_3(d) \tag{5.1}$$

and hence, the number of two - factor interactions which are not aliased with main effects is,

$$\sum_{i=1}^{f} m_i(d) = \binom{k}{2} - 3A_3(d) \tag{5.2}$$

EXAMPLE 1 : Consider an example of a 2^{6-2} design of resolution III with the defining relation.

$$I = ABE = ACDF = BCDEF$$

Here $2^2 - 1 = 3$ factorial effects out of $2^6 - 1 = 63$ appear in the defining relation and the remaining $2^6 - 2^2 = 60$ effects are partitioned into $g = 2^{6-2} - 1 = 15$ alias sets each of size 2^2.

In this design out of $\binom{6}{2} = 15$ two-factor interactions, the three two-factor interactions AB, AE, and BE are aliased with the main effects E, B and A respectively. Among the remaining 12 two-factor interactions, AC is aliased with DF, AD is aliased with CF, and AF is aliased with CD. Therefore, the 12-two-factor interactions which are not aliased with main effects are partitioned into nine alias sets, three of which are of size two and each of the remaining six contains one single two-factor interaction.

In this case, $g = 15$, $f = 15 - 6 = 9$, and the values of $m_i(d)$'s are, $m_1(d) = m_2(d) = m_3(d) = 1$,(the corresponding two factor interactions are aliased with main effects) and $m_4(d) = m_5(d) = m_6(d) = 2, m_7(d) = m_8(d) = m_9(d) = ... = m_{12}(d) = 1$ (the corresponding two factor interactions are not aliased with main effects) and $m_{13}(d) = ... = m_{15}(d) = 0$. Here $A_3(d) = 1$ and the number of two-factor interactions that are aliased with main effects is equal to $\sum\limits_{i=f+1}^{g} m_i(d) = 3A_3(d) = 3$ and of those which are not aliased with main effects is, $\sum\limits_{i=1}^{f} m_i(d) = 12 = \binom{6}{2} - 3A_3(d)$ □

In the next section, we discuss an another aspect of minimum aberration design.

5.2.2 AN ANOTHER ASPECT OF MINIMUM ABERRATION DESIGN

In this section we demonstrate that a minimum aberration design maximizes the number of two-factor interactions which are not aliased with main effects, (equivalently minimizes the number of two factor interactions which are aliased with main effects) consequently making a maximum number of main effects estimable; subject to this condition it minimizes $\sum_{i=1}^{g}(m_i(d))^2$. In other words, minimum aberration designs tend to make the numbers $m_i(d)$'s very uniform and more concentrated around their mean. This main result is presented in Theorem 5.1.

The following lemma will be used in the proof of Theorem 5.1.

LEMMA 5.1:

For a 2^{k-p} design of resolution III or higher, the number of words of length 4 in the defining relation is given by,

$$A_4(d) = \frac{1}{6}\left[\sum_{l=1}^{g}(m_i(d))^2 - \binom{k}{2}\right] \tag{5.3}$$

PROOF :

Let $A_4(d,i,j)$ denote the number of words of length 4 in the defining relation which include both factors i and j. Since each word of length 4 includes $\binom{4}{2} = 6$ pairs of factors, we get

$$A_4(d) = \frac{1}{6}\sum_{i<j} A_4(d,i,j)$$

Note that, two two-factor interactions ij and lk are aliased with each other if and only if the word $ijlk$ of length 4 appears in the defining relation. Therefore, l^{th} alias set containing $m_l(d)$ two-factor interactions generates $m_l(d)$ pairs of factors (i,j) (the interaction ij in the alias set gives the pair (i,j)) for which $A_4(d,i,j) = m_l(d) - 1$ (obtained by taking generalized interaction of ij with the remaining $m_l(d) - 1$ two -

factor interactions in that alias set). Then, taking the sum of $A_4(d,i,j)$ over all the alias sets, we get

$$A_4(d) = \frac{1}{6} \sum_{i<j} A_4(d,i,j) = \frac{1}{6}\left[\sum_{l=1}^{g} m_l(d)(m_l(d)-1)\right]$$

$$= \frac{1}{6}\left[\sum_{l=1}^{g}(m_l(d))^2 - \sum_{l=1}^{g} m_l(d)\right]$$

Therefore, from (5.1) and (5.2), we get

$$A_4(d) = \frac{1}{6}\left[\sum_{l=1}^{g}(m_i(d))^2 - \binom{k}{2}\right]$$

□

As a consequence of equations (5.2) and (5.3), we have the following Theorem.

THEOREM 5.1 :

A minimum aberration design of resolution III or higher, maximizes $\sum_{i=1}^{f} m_i(d)$ (which is equal to the number of two factor interactions that are not aliased with main effects) and among the designs maximizing $\sum_{i=1}^{f} m_i(d)$, it mimimizes $\sum_{i=1}^{g}(m_i(d))^2$.

PROOF :

From equation (5.2),

$$\sum_{i=1}^{f} m_i(d) = \binom{k}{2} - 3A_3(d)$$

By definition, a minimum aberration design of resolution III or higher minimizes $A_3(d)$ (the number of words of length 3) hence from above identity it is clear that it maximizes $\sum_{i=1}^{f} m_i(d)$.

Further, subject to the condition that $A_3(d)$ is minimum, a minimum aberration design further minimizes the number of words of length 4, i.e. $A_4(d)$. Since from equation (5.3), we have,

$$A_4(d) = \frac{1}{6}\left[\sum_{l=1}^{g}(m_i(d))^2 - \binom{k}{2}\right],$$

it follows that it minimizes $\sum_{i=1}^{g}(m_i(d))^2$. □

REMARK :

[**1**] Since $\sum_{i=1}^{g} m_i(d) = \binom{k}{2}$ is a fixed number, minimizing $\sum_{i=1}^{g}(m_i(d))^2$ is equivalent to minimizing the variance among the numbers $m_i(d)$, $i = 1, 2, ..., g$. Thus, for minimum aberration designs, the numbers $m_i(d), i = 1, 2, ..., g$ will have smallest variance and hence are more 'uniform' and concentrated around their mean.

