CHAPTER 3

SELECTION METHODS FOR DEFINING CONTRASTS

CHAPTER - 3

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3.1 PRELIMINARIES

As discussed in the previous chapter, in fractional factorial designs, the main problem is the best possible choice of the defining contrasts such that the main effects and important interactions are estimable, that is they are not aliased with one other. The half fraction of a factorial experiment is straightforward for the selection of the defining contrasts because it requires a single defining contrast and usually in this situation the highest order interaction is the best choice. (in this dissertation we use the words defining contrasts and generators synonymously). For other fractions say, 2^{n-k}, $k > 1$, k independent generators are needed. Their generalized interactions automatically become members of the defining relation. When some specified interactions are required to be estimated, the choice of defining contrasts is not easy so as to generate a satisfactory alias structure of the resultant fractional design. We use the word 'satisfactory' in the sense that the main effects and important interactions are not aliased with each other.

Often the task of choosing an appropriate set of the defining contrasts based on mere guess work is not easy. Usually, first one has to choose a set of defining contrasts based on the judgement, then generate the alias structure and check whether the main effects and important interactions are not aliased with one another. If these are aliased with one another, again repeat the procedure with a new set of defining contrasts. Thus this process is difficult to implement as well as very time consuming. One may have to spend hours to search a suitable set of the defining contrasts by trial and error.

In this chapter, we discuss some systematic methods for selecting defining contrasts so as to enable estimation of main effects and specified interactions while certain other higher order interactions are considered negligible. In section(3.2) we discuss the method introduced by Greenfield (1976) and later improved by Franklin (1977). In section (3.3), the method introduced by Franklin and Bailey (1977) is studied. The methods are illustrated with an appropriate example. Justification for Franklin and Bailey's method is given in section 3.4. and also we discuss an another method for obtaining added factors and basic factors for given defining contrasts. Also we compare Greenfields and Franklin and Bailey algorithm in section 3.5.

3.2 THE METHOD DUE TO GREENFIELD(1976)

In this section, we discuss the method of selecting an appropriate set of the defining contrasts introduced by A. A. Greenfield (1976). First we present the algorithm and then illustrate it by an example. Later Greenfield (1977) modified the method slightly based on a suggestion by M. Franklin.

3.2.1 THE ALGORITHM

In this method, we have to specify apriori which main effects and interactions are to be estimated from the experiment. This set is known as the requirements set. Greenfield (1976) gives a step by step procedure to generate the defining contrasts and the aliasing matrix at the same time when the requirements set is specified. The algorithm for the procedure is presented below. The aim is to produce the smallest fraction of the 2^k experiment say, 2^{k-p}, (i.e. to find largest possible value of p) such that all members of the requirements set are estimable.

Step 1 : Let k be the number of factors and m be the number of interactions in the requirements set. First we decide the fraction number 'p' which is the largest integer satisfying ,

$$2^{k-p} \geq 1 + k + m \qquad (3.1)$$

This is because for estimating the k main effects and m interactions along with the overall mean requires at least $k + m + 1$ observations (If the algorithm does not produce a satisfactory alias structure with this value of p , then the algorithm is reapplied with value of p decreased by one that is a fraction which is double the size of the previous one). Let R denote the requirements set.

Step 2 : First choose a suitable set of $(k - p)$ factors and write the identity element I and these $(k-p)$ factors along with all their possible interactions in the first column in a systematic order. Thus column 1 contains 2^{k-p} elements. Then mark the elements in this column and also in the requirements set R by an asterisk which are common to both.

Step 3 : Generate the first defining contrast (generator) taking the product of the last available (unmarked) element in the first column and the last available (unmarked) element in the requirements set. The product should be modulo 2.

Step 4 : Use this defining contrast to generate the second column of the aliasing matrix by taking its generalized interactions with all the members of the first column. (Note that entries in the same row of the columns are aliases of each other). At the same time check whether any element of the requirements set is aliased with those elements, which are already marked in the first column. If not, mark those elements from the requirements set which have been newly introduced in the second column.

