CHAPTER 2

FRACTIONAL FACTORIAL EXPERIMENTS

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2.1 INTRODUCTION

In factorial experiments, when the number of treatment combinations is very large, it becomes beyond the resources and budget of the investigator to experiment with all of them. Also when there are large number of factors even at two levels each, the total number of treatment combinations becomes so large that it is very difficult to organize an experiment involving all these treatment combinations. Usually the demand on the resources is so great that it may not always be possible for the experimenter to provide them. In addition to this, non-experimental type errors may crop up while planning and conducting such a big experiment. For example, the treatment combination labeling may be wrongly noted and similar other things may happen. For such cases, Finney (1945) proposed a method in which only a fraction of total treatment combinations will be experimented with. Further, Plackett and Burman (1946) studied the problem in more detail and gave different fractional factorial designs.

Thus a large number of the treatment combinations is very difficult to test from practical view point and hence a full factorial design is infeasible when large number of factors are required to study. Also in full factorial experiment with a large number of factors, only a few degrees of freedom (d.f.) are allotted to the main effects and two factor interactions which are of prime importance and the rest large portion goes to the higher order interactions which are of less interest to the investigator. For example, a complete replicate of the 2^6 design requires 64 runs and out of 63 degrees of freedom only 6 d.f. correspond to the main effects and 15 d.f. correspond to the two factor interactions. The remaining 42 d.f. are associated with three factor and higher order interactions which are not much important to the experimenter. In 3^k series the situation is worst. For example, in a 3^5 factorial design which requires 243 runs, only 12 d.f. out of the 242 d.f. correspond to the main effects and 10 d.f. correspond to the two factor interactions while the rest large portion goes to the higher order interactions which are of least interest to the investigator.

If the experimenter can reasonably assume that certain higher-order interactions are negligible then the information on the main effects and low-order interactions may be obtained by running only a fraction of a complete factorial experiment. The design in which only a fraction of the total number of the treatment combinations are experimented is called as fractional factorial design (FFD). FFD's are widely used in industrial research, due to their optimum economy. A major use of fractional factorials is in screening experiments in which many factors are considered with the purpose of identifying those factors (if any) which have large influence on the variable under study.

In section 2.2 we explain fractions of 2^k factorial experiments. We presents fractions of 3^k factorial experiments in section 2.3. Section 2.4 deals with the concept of resolution.

In the next section, we explain fractions of 2^k and 3^k factorial experiments.

2.2. FRACTIONS OF 2^n SERIES

First, we consider the simplest fraction i.e. 1/2 fraction of the 2^k designs.

2.2.1. A 1/2 FRACTION OF 2^k EXPERIMENT (2^{k-1} FFD)

Let us consider a situation in which there are three factors each at two levels. Suppose the experimenter cannot afford to run all 8 treatment combinations. Suppose he can afford only four runs. Consider the Table 2.1 (displayed on the next page) of various main effect and interaction contrasts for a 2^3 full experiment.

Suppose the experimenter experiments on only four combinations, say, a, b, c and abc. From the above Table 2.1 we observe that

Table 2.1

Factorial	Treatment combinations							
Effect	(1)	(a)	(b)	(ab)	(c)	(ac)	(bc)	(abc)
$8I$	+	+	+	+	+	+	+	+
$4A$	−	+	−	+	−	+	−	+
$4B$	−	−	+	+	−	−	+	+
$4AB$	+	−	−	+	+	−	−	+
$4C$	−	−	−	−	+	+	+	+
$4AC$	+	−	+	−	−	+	−	+
$4BC$	+	+	−	−	−	−	+	+
$4ABC$	−	+	+	−	+	−	−	+

$$\left.\begin{aligned} 2(A+BC) &= (abc+a-b-c) \\ 2(B+AC) &= (-a+b-c+abc) \\ 2(C+AB) &= (-a-b+c+abc) \end{aligned}\right\} \qquad (2.1)$$