[**2**] In particular for a minimum aberration design of resolution IV, $\sum_{i=f+1}^{g} m_i(d) = 0$ (the number of two - factor interactions that are aliased with main effects), so that $\sum_{i=1}^{f} m_i(d) = \binom{k}{2}$.

We illustrate the above results with the help of the following example,

EXAMPLE 2

Let us consider an example of 2^{7-2} fraction. First we find a minimum aberration design of resolution R_{max} = IV for this fraction with the help of algorithm 4.4.1 discussed in section 4.4. Here the word length pattern should be of the form $W = \{w_1 \;\; w_2 \;\; w_{12}\}$. The word length pattern W of a minimum aberration design must satisfy the conditions (4.4) and (4.5) discussed in the section 4.3, namely, $\sum_{i=1}^{m} w_i = 2^{p-1}k$, where $m = 2^p - 1$ and either the $w's$ all are even or exactly 2^{p-1} of them are odd. The word length patterns satisfying these conditions are $W_1 = (4\ \ 5\ \ 5)$, $\ W_2 = (4\ \ 4\ \ 6)$. Here $m = 3$, $w_i \geq 4$ and $\sum_{i=1}^{m} w_i = 14$ for both W_1 and W_2. Further, since W_1 has less number of the words of length four, it is the best word length pattern of resolution IV. Thus the design d_1 having the word length pattern given by W_1, with the following defining relation is a minimum aberration design.

$$d_1 : I = ABCF = ABDEG = CDEFG \tag{5.2.1}$$

Now consider an another design d_2 with the defining relation

$$d_2 : I = ABCD = ABDEFG = CEFG \tag{5.2.2}$$

For these designs, the complete alias structure is given in Appendix C. For these designs, $g = 31, f = 24$.

The vectors $\underline{m}(d_1)$ and $\underline{m}(d_2)$ for these designs (where the i^{th} component of $\underline{m}(d_j), j = 1, 2$ is $m_i(d_j)$, the number of two factor interactions in the i^{th} alias set, $i = 1, 2, ..., 24$) are given by,

For design d_1, $\underline{m}(d_1) = (2\ 2\ 2\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0)$

For design d_2, $\underline{m}(d_2) = (2\ 2\ 2\ 2\ 2\ 2\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$

For these designs $\sum_{i=1}^{f} m_i(d)$ and $\sum_{i=1}^{g} (m_i(d))^2$ are respectively given in the following Table 5.1.

Table 5.1

	d_1	d_2
$\sum_{i=1}^{f} m_i(d)$	21	21
$\sum_{i=1}^{g} m_i(d)^2$	27	33

According to the above discussion, since design d_1 is a minimum aberration design, it should maximize $\sum_{i=1}^{f} m_i(d)$ and among the designs having the same maximum value of $\sum_{i=1}^{f} m_i(d)$, it should minimize $\sum_{i=1}^{g}(m_i(d))^2$. This expected behaviour of d_1 is reflected in the Table 5.1. i.e. $\sum_{i=1}^{f} m_i(d_1) = \sum_{i=1}^{f} m_i(d_2) = 21$ and $\sum_{i=1}^{g}(m_i(d_1))^2 < \sum_{i=1}^{g}(m_i(d_2))^2$. $\square$

The next section deals with some criteria for assessment of model robustness.

5.3 SOME CRITERIA FOR ASSESSMENT OF MODEL ROBUSTNESS

In this section, we discuss two - different criteria for assessing the idea of model robustness : i) Estimation Capacity ii) The expected number of suspect two - factor interactions. These concepts are discussed in more detail in the following subsections.

5.3.1 THE CONCEPT OF ESTIMATION CAPACITY

If all the effects included in a model are estimable under a design 'd', then the model is said to be estimated by the design 'd'. For any $1 \leq n \leq \binom{k}{2}$, let $E_n(d)$ denote the number of models containing all main effects and exactly n two-factor interactions that are estimated under design 'd'. Note that any two factor interaction that is aliased with a main effect is not estimable. In a design 'd', there are f alias sets which contain no main effect. From one such set at most one two factor interaction can be estimated. Therefore, at most f-two factor interactions are estimated under design 'd' and for $n > f$, $E_n(d) = 0$.

Let us try to enumerate the number of models containing all main effects and exactly n two factor interactions that can be estimated under design 'd'. Note that from one alias set (containing at least one two - factor interaction) only one two factor interaction can be estimated. Therefore, for estimating n $(n \leq f)$ two-factor interactions we need n alias sets. Consider the n alias sets namely, $i_1^{th}, i_2^{th}, ..., i_n^{th}$ set out of the f not containing main effects such that each of them contains at least one two factor interaction. We are to select one two - factor interaction from each of these alias sets. Note that the i_j^{th} alias set contains $m_{i_j}(d)$ two factor interactions. Therefore, there are $m_{i_j}(d)$ ways in which a two - factor interaction can be selected from the i_j^{th} alias set, $j = 1, 2, ..., n$. Hence the total number of models that are estimable under d, which include all main effects and exactly n - two factor interactions is given by,

$$E_n(d) = \sum_{1 \leq i_1 \leq \ldots \leq f} \prod_{j=1}^{n} m_{i_j}(d), \qquad if \;\; n \leq f \tag{5.4}$$

In particular, for $n = 1$, this gives,

$$E_1(d) = \sum_{i=1}^{f} m_i(d) = \binom{k}{2} - 3A_3(d) \qquad (by \;\; equation \;\; (5.2))$$

Note that, there are in all $\binom{k}{2} = \frac{\mathrm{k(k-1)}}{2} = u$ (say) two factor interactions. Out of these we can choose exactly n of them in $\binom{u}{n}$ ways. Thus, the total number of models that include exactly n two-factor interactions is $\binom{u}{n}$, and $E_n(d)$ out of them are estimable under d.