Step 5 : If any unwanted aliasing (as described in Step 4) has occurred in step 4, then discard the chosen generator (and also discard the element of the first column used in the previous step) and go to step 3 to generate a new defining contrast by taking the product of the next last unmarked element of the first column and the last available element of the requirements set. Everytime when a new generator is determined, create new columns corresponding to itself and its generalized interactions with all the previous ones by taking the product modulo 2 of the elements of the first column with the corresponding generalized interaction.

Step 6 : Repeat the procedure until 'p' independent generators are chosen.

Step 7 : If there are no more available elements in the first column and the number of generators selected is less than p then the current value of p is not suitable. Then decrease the value of p by one and start the procedure afresh by returning to step 2.

We illustrate the algorithm by an example presented below:

3.2.2 EXAMPLE 1

Consider an example of a 2^5 experiment. i.e. there are 5 factors, namely, $A \quad B \quad C \quad D \quad E$ each at two levels. Suppose we have to estimate all the main effects and the two-factor interactions AC and CD, assuming that all other interactions are negligible. Hence the requirements set R is, $\{A, \quad B, \quad C, \quad D, \quad E, \quad AC, \quad CD\}$

Step 1 : Here $k = 5$ and m $= 2$ and $1 + k + m = 8$ i.e. 2^3. Therefore, the maximum value of p satisfying (3.1) is 2. So the smallest fraction to get a suitable design is a quarter fraction.

Step 2 : Choose the factors A, B and C to generate the first column. Thus the first column with the elements common to the requirements set marked with an asterisk is as follows.

Column 1	The requirement Set:- R
I	$\{A^* \; B^* \; C^* \; D \; E \; AC^* \; CD\}$
A^*	
B^*	
AB	
C^*	
AC^*	
BC	
ABC	

Step 3 :- The first defining contrast is obtained by taking the product modulo 2 of the last unmarked element from the first column and the last unmarked element of the requirements set. In this example, these elements are ABC and CD respectively. Taking the product (mod 2) of these elements, we get $ABC \times CD = ABD$. This is the first defining contrast which is used to generate the second column by taking the generalized interactions with each element of the first column. We check whether any members of the requirements set are aliased with those already marked in the first column. In this example, thus we obtain the second column as,

Column 1	Column 2
I	ABD
A^*	BD
B^*	AD
AB	D^*
C^*	$ABCD$
AC^*	BCD
BC	ACD
ABC	CD^*

The requirement Set :- R

$\{A^* \ B^* \ C^* \ D^* \ E \ AC^* \ CD^*\}$

Here no two effects from the requirements set are aliased with each other. Therefore we proceed further. The second defining contrast is the generalized interaction of the next last unmarked element in the first column and the next unmarked element in the requirements set. Here, these elements are BC and E respectively. The second defining contrast is then $BC \times E = BCE$. This automatically leads to the third defining contrast which is the product (mod 2) (i.e. generalized interaction) of the first and the second defining contrast. that is, $ABD \times BCE = ACDE$. Then generate the third and fourth columns respectively by taking the generalized interactions of BCE and $ACDE$ with the elements of first column. This gives the full aliasing matrix. Everytime we check the aliasing condition as in step 4 and mark with an asterisk the elements of the respective column common with the requirements set. The aliasing matrix is given below.

Column 1	Column 2	Column 3	Column 4
I	ABD	BCE	$ACDE$
A^*	BD	$ABCE$	CDE
B^*	AD	CE	$ABCDE$
AB	D^*	ACE	$BCDE$
C^*	$ABCD$	BE	ADE
AC^*	BCD	ABE	DE
BC	ACD	E^*	$ABDE$
ABC	CD^*	AE	BDE

Here no unwanted aliasing has occurred (that is no two elements of the requirements set are aliases of each other). Thus we have obtained the complete alias structure with $p = 2$ independent generators.