From these expressions we see that, with only four treatment combinations a, b, c, abc, one can estimate $A + BC, B + AC$, and $C + AB$. That is, the effects in the pairs $\{A, BC\}$; $\{B, AC\}$; $\{C, AB\}$ get totally confounded or mixed with each other. Further, if all two-factor interactions are known to be absent then the contrasts in LHS of (2.1) will be estimates of $2A, 2B$, and $2C$ respectively. In fact while estimating the main effects A, B and C with only four treatment combinations

a, b, c, and abc , we are really estimating $A+BC$, $B+AC$ and $C+AB$. Then we say that within the parenthesis $\{A, BC\}$, $\{B, AC\}$, and $\{C, AB\}$ are aliases of each other. The contrast ABC is not estimable. The above alias structure for this design can be easily obtained from the equation, $I = \pm ABC$ known as defining relation or generating relation and ABC is called the generator. Given $I = ABC$, we generate the alias sets

$$A = BC, \quad B = AC, \quad C = AB$$

by multiplying both sides of the generating equation modulo 2 by A, B and C respectively. One-half fraction with $I = +ABC$ is usually called the principal fraction which consists of the treatment combinations appearing with plus sign in the contrast ABC. One can also experiment with the treatment combinations $1, ab, bc, ac$ that appear with a minus sign in the contrast ABC. Then the alias structure is given by,

$$A = -BC, \quad B = -AC, \quad C = -AB$$

and the generating equation in this case is $I = -ABC$. From the analysis viewpoint, it doesn't matter which fraction is actually used. Since only half the total number of treatment combinations are used namely, a, b, c, abc or (1), ac, ab, bc the resultant design is called a half fraction of the original design.

In general a half fraction of a 2^k design is called a 2^{k-1} fractional factorial design(FFD). This design contains 2^{k-1} runs and requires one independent generator. Let us denote this generator by P. Then $I = P$

is called the defining relation for the design. The sign of P is either + or −, it depends on which one of the half fractions it produces. The fraction for which P is positive is called a principal fraction.

The alias structure is obtained by multiplying each effect (mod 2) to the both sides of the defining relation. Usually the highest order interaction is preferred to be the generator. We say more about the choice of the generators in section 3.2.

2.2.2. THE ONE-QUARTER FRACTION OF 2^k EXPERIMENT (2^{k-2} FFD)

The one-quarter fraction of the 2^k design is called a 2^{k-2} fractional factorial design and this design contains 2^{k-2} runs. A one-quarter fraction of the 2^k has two generators, say X_1 and X_2. The signs of X_1 and X_2 are either + or − which produces one of the one-quarter fraction. There are in all four fractions associated with the choice of $\pm X_1$ and $\pm X_2$ which produce the same aliasing pattern (apart from the signs attached to the members of the alias group). Their generalized interaction X_1X_2 is also a member of defining relation. The treatment combinations which have a common sign either + or − in X_1 as well as X_2 will be selected for conducting the fraction (The signs in the different generators may be different). Then automatically these combinations will have the same sign in X_1X_2 and hence X_1X_2 will also be not estimable. The corresponding defining relation for the design is given as,

$$I = \pm X_1 = \pm X_2 = \pm X_1X_2$$

The sign of X_1 and X_2 is either positive $(+)$ or negative $(-)$, it depends on which one of the four possible one-quarter fractions is produced. The fraction for which both X_1 and X_2 are positive is the principal fraction. The aliases of any particular effect are obtained by multiplying that effect to all members of the defining relation. The multiplication should be modulo 2. Each effect has three aliases and the experimenter should be careful in choosing generators so that important effects are not aliased with each other.

Let us consider an example of a 2^{6-2} design. Let $X_1 = ABCE$ and $X_2 = ACDF$ be the generators, then $X_1X_2 = BDEF$ is their generalized interaction and the defining relation for this design is given by,

$$I = ABCE = ACDF = BDEF$$

To find the aliases of any effect (e.g. A), multiply (modulo 2) by that effect to each member of the defining relation, in this case, the effect A has three aliases.

$$A = BCE = CDF = ABDEF$$

Here each main effect is aliased with two 3-factor and one 5-factor interactions, while 2-factor interactions are aliased with each other and four factor interactions e.g.

$$AB = CE = BCDF = ADEF$$

Thus, while estimating the main effect A ,we are really estimating $A+BCE+CDF+ABDEF$. i.e. the effects A, BCE , CDF, $ABDEF$

get totally confounded or mixed with each other. Practically, if we assume the third and higher order interactions to be negligible then the main effect A is estimated. The complete alias structure of this design is shown in the Table 2.2. on the next page

In this design, both the generators $ABCE$ and $ACDF$ are positive, so this is the principal fraction.

