CHAPTER 1

FACTORIAL EXPERIMENTS

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1.1 INTRODUCTION

When many factors influence a character under study, it is always desirable to test different combinations of factors at various levels. Such experiments are called factorial experiments. Factorial experiments are widely used to study the joint effect of many factors on a response.

Prior to 1926, a factorial experiment was called as a 'complex experiment'. Fisher and Yates have developed and anaylzed the factorial experiments. Besides them , Barnard (1936), Bose (1938, 1942, 1947 etc.) Cochran and Cox (1959), Kempthrone (1952), Fedrer (1955) among the others have contributed significantly. A systematic development of this topic is available in the books by Kempthrone (1952), Fedrer (1955), Cochran and Cox (1959), John (1971), Ogawa(1981), Das and Giri (1979), Fedrer, Hedayat and Raktoe (1981), Montogomory (1984), Wu and Hamada (2000).

In full factorial experiments, all combinations of levels of all factors are considered as the treatments. If many single factor experiments are carried out varying the levels of only one factor at a time and keeping the levels of other factors fixed, it will not be possible to estimate interactions among different factors. Also the procedure is more expensive and time consuming and requires more resources. On the other hand, factorial experiments are very economical, speedy and provide a more reliable result with less experimental material and enable to estimate the interactions between the factors.

If all the factors have same number of levels, then it is called a symmetric factorial experiment, otherwise it is called an asymmetric factorial experiment. If there are k factors each at 2 levels then it is called by a "2^k experiment".

In general for a s^k factorial experiment, there are k factors each at s levels. The levels are denoted by the integers 0,1,2,. . ., $s-1$ and the factors are labeled A,B,C, . . . or 1,2,3, ... The treatment combinations consisting of the level x_1 of A, the level x_2 of B, the level x_3 of C, . . . is denoted by $a^{x_1}\ b^{x_2}\ c^{x_3}\ d^{x_4}$. . . or by $(x_1\ x_2\ x_3 \ldots x_k)$, where $x_1\ x_2\ x_3 \ldots$ are all integers taking values between 0 and s-1. If all x_i's , $i = 1, 2, ..., k$ are zero, then the treatment combination is denoted by 1. In all there are s^k treatment combinations.

Various factorial effects to be defined in the next section are denoted by $A^{\alpha_1}\ B^{\alpha_2}\ C^{\alpha_3}\ D^{\alpha_4} \ldots$ .. where α_i is the exponent of i^{th} factor in the effect and α_i's are all integers between 0 and $s-1$. For example , in a 3^2 experiment, there are 2 factors each at three levels 0,1,2 and the treatment combinations are denoted by ,

$$1 \quad a \quad a^2 \quad b \quad ab \quad ab^2 \quad b^2 \quad a^2b \quad a^2b^2$$

or

$$00 \quad 01 \quad 02 \quad 10 \quad 11 \quad 12 \quad 20 \quad 21 \quad 22.$$

Various factorial effects studied are the main effects A, B and the two-factor interactions AB , AB^2.

In a s^k experiment there are $s^k - 1$ different effects and interactions. These are nothing but some well defined treatment contrasts. The analysis of a factorial experiment is basically concerned with testing significance of these treatment contrasts.

In this chapter we present definitions of these contrasts and their tests of significance. In section 1.2, we discuss the analysis of full factorial experiments at two levels and three levels. The section 1.3 deals with the technique of confounding and the construction of confounded factorial experiments for 2^k and 3^k factorial experiments. In section 1.5 we present a chapterwise summary.

In the next section, we describe in brief the analysis of a full factorial experiment.

1.2 ANALYSIS OF FULL FACTORIAL EXPERIMENT

1.2.1. 2^K FULL FACTORIAL EXPERIMENT

In a 2^k full factorial experiment, the treatment sum of squares carries 2^k-1 degrees of freedom. This is further partitioned into the sum of squares due to 2^k-1 mutually orthogonal treatment contrasts which are interpreted as main effects and interaction contrasts. These contrasts are explained below, first with the help of a 2^2 experiment and then generalized.

Main Effects And Interaction Contrasts

Let us consider a 2^2 factorial experiment. There are 4 treatments i.e. 1, a, b and ab with abuse of notation, these letters also denote the unknown effects due to the treatment combinations. The treatment sum of squares (Tr.SS.) carries 3 d.f. This is further partitioned into three components each carrying 1 d.f. These components are respectively the sum of squares due to main effects A, B and the interaction AB. Here the main objective is to test the significance of the main effects A, B and interaction AB.