In the next example, we illustrate the situation where the largest value of p given by equation (3.1) does not produce a satisfactory alias structure and a larger fraction has to be chosen.

3.2.3 EXAMPLE 2

Consider again the example of a 2^5 experiment with slightly different requirements set , $R = \{A, \ B, \ C, \ D, \ E, \ AB, \ CE\}$. The maximum value of p obtained from (3.1) is again 2 which suggests a quarter fraction in the first instance. But the algorithm later reveals that a satisfactory quarter design does not exist and it needs a half fraction of the design. The first column of the aliasing matrix is obtained as explained in the first example and is given by,

Column 1	The requirement Set:- R
I	$\{A^* \ B^* \ C^* \ D \ E \ AB^* \ CE\}$
A^*	
B^*	
AB^*	
C^*	
AC	
BC	
ABC	

As before the elements common to the requirements set are marked by an asterisk. The first defining contrast is the generalized interaction of the last unmarked element from the first column and the last unmarked element of the requirements set. They are ABC and CE respectively, which give the first generator $ABC \times CE = ABE$. The second column of the aliasing matrix consists of the generalized interactions of ABE with each member of the first column. Thus, we obtain the second column as,

Column 1	Column 2
I	ABE
A^*	BE
B^*	AE
$\underline{AB^*}$	$\underline{E^*}$
C^*	$ABCE$
AC	BCE
BC	ACE
ABC	CE^*

The requirement Set :- R

$\{A^* \;\; B^* \;\; C^* \;\; D \;\; E^* \;\; AB^* \;\; CE^*\}$

Here, the two elements of the requirements set namely, AB and E (underlined) are aliased with each other and therefore we discard the generator ABE and return to the step 3. Then take the next unmarked element from the first column i.e. BC and the last unmarked element of the requirements set i.e. CE which leads to the generator $BC \times CE = BE$. Applying the same procedure as in example 1, we get

Column 1	Column 2
I	BE
A^*	ABE
$\underline{B}^*$	$\underline{E}^*$
AB^*	AE
C^*	BCE
AC	$ABCE$
BC	CE^*
ABC	ACE

The requirement Set :-R

$\{A^* \ B^* \ C^* \ D \ E^* \ AB^* \ CE^*\}$

Here also two effects namely, B and E (underlined) from the requirements set are aliased with each other. Hence, this is also not a suitable defining contrast. We discard this defining contrast. Again take the next unmarked element AC of the first column and the last unmarked element CE of the requirements set, giving rise to the generator $AC \times CE = AE$. Here also it is easy to see that the main effects A and E become aliases of each other. Further there are no more unmarked elements available in the first column, which suggests that $p = 2$ is not possible. So we decrease the value of p by one and hence the design requires a half fraction. Obviously the generator for this half fraction should be the largest order interaction. i.e. $ABCDE$.

Next we discuss a modification of algorithm 3.2.1.

3.2.4 MODIFACTION OF ALGORITHM

Greenfield (1977) gave a modification of the above algorithm (on the basis of a suggestion made by M.Franklin (1977)) which suggests that the $(k-p)$ factors chosen previously as basic factors to generate the first column should be majority factors. That is, those factors which are represented maximum number of times in the requirements set. This is explained with the help of an example of a 2^5 design with the requirements set, $R = \{\ A,\ \ B,\ \ C,\ \ D,\ \ \ E,\ \ \ AD,\ \ AE\ \}$. For this requirements set it can be easily verified that the choice of factors A, B, C for step 2 of the algorithm will not generate a 2^{5-2} design. But if we write the first column of the aliasing matrix in terms of $(k-p)$ majority factors in the requirements set, namely, A, D and E instead of taking the first $(k-p)$ factors the algorithm works well. For this choice of factors, the first column is given by,

