Appendix 1

1. ZERO INFLATED POWER SERIES DISTRIBUTION

A1.1. Let $X_1, X_2, ..., X_m$ be independently and identically distributed (iid) random variables (r. v.) with ZIPSD then,

i. $$E(m_0) = m\left(1-\pi+\pi\frac{a_0}{f(\theta)}\right).$$

ii $$E\left(\sum_{i=1}^{\infty} m_i\right) = m\pi\left(1-\frac{a_0}{f(\theta)}\right).$$

iii $$E\left(\sum_{i=1}^{\infty} i\,m_i\right) = \frac{m\pi\theta f'(\theta)}{f(\theta)}.$$

Proof.-

i. Define ,

$$Y_i = \begin{cases} 1, & if \quad X_i = 0 \\ 0, & \text{otherwise} \end{cases}$$

$$E(m_0) = E\left(\sum_{i=1}^{m} Y_i\right),$$

$$= \sum_{i=1}^{m} E(Y_i) \quad = \sum_{i=1}^{m} 1 * P(X_i = 0) \;,$$

$$= \sum_{i=1}^{m}\left(1-\pi+\pi\frac{a_0}{f(\theta)}\right) \quad = m\left(1-\pi+\pi\frac{a_0}{f(\theta)}\right),$$

ii. To find $E\left(\sum_{i=1}^{\infty} m_i\right)$.

Define, $Z_{ij} = \begin{cases} 1, & \text{if } X_i = i \\ 0, & \text{otherwise} \end{cases}$

$\sum_{i=1}^{\infty} Z_{ij} = m_i$ = number of i's.

$$E(m_i) = E\left(\sum_{j=1}^{m} Z_{ij}\right) = \sum_{j=1}^{m} E(Z_{ij}) \quad ,$$

$$= \sum_{j=1}^{m} 1 * P(X_j = i) \quad = \sum_{j=1}^{m} \frac{\pi\, a_i\, \theta^i}{f(\theta)},$$

$$= \frac{m\pi\, a_i\, \theta^i}{f(\theta)},$$

Now, $E\left(\sum_{i=1}^{\infty} m_i\right) = \sum_{i=1}^{\infty} E(m_i) \quad = \sum_{i=1}^{\infty} \frac{m\pi\, a_i\, \theta^i}{f(\theta)},$

$$= \frac{m\pi}{f(\theta)} \sum_{i=1}^{\infty} a_i\, \theta^i \quad = \frac{m\pi}{f(\theta)} \left(f(\theta) - a_0\right) \quad ,$$

Therefore, $E\left(\sum_{i=1}^{\infty} m_i\right) = m\pi \left(1 - \frac{a_0}{f(\theta)}\right)$.

iii. Consider $E\left(\sum_{i=1}^{\infty} i\, m_i\right) \; = \sum_{i=1}^{\infty} iE(m_i) \; = \sum_{i=0}^{\infty} iE(m_i),$

$$= \sum_{i=0}^{\infty} i\, m\, P(X = i) \; = m\pi \sum_{i=0}^{\infty} \frac{i\, a_i\, \theta^i}{f(\theta)} \quad ,$$

$$= \frac{m\pi\, \theta\, f'(\theta)}{f(\theta)} \; .$$

2. ZERO INFLATED POISSON DISTRIBUTION

A1.2. Let $X_1, X_2, ..., X_m$ be iid r. v. with ZIPD then,

i. $E(m_0) = m\left(1 - \pi + \pi e^{-\theta}\right)$

ii $E\left(\sum_{i=1}^{\infty} m_i\right) = m\pi\left(1 - e^{-\theta}\right)$

iii $E\left(\sum_{i=1}^{\infty} i\, m_i\right) = m\pi\theta$

Proof.- Define,

$$Y_i = \begin{cases} 1, & \text{if } X_i = 0 \\ 0, & \text{otherwise} \end{cases}$$