The difference $(a-1)$ is called as the effect of factor A at the lower levels of B and $(ab-b)$ is interpreted in the similar manner. The main effect of factor A is the average effect of factor A over the two levels of factor B and is given by,

$$A = \frac{1}{2}\{(a-1)+(ab-b)\} = \frac{1}{2}(ab-b+a-1)$$

which for notational convenience represented as $A = \frac{1}{2}(a-1)(b+1)$ where the RHS is to be expanded algebraically and the terms are to be interpreted as effects due to the particular treatment combinations. Similarly , the main effect of factor B is defined as

$$B = \frac{1}{2}\{ab+b-a-1\} = \frac{1}{2}(a+1)(b-1).$$

The interaction effect AB is the average difference between the effect of A at the high level of B and the effect of A at the low level of B and is given by

$$AB = \frac{1}{2}\{ab - b - a + 1\} = \frac{1}{2}(a-1)(b-1)$$

In general for a 2^k experiment the main effect of a factor A is the average of the 2^{k-1} effects of A computed at each of the possible combinations of the remaining $(k-1)$ factors and is given by

$$A = \frac{1}{2^{k-1}}\{(a-1)(b+1)(c+1)...\},$$

where the RHS is to be expanded algebraically and the terms are to be interpreted as effects due to particular treatment combinations.

Similarly a p - factor interaction $ABC...$ is defined by

$$ABC... = \frac{1}{2^{k-1}}\{(a \pm 1)(b \pm 1)(c \pm 1)...\},$$

where the sign inside a bracket is negative if the factor is included in the effect and positive if the factor is not included in the effect and a similar interpretation as explained above is to be given to the brackets.

Estimates (BLUE) of various main effects and interaction contrasts are obtained by replacing a particular treatment combination in the expression of contrasts by the average yield of that a particular treatment combination.

Sum Of Squares Due To Factorial Effects

The sum of squares for testing the significance of a factorial effect X is given by

$$SS(X) = \frac{\left[X\right]^2}{2^k r} \tag{1.1}$$

where $\left[X\right]$ is the factorial effect total obtained by replacing treatment combinations in the contrast X by the total yield obtained from the respective treatment combinations. $SS(X)$ carries a single d.f. The sum of squares due to various contrasts can be easily obtained by Yates algorithm. (cf. Montogomery (1984), pp.280)

ANOVA (Analysis of Variance)

Suppose the 2^k factorial experiment is conducted in an RBD having r replicates under the usual assumption of normality and independence. The total sum of squares is given by

$$SST = \sum_i \sum_j y_{ij}^2 - \frac{y_{..}^2}{2^k r} \tag{1.2}$$

where $y_{..}$ = grand total, y_{ij} = observation on the plot receiving i^{th} treatment in the j^{th} block $i = 1, 2, ..., k, j = 1, 2, ..., r$. This carries $(2^k r - 1)$ d .f.

The block sum of squares is given by

$$SSR = \frac{\sum_j B_j^2}{2^k} - \frac{y_{..}^2}{2^k r} \tag{1.3}$$

where B_j is the total yield from the j^{th} block and carries $(r - 1)$ d.f. Treatment S.S.(Tr.SS.), is the S.S. due to all $2^k - 1$ orthogonal main effect/interaction contrasts and carries $2^k - 1$ d.f. The error sum of squares is computed by substraction, $ESS = SST - SSR - Tr.SS$ and has $(r-1)(2^k-1)$ d.f. The equality of block effects and the significance of factorial effects (main /interaction) can be tested by comparing their mean sum of squares with the mean error sum of squares using F test. (Mean sum of squares is obtained by dividing the sum of squares by corresponding degrees of freedom). Various tests of hypothesis are given below :

1) F-statistic for testing the equality of block effects is

$$F_R = \frac{SSR/(r-1)}{SSE/(2^k-1)(r-1)} \sim F_{(r-1),(2^k-1)(r-1)} \tag{1.4}$$

2) F-statistic for testing the significance of the main effect/interaction (X) is given by,

$$F_X = \frac{SS(X)}{SSE/(2^k-1)(r-1)} \sim F_{1,(2^k-1)(r-1)} d.f. \tag{1.5}$$

Let α be the level of significance and F_{α,ν_1,ν_2} represents the $100(1-\alpha)^{th}$ percentile of F-distribution with ν_1, ν_2 d.f. The hypothesis of equality of block effects is rejected at α % level of significance if the corresponding $F_R > F_\alpha, \nu_1, \nu_2$ with $\Big((r-1),(2^k-1)(r-1)\Big)$ d.f. otherwise the hypothesis is accepted. Similarly, if $F_X > F_\alpha, \nu_1, \nu_2$ with $\Big(1,(2^k-1)(r-1)\Big)$ d.f., reject the hypothesis that the effect X is significant at α % level of significance, otherwise the hypothesis is accepted.