Column 1	The requirement Set:- R
I	$\{A^*\ B\ C\ D^*\ E^*\ AD^*\ AE^*\}$
A^*	
D^*	
AD^*	
E^*	
AE^*	
DE	
ADE	

where, an asterisk denotes the common elements between the first column and the requirements set. According to the algorithm the first defining contrast is the generalized interaction of ADE and C i.e. $ACDE$. Repeating the same procedure two other independent defining contrasts are found to be BDE and $ACDE$ and their generalized interaction is ABC. Thus the full aliasing matrix is generated and is given below:

Column 1	Column 2	Column 3	Column 4
I	$ACDE$	BDE	ABC
A^*	CDE	$ABDE$	BC
D^*	ACE	BE	$ABCD$
AD^*	CE	ABE	BCD
E^*	ACD	BD	$ABCE$
AE^*	CD	ABD	BCE
DE	AC	B^*	$ABCDE$
ADE	C^*	AB	$BCDE$

Here no unwanted aliasing has occurred. Thus we have obtained the complete alias structure with $p = 2$ independent generators, i.e. a quarter fraction.

In the next section, we discuss the method of selecting an appropriate set of defining contrasts by Franklin and Bailey.

3.3 THE METHOD BY FRANKLIN AND BAILEY (1977)

3.3.1 NOTATIONS AND PRELIMINARIES

In this section, we discuss the method of selecting an appropriate set of defining contrasts introduced by Franklin and Bailey (1977) for a fractional factorial design. This method can also be used for obtaining a confounded design. The method by Greenfield (1976) discussed in section 3.2. gives at most one suitable set of defining contrasts for obtaining a fractional factorial design. On the other hand, the method suggested by Franklin and Bailey (1977) searches all possible suitable sets of defining contrasts each one of them giving rise to a different fractional factorial design. Before we present the actual algorithm, we will discuss certain related terms below.

For a 2^k factorial design whose suitable fraction is to be decided, the set of all main effects and interactions (which are 2^k-1 in number) is partitioned into two subsets,

1) Ineligible effects 2) Eligible effects

An ineligible effects set is the set of effects which are not appropriate for choice as a defining contrast and an eligible effects set is the set of effects which are eligible for choice as a defining contrast. For example, the effects belonging to the requirements set (defined in section 3.2.3) and all their pairwise with respect to the operation (*)). Also, for a 2^{k-p} factorial experiment, there are $2^p - 1$ defining contrasts together with the identity element I which form a group of size 2^p . Thus, if we can find a suitable group (i.e. containing all eligible effects) of 2^p defining contrasts, then we can generate a 2^{k-p} fraction. Obviously the smallest possible fraction is determined by the size of the largest group in the eligible effects set. The search procedure of such a group is outlined in the algorithm given in the next subsection.

If there are several groups which have this largest size, the algorithm finds all of them. Suppose the largest groups in the eligible and ineligible effects set are of size 2^r and 2^s respectively. Since the group formed by the product of these two groups is a subset of the group of size 2^k containing all 2^k effects, we have $r + s \leq k$.

First we describe the algorithm and then illustrate it by an example.

3.3.2 THE ALGORITHM

In this method, the aim of the procedure is to search all possible smallest fractions of the 2^k experiment. Franklin and Bailey (1977) give a stepwise algorithm to generate p independent defining contrasts. To start with, we have to decide $(k - p)$ basic factors and the remaining p added factors. The basic effects group formed by the effects and interactions among the basic factors contains 2^{k-p} elements.

Step 1 : First define the requirements set and then generate the set of ineligible effects as explained in the beginning of the section 3.3.

Step 2 : Choose the value of p such that the size of the largest group in the ineligible effects set contains not more than 2^{k-p} members.

Step 3 : Select a (new) set of $(k-p)$ basic factors. The group formed by all main effects and interactions among these $(k-p)$ basic factors is called basic effects group.