Thus,

$$E_n^{'}(d) = \frac{E_n(d)}{\binom{u}{n}} \tag{5.5}$$

is the proportion of models containing exactly n two-factor interactions that are estimable under d. The vector $\underline{E}^{'}(d) = (E_1^{'}(d), E_2^{'}(d), \ldots, E_n^{'}(d))$ is defined to be the 'Estimation Capacity' of the design 'd'.

Note that, for a design d, it is desirable that every component of the vector of estimation capacity should be as large as possible. Therefore, we have the following definition.

DEFINITION 5.1 :

For any two designs d_1, d_2, if $E_n^{'}(d_1) \geq E_n^{'}(d_2)$, for all n, with strict inequality for at least one n, then the design d_1 is said to dominate d_2 with respect to the estimation capacity. □

We discuss here a sufficient condition for a design d_1 to dominate an another design d_2. The concept of majorization discussed below, is used for developing the condition. A detailed discussion about majorization can be found in Marshall and Olkin (1979).

5.3.2 THE CONCEPT OF MAJORIZATION

DEFINITION 5.2:

A vector $\underline{x} = (x_1, x_2, ..., x_t)$ is said to be majorized by another vector $\underline{y} = (y_1, y_2, ..., y_t)$ if and only if $\sum_{i=1}^{t} x_i = \sum_{i=1}^{t} y_i$ and $\sum_{i=1}^{n} x_{[i]} \geq \sum_{i=1}^{n} y_{[i]}$, for all $1 \leq n \leq t-1$, where $x_{[1]} \leq x_{[2]} \leq ... \leq x_{[t]}$ and $y_{[1]} \leq y_{[2]} \leq ... \leq y_{[t]}$ are the ordered components of $\underline{x}$ and $\underline{y}$. This is denoted by $\underline{x} \prec \underline{y}$. □

e.g. Consider the two vectors $\underline{x} = (\frac{1}{2} \;\; \frac{1}{2} \;\; 0 \;\; 0 \;\; 0)$ and $\underline{y} = (1 \;\; 0 \;\; 0 \;\; 0 \;\; 0)$. From definition, we have $\sum_{i=1}^{5} x_i = \sum_{i=1}^{5} y_i = 1$. For $1 \leq n \leq 4$. The corresponding vectors with the components ordered in ascending order are $\underline{x}_o = (0 \;\; 0 \;\; 0 \;\; \frac{1}{2} \;\; \frac{1}{2})$ and $\underline{y}_o = (0 \;\; 0 \;\; 0 \;\; 0 \;\; 1)$. From this it is easy to verify that $\sum_{i=1}^{n} x_{[i]} \geq \sum_{i=1}^{n} y_{[i]} \Rightarrow \underline{x} \prec \underline{y}$.

DEFINITION 5.3:

For $\underline{x}$, $\underline{y} \in \Re^t$ $\left(\Re^t : \{(x_1, x_2, ..., x_t) : x_i \in \Re \;\; \forall i\}\right)$, $\underline{x}$ is said to be upper weakly majorized by $\underline{y}$, if and only if $\sum_{i=1}^{t} x_i \geq \sum_{i=1}^{t} y_i$ and $\sum_{i=1}^{n} x_{[i]} \geq \sum_{i=1}^{n} y_{[i]}$, for all $1 \leq n \leq t-1$. This is denoted by $\underline{x} \prec^{w} \underline{y}$. □

e.g. Consider the two vectors $\underline{x} = (\frac{1}{4} \;\; \frac{1}{4} \;\; \frac{1}{4} \;\; \frac{1}{4} \;\; \frac{1}{4})$ and $\underline{y} = (\frac{1}{5} \;\; \frac{1}{5} \;\; \frac{1}{5} \;\; \frac{1}{5} \;\; \frac{1}{5})$. Here, $\sum x_i = \frac{5}{4}$ and $\sum y_i = \frac{5}{5} = 1$, therefore from definition, we have

$\sum_{i=1}^{5} x_i \geq \sum_{i=1}^{5} y_i$ and for $1 \leq n \leq 4$, $\sum_{i=1}^{n} x_{[i]} \geq \sum_{i=1}^{n} y_{[i]} \Rightarrow \underset{\sim}{x} \prec^{w} \underset{\sim}{y}$.

DEFINITION 5.4:

A real valued function f of $\underset{\sim}{x}$ is called Schur Concave, if $\underset{\sim}{x} \prec \underset{\sim}{y}$ $\Rightarrow f(\underset{\sim}{x}) \geq f(\underset{\sim}{y})$. □

DEFINITION 5.5:

$S_n(x)$ is called the n^{th} elementary symmetric function of $x_1, x_2, ..., x_t$, if

$$S_n(x) = \sum_{i=1}^{t} \prod_{j=1}^{n} x_{i_j} \tag{5.6}$$

In particular $S_0(x) \equiv 1$ $\quad S_1(x) = \sum_{i=1}^{t} x_i \quad S_2(x) = \sum_{i}^{t} x_{i_1} x_{i_2}$ □

Before proving the main result, we list certain lemmas (without proof) from Marshall and Olkin (1979), which are used in the seqnel.