Analysis of variance table for a 2^k experiment in r blocks is given in Table 1.1.

Table 1.1

ANOVA Table for 2^k Experiment in r Blocks

S.V.	D.F.	S.S.	F Statistic
Blocks	r-1	SSR	F_R
Treatments	$2^k - 1$		
Main effects			
A	1	$SS_A = \frac{[A]^2}{2^k r}$	F_A
B	1	$SS_B = \frac{[B]^2}{2^k r}$	F_B
.	.	.	.
.	.	.	.
2-factor Int.			
AB	1	$SS_{AB} = \frac{[AB]^2}{2^k r}$	F_{AB}
BC	1	$SS_{BC} = \frac{[BC]^2}{2^k r}$	F_{BC}
.	.	.	.
.	.	.	.
3-factor Int.			
ABC	1	$SS_{ABC} = \frac{[ABC]^2}{2^k r}$	F_{ABC}
.	.	.	.
.	.	.	.
k-factorInt.			
$ABC...k$	1	$SS_{ABC...k} = \frac{[ABC...k]^2}{2^k r}$	$F_{ABC...k}$
Error	(r-1) $(2^k - 1)$	SSE = by subtraction	-
Total	$2^k r - 1$	$SST = \sum_i \sum_j y_{ij}^2 - \frac{y^2}{2^k r}$	-

NOTE : F-statistics are computed as given in (1.4) and (1.5).

1.2.2. A 3^k FULL FACTORIAL EXPERIMENT

In a 3^k factorial experiment , there are k factors each at three levels. There are 3^k treatment combinations and the treatment sum of squares carries $3^k - 1$ d.f. The treatment sum of squares is partitioned into $(3^k - 1)/2$ sets corresponding to various main effects and interactions each carrying 2 d.f. Each main effect carries 2 d.f and a two-factor interaction carries 4 d.f. In general a k-factor interaction carries 2^k d.f.

Effects, Interactions And Their Sum Of Squares

Let us consider an example of a 3^2 experiment. There are 2 factors each at 3 levels. The nine treatment combinations are, 1, a, a^2, b, ab, a^2b, ab^2, b^2, a^2b^2 . The treatment sum of squares carries 8 d.f. The main effects A and B each have 2 d.f. and interaction AB has 4 d.f. Further this two-factor interaction AB can be decomposed into two components AB and AB^2 each with 2 d.f. Thus there are 4 groups of effects/interactions each with 2 d.f.

For obtaining sum of squares due to a particular set carrying 2 d.f. the nine treatment combinations are divided into three groups where the grouping is based on a particular polynomial associated that particular set, namely, polynomial associated with $A^l B^m$ is $lx_1 + mx_2$. The following table shows the polynomials and groups associated with the sets for a 3^2 experiment. Here all the calculations are done modulo 3.

Table 1.2

Effect set carrying 2 d.f.	Polynomial p($\underline{x}$)	Grouping		
		G_1 :p($\underline{x}$)= 0	G_2 :p($\underline{x}$) = 1	G_3 :p($\underline{x}$) = 2
A	x_1	00 01 02	10 11 12	20 21 22
B	x_2	00 10 20	01 11 21	02 12 22
AB	$x_1 + x_2$	00 12 21	01 10 22	20 02 11
AB^2	$x_1 + 2x_2$	00 11 22	02 10 21	01 12 20

If $G_i(X)$ denotes the total yield on the combinations from the i^{th} group satisfying (p($\underline{x}$) $= i$) for the effect 'X', then the sum of squares for testing the significance of the effect X is given by,

$$SS(X) = \frac{\sum_{i} G_i^{2}(X)}{3r} - \frac{G^2}{3^2 r} \tag{1.6}$$

where G = Grand total and r = number of replications. Each group contains 3 treatment combinations. In general for a 3^k experiment, calculation of sum of squares due to an effect/interaction, A^{α_1} B^{α_2} C^{α_3} D^{α_4} ... ,where α_i takes values 0 ,1,2 depends on the grouping of treatment combinations given by p($\underline{x}$) $= i$ (mod 3), i = 0,1,2 where p($\underline{x}$) $= \sum_{i=1}^{k} \alpha_i x_i$. The sum of squares due to a particular effect set X carrying 2 d.f. is given by ,

$$SS(X) = \frac{\sum_{i} G_i^{2}(X)}{3^{k-1} r} - \frac{G^2}{3^k r} \tag{1.7}$$

Suppose the factorial experiment is conducted in an RBD having r replicates. The total sum of squares and error sum of squares are computed as given in previous section.