Step 4 : Arrange a two-dimensional table with 2^{k-p} rows corresponding to the elements of basic effects group and 'p' columns corresponding to the added factors denoted by numbers 1 to p. The $(i,j)^{th}$ cell entry is the generalized interaction X_iY_j between the i^{th} row basic effect (X_i) and j^{th} column added effect (Y_j) of the table and should be an eligible effect. That is, it should not be present in the set of ineligible effects. An ineligible effect is denoted by '-' in the table.

Step 5 : Begin to search a set of p defining contrasts, one contrast being selected from each of the p columns of the table, which generates the largest group in the eligible effects set.

Step 6 : Initialize a starting position for a search of the table i.e. column number 0 and a defining contrasts group containing only the mean effect I.

Step 7 : Increase the column number by one. i.e. start from the next column.

Step 8 : Select the next available effect from the current column. If all the elements in the current column have been exhausted then go to step 11.

Step 9 : Check whether all generalized interactions between the effect selected and each member of the defining contrasts group are eligible effects. If not, return to step 8.

Step 10 : Extend the defining contrasts group by the selected effect and its interactions with all the members of the defining contrasts group. If the last column has been reached, an acceptable set of defining contrasts has been found. Therefore output this set and return to step 8, otherwise go to step 7.

Step 11 : If the current column is the first column move to step 12, otherwise move to the previous column (do not reinitialize the pointer) and go to step 8.

Step 12 : The search procedure has been completed by using the current basic factors set. If a new set of basic factors is available then go to step 3 and repeat the procedure.

Step 13 : If a suitable design i.e. a suitable set of defining contrasts has been found or $p = 1$ then terminate the search, otherwise decrease p by one (that is double the fraction size) and return to step 3.

The example given next illustrates the above algorithm.

3.3.3 EXAMPLE 3

Consider an example of a 2^5 experiment with the requirements set, $\{A \quad B \quad C \quad D \quad E \quad BD \quad BE\}$.
The ineligible effects contains the members of the requirements set and all their pairewise interactions and is given by,

$$\{I,\ A,\ B,\ AB,\ C,\ AC,\ BC,\ D,\ AD,\ BD,\ CD,\ ABD,$$

$$E,\ AE,\ BE,\ CE,\ DE,\ BDE,\ ABE,\ BCD,\ BCE\}$$

The value of p is chosen according to step 2 of the algorithm. In this example, the group generated by the factors A, B and D is the largest group in the ineligible effects set which contains $2^3 = 8$ elements. Thus it suggests the initial value of p equal to $5 - 3 = 2$. We select the factors A, B and D as basic factors and C, E as added factors. The basic effects group generated by these factors is written in the first column of the table. We arrange a two-way table with 2^{5-2} rows headed by the members of the basic effects group and two columns headed by the added factors. The entries in the table are the generalized interactions of the basic effects with the corresponding added factors. The table is presented below :

Table 3.3.1

Basic effects	Added factors	
	C	E
I	-	-
A	-	-
B	-	-
AB	ABC	-
D	-	-
AD	ACD	ADE
BD	-	-
ABD	$ABCD$	$ABDE$

In the table, '-' denotes an ineligible effect.

According to the algorithm, we start from the first column of the added factors. The first eligible effect in this column ABC is selected as a generator. Then an eligible effect from the second column is selected in such a way that its generalized interaction with ABC is also eligible. Here two such effects are available, ADE and $ABDE$ and their generalized interactions with ABC are $BCDE$ and CDE respectively which are eligible. As there are no more columns to search, we get the corresponding generating equations given below:

$$I = ABC = ADE = BCDE$$

and

$$I = ABC = ABDE = CDE$$

Each of these equation generates a different fraction meeting the estimability requirements. Then we come to step 11 and select the next available eligible effect in the first column, ACD and the search of the second column is repeated. The generalized interactions of ACD with ADE and $ABDE$ are CE and BCE respectively which are not eligible effects. Hence we discard the contrast ACD. Again the next eligible effect in the first column, $ABCD$ is selected and the procedure is repeated. It does not produce any suitable group of defining contrasts and we discard $ABCD$. As the first column is exhausted, the search procedure is terminated. There is no need to choose another set of basic factors, because if we choose any set of basic factors, it produces again the same groups of the defining contrasts.