2.2.3. A $(1/2)^p$ FRACTION OF 2^k EXPERIMENT (2^{k-p}FFD)

A $(1/2)^p$ fraction of 2^k factorial design is called 2^{k-p} fractional factorial design ($p \leq k$). There are 2^{k-p} runs. This design has 'p' prechosen independent generators say $X_1, X_2, ...X_p$ and the defining relation consists of the $2^p - p - 1$ generalized interactions among these p generators in addition to the chosen generators.

Here independent means that none of the effect chosen is the generalized interaction of the others. The defining relation is given by,

$$I = \pm X_1 = \pm X_2 = \pm X_1 X_2 = \pm X_3 = \pm X_1 X_3$$

$$= \pm X_2 X_3 = ... = \pm X_1 X_2 ... X_p.$$

If all chosen generators have positive sign, then the fraction is called the principal fraction. Thus there are 2^p-1 interactions/effects which are inseparable from the mean effect and the remaining $2^k - 2^p$ interactions are mutually inseparable in sets of 2^p. The interactions belonging to the same set are called aliases of each other.

Table 2.2

Alias structure for the 2^{6-2} experiment with

$$I = ABCE = ACDF = BDEF$$

Effect	Alias		
A	BCE	CDF	$ABDEF$
B	ACE	$ABCDF$	DEF
C	ABE	ADF	$BCDEF$
D	$ABCDE$	ACF	BEF
E	ABC	$ACDEF$	BDF
F	$ABCEF$	ACD	BDE
AB	CE	$BCDF$	$ADEF$
AC	BE	DF	$ABCDEF$
AD	$BCDE$	CF	$ABEF$
AE	BC	$CDEF$	$ABDF$
AF	$BCEF$	CD	$ABDE$
BD	$ACDE$	$ABCF$	EF
BF	$ACEF$	$ABCD$	DE
ABF	CEF	BCD	ADE
CDE	ABD	AEF	BCF

The aliases of a particular effect are obtained by multiplying that effect to each member of the defining relation (multiplication should be modulo 2). Note that, the generators should be chosen very carefully so that the effects of interest are not aliased with each other. We usually assume the higher order interactions to be negligible and this simplifies the alias structure.

To illustrate the consequences of an improper choice of generators, consider an example of a 2^{6-2} design with two independent generators $X_1 = ABCDE$ and $X_2 = ABCDF$. Their generalized interaction EF. The defining relation is given by,

$$I = ABCDE = ABCDF = EF$$

Then the aliases of effect E are

$$E = ABCDE = ABCDF = F$$

Here, we observe that the two main effects namely, E and F are aliased with each other which is not desirable, because every main effect must be estimable. Thus the above choice of generators gives undesirable alias structure. Hence, the generators should be very carefully selected such that none of the important effect/interaction (s) are aliases of each other.

In the next section we discuss fractions of 3^n series.

2.3 FRACTIONS OF 3^n SERIES

2.3.1. A 1/3 FRACTION OF 3^k EXPERIMENT (3^{k-1} FFD)

The largest useful fraction of the 3^k design is a one-third fraction containing 3^{k-1} runs. Such design is called a 3^{k-1} fractional factorial design. To construct a 3^{k-1} fractional factorial design, we select a two degrees of freedom component of interaction, usually, the highest order interaction as generator and partition the full 3^k design into three blocks each of size 3^{k-1} based on the chosen interaction employing the same technique which was used to confound 3^k experiment in 3 blocks as explained in section 1.3. Each of the three resulting blocks is a 3^{k-1} fractional design. If X_1 denotes the generator, the defining relation of the fractional factorial design (FFD) is $I = X_1$ and the aliases of any effect say Y are given by,

$$Y = YX_1 = YX_1{}^2 \tag{2.2}$$

where the multiplications are done modulo 3. Moreover in this case any effect / interaction X_1 is same as X_1^2 because it produces the same grouping of the treatment combinations as that given by X_1 i.e. $X_1 = 0, X_1 = 1, X_1 = 2$. Therefore as a convension, while taking the products YX_1, YX_1^2 etc. the power of the resulting interaction is so adjusted that the power of the first factor in that interaction is equal to one.