1.2.3. ALTERNATIVE METHOD OF ANALYSIS

If the factors are quantitative and equispaced, the effects can be partitioned into the linear and quadratic components, each carrying single d.f.

To illustrate this splitting, consider a 3^2 experiment, where there are 2 factors, say A ,B each at three levels a_0, a_1, a_2, and b_0, b_1, b_2 respectively and the levels are equispaced. The difference $(a_2b_0 - a_0b_0)$ can be called as the linear effect of A at the lower level b_0 of B and the effects at other levels of B be defined similarly. The linear effect A_L is average of these three and is given by,

$$A_L = \frac{1}{3}\{(a_2 - a_0)(b_2 + b_1 + b_0)\}$$

where RHS is expanded algebraically and the terms are interpreted as treatment combinations.

Next, for fixed level say, b_0 of B, the linear effect A when level of A changes from a_0 to a_1 should not be same as that when level of A changes from a_1 to a_2. The difference between these two may be called as quadratic effect of A. Thus quadratic effect A at level b_0 of B is given by, $(a_2b_0 - a_1b_0) - (a_1b_0 - a_0b_0) = (a_2 - 2a_1 - a_0)b_0$. For other two levels b_1 & b_2 of B, the quadratic effect A can be similarly defined. The quadratic effect A_Q is the average of these three and is given by,

$$A_Q = \frac{1}{3}\{(a_2 - 2a_1 + a_0)(b_0 + b_1 + b_2)\}$$

where RHS is expanded algebraically and the terms are interpreted as treatment combinations. The main effect B can be similarly partitioned and interpreted.

In a similar way, the two-factor interactions are decomposed into four mutually orthogonal contrasts namely, Linear $\times$ Linear, Linear $\times$ Quadratic, Quadratic $\times$ Linear and Quadratic $\times$ Quadratic respectively denoted by $A_L B_L$, $A_L B_Q$, $A_Q B_L$, and $A_Q B_Q$ (each carrying single d.f.) and are defined below,

$$A_L B_L = \frac{1}{3}\{(a_2 - a_0)(b_2 - b_0)\}$$

$$A_L B_Q = \frac{1}{3}\{(a_2 - a_0)(b_2 - 2b_1 + b_0)\}$$

$$A_Q B_L = \frac{1}{3}\{(a_2 - 2a_1 + a_0)(b_2 - b_0)\}$$

$$A_Q B_Q = \frac{1}{3}\{(a_2 - 2a_1 + a_0)(b_2 - 2b_1 + b_0)\}$$

The splitting of sum of square due to various main effects and higher order interactions in terms of their linear and quadratic effects in a general 3^k experiment can also be done in a similar manner.

Estimates of various main effects and interaction contrasts are obtained by replacing a particular treatment combination in the expression of a contrast by the average yield of that particular treatment combination.

The sum of squares due to each contrast say X is given by,

$$SS(X) = \frac{[X]^2}{2^m 3^{k-p} r} \tag{1.8}$$

has single d.f. where, m = number of factors in the effect/interaction, k= total number of factors in the experiments, p = number of linear terms in the effect/interaction, r = Number of replications.

The sum of squares due to various effects can be suitably determined by using Yate's procedure(cf.Montogomery (1984) pp.291).

The total sum of squares with $(3^k r - 1)$ d.f. and error sum of squares with $3^k(r-1)$ d.f. are obtained in the usual manner (explained in section1·2). Analysis of variance table for a 3^2 experiment in r blocks is given in Table 1.3 displayed on the next page.

In the next section, we discuss the technique of confounding and construction of confounded factorial experiments for 2^k and 3^k factorial experiments.