$$E(m_0) = E\left(\sum_{i=1}^{m} Y_i\right)$$

$$= \sum_{i=1}^{m} E(Y_i)$$

$$= \sum_{i=1}^{m} 1\, P(X_i = 0)$$

$$= \sum_{i=1}^{m} \left(1 - \pi + \pi e^{-\theta}\right)$$

$$= m\left(1 - \pi + \pi e^{-\theta}\right)$$

ii. To find $E\left(\sum_{i=1}^{\infty} m_i\right)$

Define $Z_{ij} = \begin{cases} 1, & \text{if } X_i = i \\ 0, & \text{otherwise} \end{cases}$

$\sum_{j=1}^{\infty} Z_{ij} = m_i$ = number of i's.

$$E(m_i) = E\left(\sum_{j=1}^{m} Z_{ij}\right) = \sum_{j=1}^{m} E(Z_{ij}) \ ,$$

$$= \sum_{j=1}^{m} 1 * P(X_j = i) \ = \sum_{j=1}^{m} \frac{\pi e^{-\theta} \theta^i}{i!} \ ,$$

$$= \frac{m\pi e^{-\theta} \theta^i}{i!} .$$

Now, $E\left(\sum_{i=1}^{\infty} m_i\right) = \sum_{i=1}^{\infty} E(m_i)$

$$= \sum_{i=1}^{\infty} \frac{m\pi e^{-\theta} \theta^i}{i!} \ ,$$

$$= m\pi e^{-\theta} \sum_{i=1}^{\infty} \frac{\theta^i}{i!}$$

$$= m\pi (1 - e^{-\theta}) .$$

iii. Consider $E\left(\sum_{i=1}^{\infty} i\, m_i\right) \ = \sum_{i=1}^{\infty} iE(m_i)$

$$= \sum_{i=0}^{\infty} iE(m_i)$$

$$= \sum_{i=0}^{\infty} i\, m\, P(X = i)$$

$$= m\pi \sum_{i=0}^{\infty} \frac{i\ e^{-\theta} \theta^i}{i!}$$

$$= m\pi\theta$$

3. ZERO INFLATED NEGATIVE BINOMIAL DISTRIBUTION

A1.3. Let $X_1, X_2, \ldots, X_m$ iid r. v. with ZINBD then,

i. $$E(m_0) = m(1 - \pi + \pi Q^{-r}).$$

ii $$E\left(\sum_{i=1}^{\infty} m_i\right) = m\pi(1 - Q^{-r}).$$

iii $$E\left(\sum_{i=1}^{\infty} i\, m_i\right) = m\pi r P.$$

Proof.- Define ,

$$Y_i = \begin{cases} 1, & \text{if } X_i = 0 \\ 0, & \text{otherwise} \end{cases}$$

i. $$E(m_0) = E\left(\sum_{i=1}^{m} Y_i\right),$$

$$= \sum_{i=1}^{m} E(Y_i),$$

$$= \sum_{i=1}^{m} 1 * P(X = 0),$$

$$= \sum_{i=1}^{m} (1 - \pi + \pi Q^{-r}),$$

$$= m(1 - \pi + \pi Q^{-r}).$$

ii. To find $E\left(\sum_{i=1}^{\infty} m_i\right)$,

Define $$Z_{ij} = \begin{cases} 1, & \text{if } X_i = i \\ 0, & \text{otherwise} \end{cases}$$

$\sum_{j=1}^{\infty} Z_{ij} = m_i$ = number of i's.