LEMMA 1: (cf. Theorem A.8, Marshall and Olkin (1979), p.59)

A real valued function f satisfies, $\underset{\sim}{x} \prec^{w} \underset{\sim}{y} \Rightarrow f(\underset{\sim}{x}) \geq f(\underset{\sim}{y})$ if and only if f is non - decreasing in each argument and Schur Concave. □

LEMMA 2: (cf. Proposition F.1, Marshall and Olkin (1979),p.78)

The function S_n (defined in (5.6)) is increasing and Schur Concave on $\Re^t_+ = \left(\Re^t_+ : \{(x_1, x_2, ..., x_t) : x_i \geq 0 \;\; \forall i\}\right)$. If $n \neq 1$, S_n is strictly Schur Concave on $\Re^t_{++} = \left(\Re^t_{++} : \{(x_1, x_2, ..., x_t) : x_i > 0 \;\; \forall i\}\right)$. □

5.3.3 CONDITION FOR DOMINANCE

In the following theorem, we present the main result of this section which gives a sufficient condition for a design d_1 to dominate an another design d_2.

THEOREM 5.2 :

If $\underline{m}(d_1)$ is upper weakly majorized by $\underline{m}(d_2)$ and $\underline{m}(d_1)$ can not be obtained from $\underline{m}(d_2)$ by permutting its components, then d_1 dominates d_2 with respect to the criterion of estimation capacity.

$$i.e. \quad \underline{m}(d_1) \prec^w \underline{m}(d_2) \Rightarrow E_n'(d_1) \geq E_n'(d_2) \ \ \forall \ \ n = 1, 2, ..., f$$

PROOF :

By Definition 5.5 and from equation (5.4), it follows that $E_n(d)$ is n^{th} elementary symmetric function of $\underline{m}(d) = (m_1(d), ..., m_f(d))$ and non-decreasing in each component of $\underline{m}(d)$, for all $n = 1, 2, ..., f$. Therefore by Lemma 2, the function $E_n(d)$ is Schur Concave.

Therefore from Lemma 1, $E_n(d)$ satisfies,

$$\underline{m}(d_1) \prec^w \underline{m}(d_2) \Rightarrow E_n(d_1) \geq E_n(d_2) \qquad \forall \ \ n = 1, 2, ..., f.$$

$$\Rightarrow E_n'(d_1) \geq E_n'(d_2) \qquad \forall \ \ n = 1, 2, ..., f. \qquad (by \ \ equation \ (5.5))$$

Hence by definition 5.1. d_1 dominates d_2. □

The following example illustrates the result of the above theorem.

EXAMPLE 3

We now reconsider the designs (5.2.1) and (5.2.2) discussed in the previous section 5.2.2. Here $g = 31$ and $f = 24$. The vectors $\underline{m}(d_i)$,

$i = 1, 2$ are

$\underline{m}(d_1) = (2\ 2\ 2\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0)$

$\underline{m}(d_2) = (2\ 2\ 2\ 2\ 2\ 2\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$.

Note that, here under design d_1, the 21 two factor interactions are distributed among 18 alias sets, fifteen of the alias sets contain a single two factor interaction and three of the alias sets contain two two-factor interactions each. Under design d_2, the 21 two factor interactions are distributed among 15 alias sets, nine of the alias sets contain a single two-factor interaction and six of the alias sets contain two two-factor interactions each.

We have $\sum_{i=1}^{24} m_i(d_1) = \sum_{i=1}^{24} m_i(d_2) = 21$. The corresponding ordered vectors $\underline{m}_o(d_1)$ and $\underline{m}_o(d_2)$ are,

$\underline{m}_o(d_1) = (0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 2\ 2\ 2)$

$\underline{m}_o(d_2) = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 2\ 2\ 2\ 2\ 2\ 2)$.

From this, it is easy to see that for $1 \leq n \leq 23$, $\sum_{i=1}^{n} m_{[i]}(d_1) \geq \sum_{i=1}^{n} m_{[i]}(d_2)$. Therefore by definition 5.3, $\underline{m}(d_1)$ is upper weakly majorized by $\underline{m}(d_2)$ i.e. $\underline{m}(d_1) \prec^{w} \underline{m}(d_2)$. Then from Theorem 5.2. we get,

$$E'_n(d_1) \geq E'_n(d_2) \quad \forall \;\; n = 1, 2, ..., 24. \qquad (5.3.1)$$

Therefore design d_1 dominates design d_2 with respect to the estimation capacity. The actual values of $E'_n(d)$ for the designs d_1 and d_2 are displayed in the Table 5.2.

Table 5.2

	d_1	d_2
$\sum_{i=1}^{f} m_i(d)$	21	21
$\sum_{i=1}^{g} m_i(d)^2$	27	33
$E'_1(d)$	1	1
$E'_2(d)$	0.9857	0.9714
$E'_3(d)$	0.9571	0.9143
$E'_4(d)$	0.9148	0.8311
$E'_5(d)$	0.8596	0.7268
$E'_6(d)$	0.7932	0.6086
$E'_7(d)$	0.7174	0.4851
$E'_8(d)$	0.6742	0.4000
$E'_9(d)$	0.5847	0.2569
$E'_{10}(d)$	0.4585	0.1665
$E'_{11}(d)$	0.3712	0.1658
$E'_{12}(d)$	0.2888	0.0497
$E'_{13}(d)$	0.2125	0.0367
$E'_{14}(d)$	0.1464	0.0066
$E'_{15}(d)$	0.0919	0.0012
$E'_{16}(d)$	to	$E'_{22}(d)$
	$= 0$	

Here we observe that $E'_n(d_1) \geq E'_n(d_2) \quad \forall \ n = 1, 2, ..., 24$, as expected.

Thus with the help of Theorem 5.2, two designs d_1 and d_2 can be compared based on the values of $m_i(d)$'s $i = 1, 2, ..., f$. Further using this condition inferior designs i.e. those which are dominated by others can be eliminated.