1.3 CONFOUNDING

When the number of treatment combinations and/or the number of factors in a factorial experiment increases the block size also increases accordingly. But in practice, large homogenous blocks will be rarely available and one is forced to use incomplete blocks. For example,a 2^5 experiment has 32 treatment combinations. Therefore the block size will be 32 plots ,2^6 experiment has 64 treatment combinations and it will require block of size 64 plots and so on. In such cases take blocks of size smaller than the number of treatments and use these smaller blocks for each replication i.e. have more than one blocks per replication. The treatments are then partitioned into as many groups as the number of blocks per replication. The different groups of treatments are allotted to the blocks. However, in the process of grouping the treatment combinations, it may happen unknowingly that an effect/interaction of experimenters interest may become identical with block contrasts. Then it is not possible to distinguish the effect due to that particular effect/interaction from the block effects.

Table 1.3

ANOVA Table for 3^2 Experiment in r Blocks

S.V.	D.F.	S.S.	F
Replicates	r-1	SSR	F_B
Treatments	8		
Main effects			
$A : A_L$	1	$SS_{A_L} = \frac{[A_L]^2}{6r}$	F_{A_L}
$: A_Q$	1	$SS_{A_Q} = \frac{[A_Q]^2}{18r}$	F_{A_Q}
$B : B_L$	1	$SS_{B_L} = \frac{[B_L]^2}{6r}$	F_{B_L}
$: B_Q$	1	$SS_{B_Q} = \frac{[B_Q]^2}{18r}$	F_{B_Q}
AB : $A_L B_L$	1	$SS_{A_L B_L} = \frac{[A_L B_L]^2}{4r}$	$F_{A_L B_L}$
$: A_Q B_L$	1	$SS_{A_Q B_L} = \frac{[A_Q B_L]^2}{12r}$	$F_{A_Q B_L}$
$AB^2 : A_L B_Q$	1	$SS_{A_L B_Q} = \frac{[A_L B_Q]^2}{12r}$	$F_{A_L B_Q}$
$: A_Q B_Q$	1	$SS_{A_Q B_Q} = \frac{[A_Q B_Q]^2}{36r}$	$F_{A_Q B_Q}$
Error	(r-1)8	SSE=by substraction	-
Total	$9r - 1$	$SST = \Sigma_i \Sigma_j y_{ij}^2 - \frac{y_{..}^2}{3^k r}$	-

Note that F-Statistics are computed as given in (1.4) and (1.5).

As an example, let us consider a 2^2 experiment. There are four treatment combinations. If we decide to use blocks of size 2 each, then we have to divide a replication into two blocks. Let the blocks are B_1 and B_2 such that $B_1 = \{a \quad ab\} \qquad B_2 = \{1 \quad b\}$

Here $B_1 - B_2 = \Big[(a) + (ab) - 1 - (b)\Big]$ is block effect.

where (.) represents the total yield from the plots receiving the treatment combinations. Note that this is same as the main effect A, i.e. main effect A is mixed up or confounded with block effects and we are not able to separate out the block effect from the main effect of A. This is not desirable because the main effects are most important for the experimenter. To avoid such undesirable situation one or more effect/ interaction(s) (contrasts) which are of least importance to the experimenter are chosen and the grouping of treatments is carried out based on the expressions of chosen contrasts. Generally, the highest order interactions are of least interest. This technique is called 'Confounding'. There are two types of confounding. 1) Total confounding 2) Partial confounding. In total confounding, the same set of interactions is confounded in all replicates and this set of interactions is not at all estimable whereas in partial confounding, different sets of interactions are confounded in different replications. In this type of confounding no interactions are lost because all confounded interactions are estimated from those replicates in which they are not confounded.

To construct such experiments, first the experimenter must decide which effect/interaction(s) to be confounded, these are called as 'generators' or 'defining contrasts'. Once the generators or defining contrasts have been decided then the blocking can be easily done. For arranging an s^k experiment in s^p blocks each of size s^{k-p}, p independent 'generators' are to be chosen.

In the following we explain the technique of confounding and also construction of confounded factorial experiments for 2^k and 3^k factorial experiments.

1.3.1. **COUNFOUNDING IN THE 2^k FACTORIAL EXPERIMENT**

In general, a 2^k factorial experiment can be arranged in 2^p incomplete blocks, $(p \leq k)$ each of size 2^{k-p}. To construct such experiment, we select p independent effects (which are of least importance to the investigator) to be confounded with blocks. By 'independent' we mean that none of the chosen interaction is a generalized interaction among others. (The generalized interaction of $x_1, x_2, ..., x_p$ is the product modulo 2 $x_1x_2...x_p$) Along with these p interactions, the $2^p - p - 1$ other effects are also automatically get confounded with blocks which are all possible generalized interactions among the previously chosen interactions.