In the next section we show that the algorithm generates all suitable smallest fractions of a factorial design. That is, it generates at least one fraction and also given an acceptable set of generators, we show that the algorithm generates an equivalent set of generators which produces the same fractional design as that of the given set.

3.4 JUSTIFICATION FOR AN ALGORITHM (3.3.1)

In this section, we show that the algorithm described in section 3.3 produces all possible appropriate fractions. i.e. it produces at least one suitable fraction (if exists) and any appropriate fraction can be generated through this algorithm.

We observe that, when the search of all the columns has been completed, an appropriate set of defining contrasts has been found (if at least one exists). To see this, note that, each of the p selected defining contrasts contains exactly one added factor A_j among all the added factors (because one defining contrast is selected from each of the columns corresponding to the added factors A_j). Therefore, if any two particular defining contrasts contain the added factors A_i and A_j respectively then their generalized interaction contains both factors A_i and A_j while any of the remaining $p-2$ contrasts do not contain any of these two factors. Thus, it follows that none of the p defining contrasts is a generalized interaction of any other selected contrasts and hence they all are independent. Moreover, we have checked in step 9 that all the generalized interactions among these defining contrasts are eligible effects. Thus, a group of size 2^p formed by these selected defining contrasts, their generalized interactions and mean effect I is suitable for forming the generating equation giving rise to an appropriate fraction. Thus, it is clear that the algorithm produces at least one suitable fraction.

Further, we show that any given fraction can be generated through this algorithm.

Consider a fraction with a particular set of p independent defining contrasts, D_i, $i = 1, 2, ...p$.

Case 1: Suppose we are able to find p added factors A_i, $(i = 1, 2, ...p)$ such that the D_i's can be written in the form

$$D_i = B_i A_i \tag{3.2}$$

where A_i are the added factors and B_i represents a set of basic factors i.e. exactly one added factor A_i occurs in each of the defining contrast D_i, $(i = 1, 2, ..., p)$. Since the algorithm generates all acceptable fractions arising from a particular set of added factors, obviously, it will generate this particular fraction.

Case 2: Suppose we are not able to find p added factors A_i, $(i = 1, 2, ...p)$ such that (3.2) holds. Then using the following algorithm we can generate an equivalent set of generators, say D_i^*, $(i = 1, 2, ...p)$ such that (3.2) holds for D_i^* , $(i = 1, 2, ...p)$. The word 'equivalent' means that these D_i^*'s will generate the same fraction as the one generated by given D_i's.

3.4.1 THE ALGORITHM

We input a particular set of p independent defining contrasts corresponding to the given fraction D_i, $(i = 1, 2, ...p)$ into the following algorithm to find a set of added factors A_i $(i = 1, 2, ...p)$ as explained in the previous paragraph.

Step 1: $i = 0$.

Step 2: Increase i by one. Select a factor A_i from D_i. If $i = p$ then move to step 4.

Step 3: For $j = i+1, ...p$, if A_i appears in D_j then replace D_j by D_iD_j. This gives again an equivalent set of p independent defining contrasts and A_i does not occur in D_j for$j > i$. Return to step 2.

Step 4 : For $j = 1, ..., i-1$, if A_i appears in D_j then replace D_j by D_iD_j . This ensures that A_i does not occur in D_j unless $i = j$.

Step 5 : If $i > 2$ decrease i by one and return to step 4. If $i = 2$ stop the process.

At the end, we get an equivalent set of defining contrasts say, D_i^*, $(i = 1, 2, ...p)$ and p added factors A_i $(i = 1, 2, ...p)$ such that exactly one of the p added factors appears in each of the defining contrasts, remaining are the basic factors, and case 1 applies. Note that, D_i and D_i^* are the members of a defining relation, hence generate the same defining relation.