Let us consider an example of a 3^{3-1} design. There are 3 factors each at 3 levels. Let $X_1 = AB^2C$ be the generator for this design. Then the defining relation is $I = AB^2C$. The alias structure obtained using (2.2) is,

$$\begin{aligned} A &= ABC^2 = BC^2 \\ B &= AC = ABC \\ C &= AB^2C^2 = ABC^2 \\ AB &= A^2C = BC \end{aligned}$$

{ Here e.g. $(A = A(AB^2C) = (A^2B^2C)^2 = ABC^2$ and $A(AB^2C)^2 = (A^3BC^2) = BC^2)$}

If the first non-zero exponent is 2, then the entire expression is squared (modulo 3) to make the first non-zero exponent equal to one. The squared contrast represents the same original contrast. i.e. X and X^2 represent one and the same contrast.

Note that with the above 3^{3-1} experiment we can estimate $A + ABC^2 + BC^2$,. . . etc. In case the interactions ABC^2 and BC^2 are negligible then the effect A will be estimable.

2.3.2. A $(1/3)^p$ FRACTION OF 3^k EXPERIMENT (3^{k-p} FFD)

In general one can have a $(1/3)^p$ fraction of a 3^k factorial design for $(p \leq k)$, where a fraction contains 3^{k-p} runs. Such design is called 3^{k-p} fractional factorial design. Thus, a 3^{k-2} design is a one- ninth fraction, a 3^{k-3} design is a one-twenty seventh fraction and so on.

To construct a 3^{k-p} design we select p independent components of interaction and use these effects to partition 3^k treatment combinations in 3^p blocks. Then each block is a 3^{k-p} FFD. The defining relation consists of the p effects initially selected and their $(3^p - 2p - 1)/2$ generalized interactions (each carrying 2 d.f.). The alias of any main effect or component of interaction is obtained by multiplying modulo 3 the effect with the members of the defining relation. For example, consider a 3^{4-2} design i.e.one-ninth fraction of 3^4. Suppose AB^2C and BCD are chosen as generators. Here $X_1 = AB^2C$, $X_2 = BCD$, $X_1X_2 = AC^2D$, $X_1X_2^2 = ABD^2$. Therefore the defining relation is,

$$I = X_1 = X_2 = X_1^2 = X_2^2 = X_1X_2 = X_1X_2^2 = (X_1X_2)^2 = (X_1{X_2}^2)^2$$

$$I = AB^2C = BCD = AC^2D = ABD^2 = AB^2C^2 = (BCD)^2$$

$$= (AC^2D)^2 = (ABD^2)^2$$

The alias structure for this design is given as,

$$A = ABC^2 = ABCD = ACD^2 = AB^2D = BC^2 = AB^2C^2D^2 = CD^2 = BD^2$$

$$B = AC = BC^2D^2 = ABC^2D = AB^2D^2 = ABC = CD = AB^2C^2D = AD^2$$

$$C = AB^2C^2 = BC^2D = AD = ABCD^2 = AB^2 = BD = ACD = ABC^2D^2$$

$$D = AB^2CD = BCD^2 = AC^2D^2 = AB = AB^2CD^2 = BC = AC^2 = ABD$$

2.3.3. A s^{k-p} FRACTIONAL FACTORIAL EXPERIMENT

In general, a $(1/s)^p$ fraction of s^k design is called s^{k-p} FFD, $(p \leq k)$. This design has s^{k-p} runs with k factors at s levels $(s = 2, 3, ...)$. It has 'p' prechosen independent generators, say $X_1, X_2, ...X_p$ and $s^p - p - 1$ generalized interactions. Then the defining relation is given by,

$$I = X_1 = X_2 = X_1^2 = X_2^2 = X_1X_2 = X_1X_2^2 = (X_1X_2)^2 = (X_1X_2^2)^2 = X_3 =$$

$$X_1X_3 = X_2X_3 = X_1X_2X_3 = \ldots = X_1X_2 \ldots X_p = (X_1X_2 \ldots X_p)^{p-1}$$

There are in all $(s^p-1)/(s-1)$ interactions in the defining relation which are inseparable from the mean. The remaining $s^k - s^p$ interactions are mutually inseparable in the sets of s^p and there are $s^{k-p} - 1$ such sets. The interactions/effects belonging to the same set are called aliases to each other. Hence the alias structure of such design is obtained by multiplying a particular effect to all the members of the defining relation modulo s.