In liue of definition of weak majorization, Chen, Steinberg, and Sun further comment that a design has large estimation capacity if $\sum_{i=1}^{f} m_i(d)$ is as large as possible and the $m_i(d)$'s are as uniform as possible. Thus, Theorem 5.1 and 5.2 together imply that a minimum aberration design should have maximum estimation capacity.

In the next section we discuss an another criterion of the expected number of suspect two-factor interactions.

5.3.4 THE CONCEPT OF THE EXPECTED NUMBER OF SUSPECT TWO-FACTOR INTERACTIONS

In Section 5.3.1, we have discussed the concept of estimation capacity. Now we focus on an another criterion 'the expected number of suspect two-factor interactions.

In a 2^{k-p} fractional design d, there are $2^{k-p} - 1$ set of alias sets. If the magnitude of the contrast associated with an alias set is large, then it indicates that one or more effects belonging that particular alias set are significant. However, it can not be decided which of the effects belonging to that alias set is significant. Usually, the lower order effect belonging to the alias set is assumed to be significant effect. Under this assumption, main effects are more likely than two-factor interactions, two-factor interactions are more likely than three factor interactions and so on.

Suppose there is a two factor interaction which is aliased with a main effect. If the contrast associated with this alias set is significant then according to the above principle, we will declare that the main effect present in this alias set is significant. Even though in reality, the two factor interaction aliased with this main effect is significant, its significance can not be identified in this situation. This two-factor interaction will be said to be a "suspect" two-factor interaction. Similar thing can happen if more than one two-factor interactions (and no main effect) are aliased with each other and the contrast associated with this alias set is significant. In such situation, it can not be identified that which of them is significant. Then all the two factor interactions belonging to this alias set will be called as "suspect".

DEFINITION 5.6

If the two factor interactions that are not clear, that is, they are either aliased with main effect or with another two factor interactions and after analysis the contrast associated with this alias set turns out to be significant, then that two factor interactions are called as suspect two factor interactions. □

Next, we obtain the expected number of suspect two-factor interactions in a 2^{k-p} fractional design.

The Expected Number Of Suspect Two-Factor Interactions

In a 2^{k-p} fractional design d, there are in all $\binom{k}{2} = \frac{k(k-1)}{2} = u$ (say) two factor interactions in t-alias sets. Suppose that there are exactly n-significant two-factor interactions. Let l^{th} alias set contains m-two factor interactions and no main effect. The remaining $(u - m)$ two factor interactions are distributed in the $(t-1)$ alias sets. Let $P(m,n)$ denotes the probability that the contrast associated with l^{th} alias set with m-two factor interactions is large (significant). Then $P(m,n)$ is given by,

$$P(m,n) = 1 - \Pr\Big[\text{the contrast associated with this alias set with } m\text{-two factor interactions is not large}\Big]$$

$$= 1 - P(A) \qquad \text{(say)}$$

Note that the event A means that the n- significant two factor interactions all come from the remaining $(u - m)$ two factor interactions distributed in other $(t - 1)$ alias sets. Noting that , there are in all u two factor interactions, out of these we can choose n-two factor interactions, in $\binom{u}{n}$ ways and out of $(u - m)$ two factor interactions, we can choose n- significant two factor interactions, in $\binom{u-m}{n}$ ways, the ratio $\binom{u-m}{n}\Big/\binom{u}{n}$ is the probability that the n significant two factor interactions belong to the set of remaining $(u-m)$ two factor interactions distributed in $(t-1)$ alias sets. In other words, it is the probability that the contrast associated with l^{th} alias set with m-two factor interactions is not large. Then

$$
\begin{aligned}
P(m,n) &= 1 - \frac{\binom{u-m}{n}}{\binom{u}{n}} \\
&= 1 - \left\{ \frac{(u-n)(u-n-1) \ \ \dots \ \ (u-n+m+1)}{u(u-1) \ \ \dots \ \ (u-m+1)} \right\} \\
&= 1 - \left(1 - \frac{n}{u}\right)\left\{1 - \frac{n}{(u-1)}\right\} \ \ \dots \ \ \left\{1 - \frac{n}{(u-m+1)}\right\}
\end{aligned}
\tag{5.7}
$$

Let the expected number of suspect two factor interactions in a design d when it contains exactly n significant two factor interactions, be denoted by $S_n(d)$. Note that, for each design d of resolution III or higher, there are $3A_3(d)$ two factor interactions which are aliased with main effects. Also there are f alias sets that do not contain any main effect but have one or more two factor interactions. According to the definition 5.6, if any of these contrasts turns to be significant then all the m-two factor interactions belonging to that set should be declared as suspect interactions. Therefore $S_n(d)$ is given by,

$$S_n(d) = 3A_3(d) + \sum_{i=1}^{f} m_i(d) P(m_i(d), n) \tag{5.8}$$

Provided that the fraction (n/u) significant two-factor interactions is not too large, $P(m,n) \approx \frac{mn}{u}$. The proof is given in the Appendix C. Then,

$$S_n(d) \approx 3A_3(d) + \frac{n}{u} \sum_{i=1}^{f} m_i(d)^2 \tag{5.9}$$

This result is exact if $n = 1$. i.e.

$$S_1(d) = 3A_3(d) + \frac{1}{u} \sum_{i=1}^{f} m_i(d)^2$$

Thus from Theorem 5.1, a minimum aberration design minimizes the expected number of two-factor interactions when n is small.

Therefore with respect to the expected number of suspect two-factor interactions, we have the following definition for the dominance of design d_1 and d_2.

DEFINITION 5.7

If $S_n(d_1) \leq S_n(d_2)$ for all n with strict inequality for at least some n, then a design d_1 is said to dominate d_2 with respect to suspect two-factor interactions. □

The following theorem gives a sufficient condition for d_1 to dominate d_2 with respect to the expected number of suspect two-factor interactions.