We illustrate the above algorithm with the help of the following example.

3.4.2 EXAMPLE 4

Consider an example of a 2^{5-2} design with the defining contrasts ABC, BDE and $ACDE$ respectively.

Step1 : Let $D_1 = ABC$ and $D_2 = BDE$ be two given independent defining contrasts.

Step 2: Select factor B from ABC.

Step 3: Since $D_2 = BDE$ contains B, replacing D_2 by $D_1D_2 = D_2{}^*$, we get $D_2{}^* = ABC \times BDE = ACDE$. Now we have an independent equivalent set of defining contrasts ABC, $ACDE$ respectively. Let $j = 2$ and go to step 2. Here, we select factor A from $D_2{}^* = ACDE$.

Step 4 : Since $D_1 = ABC$ contains A, replacing D_1 by $D_1D_2{}^* = D_1{}^*$(say)(by step 4 in the algorithm), we get $D_1{}^* = ABC \times ACDE = BDE$.

Thus, at the end, we get an independent set of defining contrasts $ACDE$, BDE with A and B as added factors and C, D and E as basic factors. Note that the defining relation

$$I = ACDE = BDE = ABC$$

is same as the one generated by the given defining contrasts ABC and BDE.

In the next section we discuss an another method of obtaining added factors and basic factors for given a defining contrasts.

3.4.3 ALTERNATIVE METHOD

The given defining contrasts D_i, $i = 1, 2, ...p$ are written in the form of a $p \times$ n matrix, where corresponding to each D_i, there is a row and corresponding to each factor there is a column. The $(i, j)^{th}$ entry equals one if j^{th} factor occurs in D_i, otherwise it is equal to zero.

To obtain the added factors, we apply elementary row operations followed by a permutation of columns to the matrix such that it reduces to the form $(I \quad X)$, where I is a $p \times p$ identity matrix and X is a $p \times (n-p)$ matrix. Since each factor occurs exactly once in the defining contrasts, the factors corresponding to the identity matrix are the added factors and the rest are the basic factors.

The above algorithm is illustrated with the following example.

EXAMPLE 5

Consider a 2^{5-2} design with the defining contrasts $D_1 = BDE$ and $D_2 = ACDE$. Then the matrix described above is,

$$\begin{array}{c} \\ D_1 \\ D_2 \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

Then, interchanging the first and second columns, the reduced matrix is ,

$$\begin{array}{c} \\ D_1 \\ D_2 \end{array} \begin{array}{c} \begin{array}{ccccc} B & A & C & D & E \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

$$= (I \quad X)$$

where I is 2×2 identity matrix and X is 2×3 matrix,

where,

$$X = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Thus, the factors B and A corresponding to the identity matrix are the added factors and the remaining namely, C , D and E are the basic factors. Then from the argument at the beginning of this section it follows that the corresponding fraction can be generated by the algorithm of section 3.3.

In next section we compare Greenfields and Franklin and Bailey algorithm which is illustrated with an example.

3.5 COMPARISON WITH GREENFIELD'S ALGORITHM

If we choose any arbitrary set of basic factors, it may not always be possible to generate all suitable designs.

In section 3.2, we have seen that in Greenfield's algorithm the first $(k-p)$ factors are always selected as the basic factors. Therefore, it can be seen that the algorithm always does not produce the smallest fraction. For an example, it is shown in section 3.2.4 that it fails to generate a suitable fraction for a 2^{5-2} design with the requirements set

$$\{A\ B\ C\ D\ E\ AD\ AE\} \tag{3.3}$$

Therefore, the algorithm is modified for the selection of alternative set of basic factors. That is, we have to select the majority factors that is those which occur most frequently in the requirements set. Further this algorithm picks out atmost one acceptable fraction.

On the other hand, in Franklin and Bailey's algorithm described in section 3.3., we have seen that the algorithm examines all sets of basic factors for generating FFD's. If the current selected set fails to give a suitable fraction, then we can select a new set of basic factors until we get all possible suitable fractions.