In fractional factorial design, the concept of resolution is quite important. In the next section, we discuss the concept of resolution of a fractional factorial design.

2.4 RESOLUTION

Box and Hunter (1961) first approach the notion of resolution as a goodness criterion. Resolution is the length of the shortest word in the defining relation. Usually the experimenters prefer a design with highest resolution.

For a 2^{k-p} design, Let A_i denote the number of words of length 'i' in its defining relation. The vector

$$W = (A_3, A_4, \ldots A_k)$$

is called the word length pattern of the design. Here W starts with A_3 because a design with a positive A_1 or A_2 is useless. If $A_1 > 0$, then main effects are aliased with the general mean effect and if $A_2 > 0$, then main effects are aliased with each other, which is not at all desirable because the main effects must be estimable.

Example : Consider a 2^{4-2} fractional factorial design with the defining relation,

$$I = ABC = BCD = AD$$

The alias structure is,

$$A = BC = ABCD = D$$

$$B = AC = CD = ABD$$

$$C = AB = BD = ACD$$

The smallest word length is two, hence the design has resolution II. Here the main effect A is aliased with the main effect D, which is not desirable.

The resolution of a 2^{k-p} design is defined as the smallest 'r' such that $A_r \geq 1$, that is , the length of shortest word in the defining relation. To denote the design resolution R, a Roman numerical script is used. For Example, in the one-half fraction of the 2^3 design with the defining relation $I = ABC$ the length of the smallest word is three. Hence it is design of resolution III and denoted as 2^{3-1}_{III}. This definition can be alternately be rephrased as follows:

DEFINITION : A design is of Resolution R if no c-factor effects are confounded with any other effect containing less than R-c factors. □

Following three types of resolutions are of interest.

1) **Resolution III**:-

In resolution III designs, no main effects are aliased with each other, but the main effects are aliased with two-factor interactions and two-factor interactions are aliased with each other. Thus here all main effects will be estimable under the assumption that the two factor interactions that are aliased with the main effects are absent.

Example : A 2^{3-1} design with $I = ABC$ has alias structure,

$$A = BC, \qquad B = AC, \qquad C = AB$$

The design has resolution III.

2) **Resolution IV** :-

In resolution IV designs, no main effects are aliased with any other main effects or two-factor interactions, but two-factor interactions are aliased with each other. Usually it is more safe to assume that three and higher order interactions are absent. Thus in this case the main effects are clearly estimable.

For example, a 2^{4-1} design with $I = ABCD$ has alias structure

$$A = BCD, \qquad AB = CD, \qquad B = ACD, \qquad AC = BD$$

$$C = ABD, \qquad BC = AD, \qquad D = ABC$$

The design has resolution IV.

3) **Resolution V** :-

In resolution V designs, no main effects or two-factor interactions aliased with any other main effects or two-factor interactions, but two-factor interactions are aliased with three factor interactions or higher order interactions. Also main effects are aliased with four or higher order interactions. Thus here one can clearly estimate all the main effects and two-factor interactions under the assumption that all three and higher order interactions are absent.

For example, a 2^{8-2} design with the defining relation

$$I = ABCDG = ABEFH = CDEFGH$$

has resolution V.

Usually for fractional factorial designs the highest possible resolution is desirable. In the next section we discuss the problem of determining R_{max}, the maximum possible value of the resolution.

2.4.1 THE PROBLEM OF DETERMINING R_{max} (THE MAXIMUM POSSIBLE VALUE OF THE RESOLUTION)

The bounds for maximum resolution of a s^{k-p} design (where s is prime) was suggested by Fujii (1976). Then setting for $s = 2$, Fries and Hunter (1980) obtained the bounds for the maximum resolution (R_{max})of a 2^{k-p} design, the bounds are given below.