THEOREM 5.3

For each design d, let $a(j, d)$ denote the number of alias sets that have j two factor interactions and no main effects. If

$$\sum_{j=1}^{h} ja(j, d_1) \geq \sum_{j=1}^{h} ja(j, d_2)$$

for all $h = 1, 2, ..., w$, where w is the size of the largest alias set under either design, and strict inequality holds for at least one h, then d_1 dominates d_2 with respect to the expected number of suspect two-factor interactions.

PROOF Suppose that there are n significant two-factor interactions.

Then by equation (5.8) the criterion value for design d_i, $i = 1, 2$ is

$$S_n(d_i) = 3A_3(d_i) + \sum_{j=1}^{h} j a(j, d_i) P(j, n),$$

where $P(j, n)$ is defined in equation (5.7). Constructing a vector $V(d_i)$ for $i = 1, 2$, of length u such that the first $a(1, d_i)$ entries are $P(1, n)$, the next $2a(2, d_i)$ entries $P(2, n)$, the next $3a(3, d_i)$ entries $P(3, n)$, etc. and the last $3A_3(d_i)$ entries are all 1. Then $S_n(d_i)$ is the sum of all the entries of $V(d_i)$. Under the given conditions, since $P(j, n)$ is increasing in j, the entries $V(d_2) - V(d_1)$ are all non-negative and at least one is non-zero. Hence $S_n(d_1) \leq S_n(d_2)$ for all n with strict inequality for at least some n. $\square$

REMARK :

1. $\sum_{j=1}^{w} j a(j, d_i) = \frac{\kappa(\kappa - 1)}{2} - 3A_3(d_i)$, since for a design d_i of resolution III or higher, there are in all $\frac{\kappa(\kappa-1)}{2}$ two factor interactions and $3A_3(d_i)$ two factor interactions which are aliased with main effects. Hence the two-factor interacticns which are not aliases of main effects are given by this equation.

EXAMPLE 4

We now reconsider example 5.3.1 with the designs (5.2.1) and (5.2.2) discussed in Section 5.2.2. Here $g = 31$ and $f = 24$. The vectors $\underline{m}(d_1)$ and $\underline{m}(d_2)$ for these designs (where the i^{th} component of $\underline{m}(d_j)$, $j = 1, 2$ is $m_i(d_j)$, $j = 1, 2, i = 1, 2, ..., 24$) are given by $\underline{m}(d_1) = (2\ 2\ 2\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0)$

$\underline{m}(d_2) = (2\ 2\ 2\ 2\ 2\ 2\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$.

The expected number of suspect two-factor interactions $S_n(d)$ by equation (5.10) for design d_1 and d_2 are displayed in the Table 5.3.

Table 5.3

	d_1	d_2		d_1	d_2
$S_1(d)$	1	2	$S_{11}(d)$	13	14
$S_2(d)$	3	3	$S_{12}(d)$	14	15
$S_3(d)$	4	5	$S_{13}(d)$	14	16
$S_4(d)$	5	6	$S_{14}(d)$	15	17
$S_5(d)$	6	7	$S_{15}(d)$	16	18
$S_6(d)$	7	9	$S_{16}(d)$	17	18
$S_7(d)$	8	10	$S_{17}(d)$	18	19
$S_8(d)$	9	11	$S_{18}(d)$	19	20
$S_9(d)$	11	12	$S_{19}(d)$	20	20
$S_{10}(d)$	12	13	$S_{20}(d)$	20	21
$S_{21}(d)$	21	21			

Here we observe that $S_n(d_1) \leq S_n(d_2) \quad \forall \ n = 1, 2, ...24$ as expected. Thus the design d_1 dominates design d_2 with respect to the expected number of two factor interactions.

In the next section , we present the summary of current literatures surveyed.

5.4 SURVEY OF CURRENT LITERATURE

In this section, we summarize the current available literatures regarding the minimum aberration criterion. Due to lack of time and space, we could not discuss these results in detail. First we give an outline overall of the reviewed literatures and in the next section give a detail summary of each article reviewed.

5.4.1 OUTLINE

As discussed in section 4.3., when there is a very little aprori knowledge about the relative sizes of factorial effects, a minimum aberration criterion selects designs with good over all properties. Franklin (1984) extended this criterion to s^{k-p} fractional designs, where s is prime power. Also he presented some tables of minimum aberration designs. Chen and Wu (1991) constructed minimum aberration 2^{k-p} designs for $p \leq 4$ and found that such designs have a periodicity property when k is large and p is fixed. Theoritical characterizations for minimum aberration designs were obtained by Chen (1992) for $s = 2, p = 5$. Using an efficient computational algorithm, Chen, Sun and Wu (1993) compiled a catalogue of regular fractions which are good under the criterion of minimum aberration. This catalogue incorporates in particular, minimum aberration designs for $s = 2$ and 3. H. Chen and Hedayat (1996) proposed the weak minimum aberration criterion i.e. a modified version of minimum aberration criterion and obtained some interesting results.

In several other situations, the theoretical study of minimum aberration can be facilitated by expressing the wordlength pattern of regular fraction in terms of the complementary set. These concepts are developed by Chen and Hedayat (1996), Tang and Wu (1996) and Suen et.al.(1997), Wu and Zang (1993) extended the idea of minimum aberration designs to the method of grouping. Using this approachSuen, Chen and Wu (1997) constructed several families of 2^{k-p} designs with minimum aberration.

5.4.2 SUMMARY OF LITERATURE SURVEYED

[1] **Franklin M. F.**, " Constructing Tables of Minimum Aberration p^{n-m} Designs", Technometrics .