The example given next illustrates the above points.

EXAMPLE 6

Consider the problem of construction of a 2^{5-2} design with the same requirements set given in (3.2.4). The corresponding ineligible set is

$\{I,\ A,\ B,\ AB,\ C,\ AC,\ BC,\ D,\ AD,\ BD,$ CD
$ABD,\ ACD\ E,\ AE,\ BE,\ CE,\ DE,\ ADE,\ ABE,\ ACE,\ \}$

Suppose we select A, B and C as the basic factors and D, E as the added factors. According to Franklin and Bailey's algorithm , we arrange a two-way table as displayed on the next page :

Table 3.3.2

Basic effects	Added factors	
	D	E
I	-	-
A	-	-
B	-	-
AB	-	-
C	-	-
AC	-	-
BC	BCD	BCE
ABC	$ABCD$	$ABCE$

'-' denotes an ineligible effect.

Applying Franklin and Bailey's algorithm, it is easy to observe that it fails to produce a suitable set of defining contrasts for generating a 2^{5-2} design. So, we choose another set of basic factors namely, A, C, D and let B, E be the added factors. Then the required two-way table is Table 3.3.3 displayed on the next page.

Again applying the algorithm, we obtain all possible sets of defining relations as follows,

$$\left.\begin{aligned} I = ABC = CDE = ABDE \\ I = ABC = ACDE = BDE \end{aligned}\right\} \qquad (3.4)$$

Note that, both the defining relations contain the generator 'ABC'.

Table 3.3.3

Basic effects	Added factors	
	B	E
I	-	-
A	-	-
C	-	-
AC	ABC	-
D	-	-
AD	-	-
CD	BCD	CDE
ACD	$ABCD$	$ACDE$

REMARK 1 :

A careful observation of the algorithm reveals that for a particular set of basic and added factors, it is not possible to generate any fraction in which a basic effect is one of the defining contrasts because one of the added factors is always appended to a basic factor to form a defining contrast. Therefore, it follows that, if basic factors are selected such that all the basic effects belong to an ineligible set, then this set of basic factors generates all possible fractions. In the above example, we observe that both the quarter fractions have ABC as one of the defining contrast. Therefore, in example 6, if we select A, B, C as the basic factors, ABC being a basic effect, it does not generate any of the fractions given in (3.4). However, if we choose the set of basic factors such that all the basic effects belong to the set of ineligible contrasts, e.g.

any of the sets $\{A, B, D\}$, $\{A, C, D\}$, $\{A, D, E\}$, $\{A, B, E\}$, $\{A, C, E\}$ then such a set produces all suitable fractions and it is not necessary to choose any other set of basic factors. Thus a criterion for choosing the set of the basic factors should be that the set for which all the basic effects belong to the set of an ineligible contrasts.

REMARK 2 :

The Greenfield's algorithm discussed in section 3.2 does not produce a suitable design which involves both fractional replication and confounding. But Franklin and Bailey's algorithm described in section 3.3 can be used to produce such designs. To generate such designs, it requires that the effects are eligible for the selection of defining contrasts as well as confounded effects. Such designs are obtained by repeating the application of Franklin and Bailey's algorithm after the required fraction has been obtained.

Consider a 2^{k-p} fraction arranged in 2^r blocks of size 2^{k-p-r}. There are $2^p - 1$ defining contrasts and $2^p(2^r - 1)$ confounded effects. These are obtained by applying Franklin and Bailey's algorithm twice, initially we have to find p defining contrasts to generate the fraction and then r effects to arrange it in 2^r blocks by applying the same procedure. Thus, there are $2^{p+r} - 1$ defining contrasts and confounded effects.

In the next Chapter, we focus on the concept of Minimum Aberration Criterion (MAC) for selecting a good fractional design, when the information regarding eligible and ineligible effects is not available.