Let, $k = q(2^p - 1) + r$, $0 \le r \le 2^p - 2$, q is an integer, $\lfloor x \rfloor$is the largest integer not exceeding x , then

$$\left.\begin{aligned} R_{max} &= k, \quad if \quad p=1 \\ &= \left[2k/3\right], \quad if \quad p=2 \\ &= 2^{p-1}q, \quad if \quad r=0,1 \\ &\leq 2^{p-1}q + [2^{p-2}(r-1)/(2^{p-1}-1)], \\ &\qquad if \quad r = 2,3,...,2^{p-1}-1 \\ &\leq 2^{p-1}q + [r/2], \\ &\qquad if \quad r = 2^{p-1},...2^{p}-2 \end{aligned}\right\} \quad (2.3)$$

Further, Fries and Hunter (1980) gave another bound for R_{max} by considering the minimum number of observations required for a design having resolution R_{max}.

$$R_{max} \leq 1 + 2H + I\left[N \geq \sum_{i=0}^{H} \binom{k}{i} + \binom{k-1}{H}\right] \quad (2.4)$$

where, $N = 2^{k-p}$ is the number of observations (experimental runs or treatment combinations), H is the largest integer such that $N \geq \sum_{i=0}^{H} \binom{k}{i}$ and I is the indicator function (proof is given is Appendix A:1). Combining these results, an improved bound for R_{max} is given by,

$$R_{max} \leq Minimum\left[R_{max}bound(1), \quad R_{max}bound(2)\right] \quad (2.5)$$

where $R_{max}^{bound}(1)$ and $R_{max}bound2$ are give in (2.3) and (2.4) respectively.

Fries and Hunter(1980) have tabulated values of R_{max} bounds obtained from (2.3), (2.4) and (2.5) for the various values of k and p which are given in Table 2.3.

2.4.2 Hicks and Turner give a lower bound for the number of factors k in a 2^{k-p} design to have a desired resolution R which is

$$k \geq \frac{R(2^p - 1)}{2^{p-1}} \tag{2.6}$$

a) Let consider a 1/2 fraction ($p = 1$) of a 2^k fractional design. Then from (2.6), $k \geq R$. Thus the minimum value of factors for half fraction of resolution III, IV or V is 3, 4 and 5 respectively.

b) Consider a one-fourth fraction ($p = 2$) of a 2^k fractional design. Then for desired resolution, k satisfy $k \geq 1.5R$. For example, for resolution III design, $k \geq (1.5)(3) = 4.5$, so the minimum number of factors is five.(cf. Hicks, C.R. and Turner,K.V (1999))

In chapter 4, we discuss a minimum aberration criterion which is used whenever there are many designs with the same resolution which are not equally good. This criterion is the natural extension of resolution. In the next chapter we discuss some systematic methods for selecting the defining contrasts.

Table 2.3 A Comparison of R_{max} and the R_{max} bounds obtained from equation (2.3), (2.4) & (2.5).

k	p	R_{max} bound (2.3)	R_{max} bound (2.4)	R_{max} bound (2.5)	R_{max}	k	p	R_{max} bound (2.3)	R_{max} bound (2.4)	R_{max} bound (2.5)	R_{max}
5	2	3	3	3	3	12	8	5	3	3	3
6	3	3	3	3	3	12	7	5	4	4	4
6	2	4	4	4	4	12	6	5	4	4	4
7	4	3	3	3	3	12	5	5	5	5	5
7	3	4	4	4	4	12	4	6	6	6	6
7	2	4	5	4	4	12	3	6	8	6	6
8	4	4	4	4	4	12	2	8	9	8	8
8	3	4	4	4	4	13	9	6	3	3	3
8	2	5	5	5	5	13	8	6	4	4	4
9	5	4	3	3	3	13	7	6	4	4	4
9	4	4	4	4	4	13	6	6	5	5	4*
9	3	4	5	4	4	13	5	6	6	6	4*
9	2	6	6	6	6	13	4	6	7	6	6
10	6	4	3	3	3	13	3	6	8	6	6
10	5	4	4	4	4	13	2	8	9	8	8
10	4	5	5	5	4*	14	10	6	3	3	3
10	3	5	5	5	5	14	9	6	4	4	4
10	2	6	6	6	6	14	8	6	4	4	4
11	7	5	3	3	3	14	7	6	5	5	4*
11	6	5	4	4	4	14	6	6	6	6	5*
11	5	5	4	4	4	14	5	6	7	6	5*
11	4	5	6	5	5	14	4	7	8	7	6*
11	3	6	7	6	6	14	3	7	10	7	7
11	2	7	9	7	7	14	2	9	11	9	9

Note that : The asterisk (*) indicates that R_{max} bound (2.5) > R_{max}