Suppose the 'p' independent interactions chosen are, $A^{\alpha_{11}}B^{\alpha_{12}}...k^{\alpha_{1k}}, A^{\alpha_{21}}B^{\alpha_{22}}...k^{\alpha_{2k}}, ..., A^{\alpha_{p1}}B^{\alpha_{p2}}...k^{\alpha_{pk}}$ where $\alpha_{ik} = 1$ if i^{th} factor is present in the interaction and $\alpha_{ik} = 0$, if it is not. Then the treatment combinations $(x_1x_2...x_k)$ which satisfy the following algebraic equations simultaneously are put in the same block.

$$\left.\begin{array}{llll} \alpha_{11}x_1 + \alpha_{12}x_2 + & \cdot \quad \cdot & \cdot & + \alpha_{1k}x_k = l_1 \\ \alpha_{21}x_1 + \alpha_{22}x_2 + & \cdot \quad \cdot & \cdot & + \alpha_{2k}x_k = l_2 \\ \cdot \qquad \cdot & & \cdot & \cdot \\ \cdot \qquad \cdot & & \cdot & \cdot \\ \cdot \qquad \cdot & & \cdot & \cdot \\ \alpha_{p1}x_1 + \alpha_{p1}x_2 + & \cdot \quad \cdot & \cdot & + \alpha_{pk}x_k = l_p \end{array}\right\} \quad (mod\ 2) \qquad (1.9)$$

where $l_i = 0$ or 1, $i = 1, 2, ..., p$. Depending on the values of $(l_1, l_2, \ldots, l_p)$ there will be 2^p such sets of equations which give rise to 2^p blocks.

Usually, it is enough to generate the principal block which contains the treatment combinations satisfying equations (1.9) with $l_i = 0$, $i = 1, 2, ..., p$. The other blocks are generated by adding (modulo 2) a treatment combination which is not present in any of the previous blocks to the contents of the principal block.

To illustrate this, consider an example of a 2^5 factorial experiment with 5 factors each at two levels. Suppose this experiment is arranged in 2^2 blocks of size 2^3. The $p = 2$ generators are, ADE and BCE. Their generalized interaction $ABCD$ also gets confounded with the blocks. The content of the principal block are those combinations satisfying the equations,

$$\left.\begin{array}{lcl} x_1 + x_4 + x_5 & = & 0 \\ x_2 + x_3 + x_5 & = & 0 \end{array}\right\} \quad .... \qquad (mod\ 2)$$

which are given below :

B_1 : (00000) (10010) (01100) (11110) (11001) (10101)(00111) (01011)

The next block is generated by adding a treatment combination, say, (10000) which is not present in the principal block and is given by,

B_2 : (10000) (00010) (11100) (01110) (01001) (00101) (10111) (11011)

Similarly, other blocks B_3 and B_4 are generated by adding a treatment combination (01000) *and* (00001) respectively which are not present in any of the previous blocks to the content of principal block. The blocks are given by ,

B_3 : (01000) (11010) (00100) (10110) (10001) (11101) (01111) (00011)

B_4 : (00001) (10011) (01101) (11111) (11000) (10100) (00110) (01010)

1.3.2. CONFOUNDING IN THE 3^k FACTORIAL EXPERIMENT

In general, a 3^k factorial experiment can be constructed in 3^p incomplete blocks , $(p \leq k)$ where each block is of size 3^{k-p}. To construct such experiment, we select p independent interactions which are of least importance to the investigator to be confounded with blocks. There are $(3^p - 2p - 1)/2$ other effects which automatically get confounded with blocks which are all possible generalized interactions (The generalized interaction of $x_1, x_2, ..., x_p$ is the product modulo 3 $x_1 x_2 ... x_p$) among the preselected p interactions.