Franklin (1984) presents some simple results that enable to improve the Fries and Hunter algorithm so that they are suitable for a wider range of k and p. Some of these results are applied to general s^{k-p} designs, s is prime ≥ 2 and therefore they enable the algorithm to be extended beyond two-level designs. He defined a class of minimum variance designs (i.e. a design which maximizes first moment and minimizes the second moment) and also optimal moments designs belonging to such class. Further, he suggested that minimum aberration designs have minimum variance among the defining contrast word lengths. Using this information he presents generators for a wide range of 2^{k-p} designs that have minimum aberration or optimal moments and also for designs with factors at three levels. He outlined a procedure for determining confounded effects.

Here, we focus on a criteria for selecting best designs, termed as optimal moments and a step by step procedure to select a best design.

To choose a good design it is important to minimize the number of defining contrasts of smaller lengths. Franklin (1984) suggested the following procedure for selecting a best design.

Step 1 : First, consider the designs with same maximum resolution R, from these designs select those which have the smallest mean length $\bar{w}$.

Step 2 : From these, select those designs for which the variance of w_i is minimized.

Step 3 : Among the designs satisfying the conditions of Step 1 and 2 above, further , select those designs for which the w_i have maximum positive skewness.

Continue in this manner as far as necessary, maximizing odd moments and minimizing even moments. so he defines the following criterions.

DEFINITION 1: Let d_1 and d_2 be two s^{k-p} fractional factorial designs. If $m = r$ is the first moment such that $M_r(d_1) \neq M_r(d_2)$ and $M_r(d_1) > M_r(d_2)$ then d_1 is better than d_2 when r is odd but d_2 is better than d_1 when r is even. If no other s^{k-p} fractional factorial design has better moments than d_1, then d_1 is said to have optimal moments. □

DEFINITION 2 : A s^{k-p} fractional factorial design is a minimum variance design if it maximizes the first moment $M_1(d)$ and minimizes the second moment $M_2(d)$. □

He derived formula for r^{th} moment as,

$$\begin{aligned} M_r(d) &= \sum_{j=1}^{\infty} j^r A_j(d) \\ &= \sum_{r} {w_i}^r, \qquad r = 1, 2, \ldots \end{aligned}$$

where $A_j(d)$ be the number of words of length j in the word length pattern $W(d)$.

AN EXAMPLE : Let d_1 and d_2 be 2^{7-2} designs of resolution $R = IV$ with the defining relation.

$$\left.\begin{aligned} d_1 &: I = ABCF = ADEG = BCDEFG \\ d_2 &: I = ABCDF = ABCEG = DEFG \end{aligned}\right\} \qquad (a)$$

The corresponding word length pattern is given as,

$W(d_1) = (0 \;\; 2 \;\; 0 \;\; 1 \;\; 0)$ and $W(d_2) = (0 \;\; 1 \;\; 2 \;\; 0 \;\; 0)$.

Since $A_4(d_1) > A_4(d_2)$, then by definition of minimum aberration criterion, the design d_2 has less aberration than d_1.

The magnitude of moments for designs d_1 and d_2 are,

$M_1(d_1) = 3, \quad M_2(d_1) = 68$ and $M_1(d_2) = 3, \quad M_2(d_2) = 66$

From definition, $M_2(d_1) > M_2(d_2)$, hence d_2 has better moments than d_1.

Thus, from both criteria, minimum aberration and optimal moments, design d_2 in (a) is the optimal 2^{7-2} design.

[2] **Chen and Wu (1991)**, "Some results on s^{n-k} fractional factorial designs with minimum aberration or optimal moments" , The Annals of Statistics.

Chen and Wu (1991) studied some periodicity properties of maximum resolution , minimum aberration criterion and optimal moments for s^{k-p} designs with large k and any fixed p.

Let $R_s(k,p)$ denote the maximum resolution of an s^{k-p} design. Define the $m - lag(\underset{\sim}{W}, m) = (\underbrace{0\ 0\ 0\ldots\ 0}_{m}, \underset{\sim}{w})$ where $\underset{\sim}{w}$ is preceded by m zeros. Chen and Wu (1991) studied the following properties :

A : <u>Periodicity property of maximum resolution</u> :

(i) For any s^{k-p} fractional factorial design d_1 with the word length pattern $\underset{\sim}{W}_1$, there exists an $s^{(k+(s^p-1)/(s-1))-p}$ design d_2 with the word length pattern $\underset{\sim}{W}_2$, such that $\underset{\sim}{W}_2 = lag(\underset{\sim}{W}, s^{p-1})$.

(ii) $R_s((k + (s^p - 1)/(s-1)) - p, p) = R_s(k,p) + s^{p-1}$.

(iii) For any fixed p, there exists a positive integer N_p, such that for $k > N_p$, $R_s(*, p) = R_s(k,p) + s^{p-1}$, $* = k + (s^p - 1)/(s-1)$?

B : <u>Periodicity property of minimum aberration criterion</u> :

(i) For any fixed p, such that there exists a positive integer M_p, such that for $k \geq N_p$, the minimum aberration property is periodic, that is, if a minimum aberration s^{k-p} design has wordlength pattern $\underset{\sim}{W}$, then there exists a minimum aberration design $s^{(k+(s^p-1)/(s-1))-p}$ with the word length pattern $lag(\underset{\sim}{W}, s^{p-1})$.

C : Periodicity property of Optimal moments :

(i) For any optimal moments [Franklin (1984)] s^{k-p} design d_1, there exists an optimal moments $s^{(k+(s^p-1)/(s-1))-p}$ design d_2, which has the same central moments as d_1.

Chen and Wu (1991) comment that the periodicity property of resolution does not hold for $p \geq 5$. Further, they give a characterization of a minimum variance s^{k-p} design. A minimum variance design is a special case of an optimal moments design.

[3] **J. Chen (1992)** "Some results on 2^{n-k} fractional factorial designs and search for minimum aberration designs", The Annals of Statistics.