$$E(m_i) = E\left(\sum_{j=1}^{m} Z_{ij}\right)$$

$$= \sum_{j=1}^{m} E(Z_{ij})$$

$$= \sum_{j=1}^{m} 1 * P(X_j = i)$$

$$= \sum_{j=1}^{m} \pi \binom{-r}{i} Q^{-r} \left(\frac{-P}{Q}\right)^i$$

$$= m\pi \binom{-r}{i} Q^{-r} \left(\frac{-P}{Q}\right)^i$$

Now, $E\left(\sum_{i=1}^{\infty} m_i\right) = \sum_{i=1}^{\infty} E(m_i)$

$$= \sum_{i=1}^{\infty} m\pi \binom{-r}{i} Q^{-r} \left(\frac{-P}{Q}\right)^i$$

$$= m\pi Q^{-r} \sum_{i=1}^{\infty} \binom{-r}{i} \left(\frac{-P}{Q}\right)^i$$

$$= m\pi \left(1 - Q^{-r}\right)$$

iii. Consider $E\left(\sum_{i=1}^{\infty} i\, m_i\right) = \sum_{i=1}^{\infty} iE(m_i)$

$$= \sum_{i=0}^{\infty} iE(m_i) = \sum_{i=0}^{\infty} i\, m P(X = i)$$

$$= m\pi \sum_{i=0}^{\infty} i \binom{-r}{i} Q^{-r} \left(\frac{-P}{Q}\right)^i = m\pi r P,$$

4. ZERO INFLATED BINOMIAL DISTRIBUTION

A1.4. Let $X_1, X_2, \ldots, X_m$ be iid r. v. with ZIBD then,

i. $E(m_0) = m\left(1 - \pi + \pi(1-\theta)^n\right)$.

ii $E\left(\sum_{i=1}^{n} m_i\right) = m\pi\left(1 - (1-\theta)^n\right)$.

iii $E\left(\sum_{i=1}^{n} i\, m_i\right) = m\, n\pi\theta$.

Proof.- Define ,

$$Y_k = \begin{cases} 1 , & \text{if } X_k = 0, \qquad k = 1,2,\ldots,m \\ 0 , & \text{otherwise} \end{cases}$$

i. $E(m_0) = E\left(\sum_{k=1}^{m} Y_k\right)$

$$= \sum_{k=1}^{m} E(Y_k) \quad = \sum_{k=1}^{m} 1 * P(X=0)$$

$$= \sum_{k=1}^{m} \left(1 - \pi + \pi(1-\theta)^n\right)$$

$$= m\left(1 - \pi + \pi(1-\theta)^n\right)$$

vii. To find $E\left(\sum_{i=1}^{n} m_i\right)$

Define ,

$$Z_{ij} = \begin{cases} 1 , & \text{if } X_j = i \qquad i = 1,2,\ldots,n,\ j = 1,2,\ldots,m. \\ 0 , & \text{otherwise} \end{cases}$$

$\sum_{j=1}^{m} Z_{ij} = m_i =$ number of i's.

$$E(m_i) = E\left(\sum_{j=1}^{m} Z_{ij}\right)$$

$$= \sum_{j=1}^{m} E(Z_{ij})$$

$$= \sum_{j=1}^{m} 1 * P(X_j = i)$$

$$= \sum_{j=1}^{m} \pi \binom{n}{i} \theta^i (1-\theta)^{n-i}$$

$$= m\pi \binom{n}{i} \theta^i (1-\theta)^{n-i}$$

Now, $E\left(\sum_{i=1}^{n} m_i\right) = \sum_{i=1}^{n} E(m_i)$,

$$= \sum_{i=1}^{n} m\pi \binom{n}{i} \theta^i (1-\theta)^{n-i} \quad ,$$

$$= m\pi \left(1-(1-\theta)^n\right),$$

iii. Consider $E\left(\sum_{i=1}^{n} i\, m_i\right) = \sum_{i=1}^{n} iE(m_i)$

$$= \sum_{i=0}^{n} iE(m_i)$$

$$= \sum_{i=0}^{n} i\, m\, P(X = i)$$

$$= m\pi \sum_{i=0}^{n} i \binom{n}{i} \theta^i (1-\theta)^{n-i}$$

$$= m\, n\, \pi\, \theta$$