Suppose the 'p' independent interactions chosen are, $A^{\alpha_{11}}B^{\alpha_{12}}...k^{\alpha_{1k}}, A^{\alpha_{21}}B^{\alpha_{22}}...k^{\alpha_{2k}}, ..., A^{\alpha_{p1}}B^{\alpha_{p2}}...k^{\alpha_{pk}}$ where $\alpha_{ik} = 1$ or 2 if i^{th} factor is present in the interaction and $\alpha_{ik} = 0$, if it is not. Then the treatment combinations $(x_1x_2...x_k)$ which satisfy the following algebraic equations simultaneously are put in the same block.

$$\left.\begin{array}{llll} \alpha_{11}x_1 + \alpha_{12}x_2 + & . & . & . + \alpha_{1k}x_k = l_1 \\ \alpha_{21}x_1 + \alpha_{22}x_2 + & . & . & . + \alpha_{2k}x_k = l_2 \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ \alpha_{p1}x_1 + \alpha_{p1}x_2 + & . & . & . + \alpha_{pk}x_k = l_p \end{array}\right\} (mod\, 3) \qquad (1.10)$$

where $l_i = 0$, 1 or 2 $i = 1, 2, ..., p$. Depending on the values of $(l_1, l_2, . . ., l_p)$ there will be 3^p such sets of equations which give rise to 3^p blocks.

Usually, it is sufficient to generate the principal block which contains the treatment combinations satisfying $l_i = 0$. The other blocks are generated by adding (modulo 3) a treatment combination which is not present in any of the previous blocks to the contents of the principal block.

Note that, in 3^k factorial experiment, if the pair of interactions, X and Y are confounded then their generalized interactions XY and XY^2 are also confounded. As a convension, the power of the first letter in the name of any interaction(s) should be one. If it is not, then its square (modulo 3) is taken the name of that interaction. e.g. $A^2B = (A^2C)^2 = A^4B^2 = AB^2(mod3)$.

To illustrate this, consider an example, suppose a 3^3 factorial experiment with 3 factors each at three levels, is constructed in three blocks of size nine. Here, $p = 1$ and we take AB^2C^2 as the generator. The contents of the principal block are those treatment combinations satisfying the equation,

$$x_1 + 2x_2 + 2x_3 = 0 \qquad (mod\ 3)$$

which are given as below :

$B_1 :$ (000) (012) (101) (202) (021) (110) (122) (211) (220)

The other blocks are obtained by adding (modulo 3) a treatment combination which is not present in any of the previous block. The contents of block B_2 and B_3 are obtained by adding (mod 3) a treatment combination (200) and (100) respectively in the principal block. The blocks are given as,

$B_2 :$ (200) (212) (001) (102) (221) (010) (022) (111) (120)

$B_3 :$ (100) (112) (201) (002) (121) (210) (222) (011) (020)

In general, a s^k factorial experiment with k factors each at s levels can be arranged in s^p incomplete blocks of size s^{k-p} where, $(p \leq k)$. The $(s^p - 1)/(s - 1)$ interactions with $(s - 1)$ d.f. get confounded with blocks. Among these interactions 'p' are independent. The remaining are the generalized interactions of these p interactions. The contents of the different blocks can be obtained from a set of equations constructed from p independent confounded interactions $A^{\alpha_{11}}B^{\alpha_{12}}\ldots k^{\alpha_{1k}}, A^{\alpha_{21}}B^{\alpha_{22}}\ldots k^{\alpha_{2k}}, \ldots, A^{\alpha_{p1}}B^{\alpha_{p2}}\ldots k^{\alpha_{pk}}$, where $\alpha_{ik} = 1, 2, \ldots s-1$ if i^{th} factor is present in the interaction and $\alpha_{ik} = 0$, if it is not. The RHS of these sets of equations is the same, where the equations are,

$$
\left.\begin{array}{l}
\alpha_{11}x_1 + \alpha_{12}x_2 + \ldots + \alpha_{1k}x_k = l_1 \\
\alpha_{12}x_1 + \alpha_{22}x_2 + \ldots + \alpha_{2k}x_k = l_2 \\
\cdot \qquad \cdot \qquad \cdot \qquad \cdot \\
\cdot \qquad \cdot \qquad \cdot \qquad \cdot \\
\cdot \qquad \cdot \qquad \cdot \qquad \cdot \\
\alpha_{p1}x_1 + \alpha_{p2}x_2 + \ldots + \alpha_{pk}x_k = l_p
\end{array}\right\} \quad \ldots. \qquad (mod\ s)
$$

where $l_i = 0, 1, \ldots, s-1$. The analysis of such experiment is as usual as if no blocking is occurred except the confounded effect/interaction(s) is not included in the ANOVA table. The blocks sum of squares is the addition of the sum of squares due to all the confounded effect/interaction(s). Sum of squares are easily obtained by using Yate's algorithm. (cf. Montogomery (1984) , pp.280)