Chen (1992) obtained minimum aberration 2^{k-p} designs for $p = 5$ and for any k. He suggests the following new method to present a defining relation. Using this new presentation, he obtains some properties of 2^{k-p} fractional factorial designs.

Method :

First construct a matrix $'H'$ as ,

$$H = \begin{pmatrix} I_p & B \\ B^t & B^tB \end{pmatrix}$$

where I_p is $p \times p$ identity matrix, b is a $p \times (2^p - p - 1)$ matrix containing all distinct and non-zero linear combinations (modulo 2) of column vectors of I_p , B^t is transpose of B. In matrix H , when 0's are replaced by -1 and 0 row and 0 column are added then a Hadamard matrix of order 2^p is obtained whose rows form a group under summation modulo

2 and a similar result holds for columns.

Denote the rows of H by $u_1, u_2, ..., u_{2^p-1}$. To define 2^{k-p} fractional design, divide k letters into $2^p - 1$ sets, namely $t_1, t_2, ..., t_{2^p-1}$. Let f_i be the number of letters in the i^{th} set such that $\sum\limits_{i=1}^{2^p-1} = k$. Then each row vector $\underline{u}_j$, $j = 1, 2, ...2^p - 1$ of matrix H forms a word w_j. The word w_j is obtained by combining all the letters in those sets for which the components of u_j is equal to one. Then (H, f) denotes a design, where $f = (f_1 \ \ f_2 \ \ ...f_{2^p-1})$ is the frequency vector of the design.

For example : Consider a 2^{5-2} fractional design with $k = 5$ factors namely, $A, B, C, DandE$, $p = 2$ and $2^p - 1 = 3$. Then the matrix H is ,

$$H = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

The five letters are divided into 3 sets t_1, t_2, t_3. Let us assign A, B to t_1, C, D to t_2 and E to t_3 respectively. Therefore the frequency vector $f = (2 \ \ 2 \ \ 1)$ which satisfies $\sum f_i = k = 5$. Let consider a row vector $u_1 = (1 \ \ 0 \ \ 1)$ which forms a word $W_1 = ABE$, similarly from other rows, we get $W_2 = CDE$ and $W_3 = ABCD$. Hence the defining relation for this design is, $I = ABE = CDE = ABCD$.

With this representation , they found a simple expression for the moments of 2^{k-p} designs d. The m^{th} moment of a 2^{k-p} design is given as,

$$M_m = \| \sum_j f_j v_j \|_m$$

where $\| v \|_m = \sum_i v_i^m$, $v = (v_1 \;\; v_2 \;\; ...v_{2^p-1})^t$, v_j are column vectors pf H.

He proves that "a 2^{k-p} fractional factorial design $(H, \underline{f})$ has minimum variance if and only if $f_j = 1$ or $q+1$ for all j, where q is determined by $k = q(2^p - 1) + r)$, where $0 \leq r < 2^p - 1$.

They also obtained the following properties of the second moment of a 2^{k-p} fractional factorial design.

1. " For any 2^{k-p} fractional factorial design, its second moment is divisible by 2^{p-1}".

2. " Let v be the second moment of a minimum variance 2^{k-p} design, $M_2 = 2^{p-2}\Big[\sum_j f_j^2 + k^2\Big]$ be the second moment of any 2^{k-p} design d and let

$$M_2 - v = m2^{p-1}$$

Then the length L of the longest word of d satisfies $L \leq q2^{p-1} + r' + m$ where $k = q(2^p - 1) + r)$, where $0 \leq r < 2^p - 1$ and $r' = min\{r, 2^{p-1}\}$".

3. Any 2^{k-p} fractional factorial design with minimum aberration is uniquely determined by its wordlength pattern when $p = 3, 4$.

Furthermore, he suggests a method to test the equivalence of fractional factorial designs and proves that minimum aberration designs for $p \leq$ are unique. A relation between 2^{k-p} and $2^{(k+1)-p}$ designs is given below. "Suppose a 2^{k-p} fractional factorial design d_1 has resolution R, and $A_r(d_1)$ is the first non-zero component of its wordlength pattern. Then there exists a $2^{(k+1)-p}$ design d_2 with $A_r(d_2) < (1/2)A_r(d_1)$, and

$A_r(d_2)$ is the first possible non-zero component of its word length pattern".

[4] **H. Chen and S. Hedayat (1996)** " 2^{n-l} designs with weak minimum aberration ", The Annals of Statistics.

In (1996) Chen and Hedayat proposed the weak minimum aberration criterion which is a modified version of the minimum aberration criterion. They construct 2^{k-p} fractional factorial designs of resolution with weak minimum aberration by using finite geometry. Also they obtained several families of 2^{k-p} fractional factorial designs of resolution III and IV with minimum aberration criterion.

The concept of weak minimum aberration is a natural and useful modification of minimum aberration and is defined below.

DEFINITION : A 2^{k-p} fractional factorial design with maximum resolution R_{max} is said to have a weak minimum aberration if it has the minimum number of words of length R_{max}.

They also studied the relationship of word length patterns between fractional factorial designs and their complementary designs in the whole factorial.

[5] **WU^1, H and WU^2, C.F.J. (2002)** "Clear Two - Factor Interactions And Minimum Aberration ", The Annals of Statistics.

H. Wu and C.F. Wu (2002) developed a method to examine whether a given design is a MaxC2 design where a MaxC2 design is a design containing maximum number of clear two-factor interactions (i.e. those two factor interactions which are not aliased with any main effects or other two-factor interactions). In particular they proved that all minimum aberration designs with resolution IV are MaxC2 designs (except in six particular cases given by them). First they develop a graphical representation and classification of length 4-words. With this representation of designs , they obtained bounds for the number of clear two-factor interactions and reduced the search for designs to a much smaller set. Together with combinatorial and group theoretic arguments , they proved some known and some new designs to be MaxC2 designs.