In the analysis of partial confounding designs, the sum of squares due to the confounded interaction is calculated by using data from those replicates in which interaction is not confounded. For example, let us consider a 2^3 experiment with 4 replicates and different interactions are confounded in each replicate. Let ABC is confounded in replicate I, AB is confounded in replicate II, BC is confounded in replicate III and AB is confounded in replicate IV respectively. To calculate the sum of squares due to ABC, information on ABC is obtained from the replicates II, III and IV. Similarly for other confounded interactions the sum of squares are obtained. i.e. $3/4^{th}$ information is obtained on the interactions because they are unconfounded in only three replicates. The sum of squares due to blocks and for the uncofounded effects are obtained in the usual manner.

1.4 ANALYSIS WITH SINGLE REPLICATE OF THE 2^k EXPERIMENT

For a moderate number of factors in the experiment, the total number of treatment combinations in a 2^k factorial experiment is large. For example, a 2^6 experiment has 64 treatment combinations, a 2^7 has 128 treatment combinations and so on. When the resources are limited, more than one replicates of the experiment are not possible. In such situation, the experimenter assumes that the random error in the process is small.

A single replicate of a 2^k experiment is also called as an unreplicated factorial. In a single replicate experiment there is no internal estimate of error. In the analysis of such experiments, certain higher order interactions are assumed to be negligible and their mean squares are combined to get an estimate of the error variance.

Daniel (1959) suggested a method of analysis of unreplicated factorial experiment. He suggests to examine a normal probability plot of the estimates of the effects. The effects which are negligible are normally distributed with mean zero and variance σ^2 and will fall along a straight line of this plot. However, significant effects which have non-zero means will lie outside the straight line. Therefore, based on normal probability plot the preliminary model can be specified for those effects which are non-zero. The sum of squares of negligible effects are combined to give an estimate of the error variance. For example, consider a single replicate of the 2^4 experiment. There are 15 factorial effects to estimate. When normal probability plot is constructed, suppose A, B, C, AC, AD lie far from the line. i.e. effects are significant. The remaining effects which are not significant lie on the line which are combined to estimate the error of variance.

1.5 CHAPTERWISE SUMMARY

We now present the Chapterwise summary of the dissertation. This dissertation includes four chapters besides Chapter 1 is introductory, describes (i) The need of factorial experiments (ii) Analysis of a full factorial experiment at two level and three level (iii) The technique of confounding (iv) Construction of confounded factorial experiments for 2^k and 3^k. (v) Analysis of single replicate factorial experiment.

In Chapter 2, we discuss (i) The need of fractional factorial experiments. (ii) The technique of constructing a fractional factorial experiments, (iii) Alias structures (iv) The concept and role of resolution in selecting an appropriate fractional 2^{k-p} factorial designs. (v) Determining the maximum possible value of resolution for 2^{k-p} design.

In Chapter 3, we explain some systematic methods for selecting defining contrasts so as to enable estimation of main effects and specified interactions while certain other higher order interactions are considered negligible. In particular we discuss (i) The method suggested by Greenfield (1976) (ii) The modification of this method by Greenfield (1977) (iii) The method by Franklin and Bailey (1977) (iv) An alternative method using Hadamard matrices (v) Comparison of this method with Greenfield's method. All these methods are illustrated with examples.

Chapter 4, deals with (i) The concept of minimum aberration criterion for distinguishing between designs of the same maximum resolution. (ii) A necessary and sufficient condition for the existence of the defining relation (iii) The algorithm suggested by Fries and Hunter(1980) for generating a best fraction of design for $N = 2^{k-p}$ runs. . Some examples are illustrated at the end of this chapter.

Chapter 5 is an extension of the concept of minimum aberration criterion discussed in Chapter 4. In this Chapter, we deal with (i) An alternative view of minimum aberration criterion (ii) Another aspect of minimum aberration design i.e. a minimum aberration design maximizes the number of two-factor interactions which are not aliased with main effects and among those designs having this property it minimizes the sum of squares of the sizes of alias sets of two factor interactions. (iii) Some criteria for assessment of model robustness a) The concept of Estimation Capacity b) The expected number of suspect two factor interactions. We discuss a sufficient condition for eliminating designs which are dominated by others. At the end, we present a summary of current literature which we could not discuss in detail.

The dissertation ends with a list of references.

In the next chapter, we discuss fractional factorial experiments of 2^k and 3^k.