Chapter - III

PARAMETER ESTIMATION FOR JM AND DE-EUTROPHICATION MODEL

§ 3.1 Introduction

In previous chapter we overviewed some of popular software reliability models. Of them JM model is one of the oldest model and is used frequently. It is extremely simple and conclusions inferred from it have only a limited applicability. Some drawbacks of this model have been pointed out by Forman and Singpurwalla (1977) and by Littlewood and Verrall (1981). We present related results in this Chapter.

Section (3.2)-(3.5) are related to JM model. In section (3.2) we study the behavior of likelihood function and parameter estimation for JM model. In section (3.3) a condition for finiteness of MLE is obtained and a situation for occurrence of finite MLE is depicted by plotting the figure. Section (3.4) is devoted to asymptotic distribution of MLE of parameters of JM model. Also asymptotic confidence intervals are given for these parameters. In section (3.6) we obtain MLE for parameters of De-eutrophication model of Moranda (1975).

§ 3.2 Parameter estimation and behavior of the likelihood function

Let T_1, T_2, ... , T_n be interoccurence times between failures of software. We assume that T_i is exponential random variable with mean $\{(N-i+1)\Lambda\}^{-1}$. That is we are assuming failure times have a JM model. The Likelihood function based upon the n observed time intervals t_1, t_2, ... ,t_n is,

$$L(N,\Lambda|\underline{t}) = \begin{cases} \prod_{i=1}^{n} (N-i+1)\Lambda \exp(-(N-i+1)\Lambda t_i) & \text{for } N \geq n \\ 0 & \text{otherwise} \end{cases} \tag{3.2.1}$$

The likelihood function has two unknown parameters N and Λ. First we fix N and obtain MLE of Λ. For any specified value of N, say $N^* \geq n$, we have,

$$L(\Lambda|N^*,\underline{t}) = \prod_{i=1}^{n} (N^*-i+1)\Lambda \ \exp\{-(N^*-i+1)\Lambda t_i\} \tag{3.2.2}$$

Let,

$$\mathbb{L}(\Lambda|N^*,\underline{t}) = \log L(\Lambda|N^*,\underline{t})$$

$$= \log \left\{ \prod_{i=i}^{n} (N^*-i+1)\Lambda \ \exp\{-(N^*-i+1)\Lambda t_i\} \right\}$$

Therefore,

$$\mathbb{L}(\Lambda|N^*,\underline{t}) = \sum_{i=1}^{n} \log(N^*-i+1) + n \log\Lambda - \sum_{i=1}^{n} (N^*-i+1)\Lambda t_i\} \tag{3.2.3}$$

Let $\Lambda(N^*)$ be the value of $\Lambda \geq 0$ for which $\mathbb{L}(\Lambda|N^*,\underline{t})$ attains the maximum. Since $\mathbb{L}(\Lambda|N^*,\underline{t})$ is strictly concave in Λ, necessary and sufficient conditions which gives us $\Lambda(N^*)$, are obtained by setting $\partial\mathbb{L}(\Lambda|N^*,\underline{t})/\partial\Lambda = 0$. Here,

$$\frac{\partial}{\partial\Lambda}\mathbb{L}(\Lambda|N^*,\underline{t}) = \frac{n}{\Lambda} - \sum_{i=1}^{n}(N^*-i+1)t_i$$

Thus $\partial\mathbb{L}(\Lambda|N^*,\underline{t})/\partial\Lambda = 0$ gives,

$$\hat{\Lambda}(N^*) = n/\sum_{i=1}^{n}(N^*-i+1)\Lambda t_i \qquad (3.2.4)$$

Thus if

$T = \sum_{i=1}^{n} t_i$ and if $k = \sum_{i=1}^{n}(i-1)t_i$ so that

$$k/T = \left\{\sum_{i=1}^{n} it_i/\sum_{i=1}^{n} t_i\right\} - 1 \qquad (3.2.5)$$

and

$$\sum_{i=1}^{n}(N^*-i+1)\Lambda t_i = N^*T - k. \qquad (3.2.6)$$

Hence,

$$\hat{\Lambda}(N^*) = n/(N^*T-k) \qquad (3.2.7)$$

Substituting this value of $\hat{\Lambda}(N^*)$ in place of Λ in the likelihood function (3.2.2), we get the likelihood function as a function of N^* only. We now search over those values of $N^* \geq n$ for which $l(N^*) = \mathbb{L}(\Lambda(N^*), N^*|\underline{t})$ is maximum.

Thus from equation (3.2.2) and (3.2.7), we get,

$$l(N^*) = \log L(\Lambda(N^*), N^* | \underline{t})$$

$$= \sum_{i=1}^{n} \log(N^*-i+1) + n \log[n/(N^*T-k)] - n \qquad (3.2.8)$$

Let $\hat{N}$ be the value of N^* for which $l(N^*)$ is maximum. Since N^* takes only discrete values necessary condition for obtaining $\hat{N}$ are, $l(N^*) \geq l(N^*+1)$ and $l(N^*) \geq l(N^*-1)$.

From equation (3.2.8) we have,

$$l(N^*+1) = \sum_{i=1}^{n} \log(N^*-i+2) + n \log[n/((N^*+1)T-k)] - n$$

and

$$l(N^*-1) = \sum_{i=1}^{n} \log(N^*-1-i+1) + n \log[n/((N^*-1)T-k)] - n \qquad (3.2.9)$$

Thus, $l(N^*) \geq l(N^*+1)$ gives,

$$\sum_{i=1}^{n}\left\{ \log(N^*-i+1) - \log(N^*-i+2)\right\} + n \log[((N^*+1)T-k)/(N^*T-k)] \geq 0$$

(3.2.10)

Expanding the sum in first term of right hand side of the above inequality, we get,

$$\sum_{i=1}^{n}\left\{ \log(N^*-i+1) - \log(N^*-i+2)\right\} = \log(N^*-n+1) - \log(N^*+1)$$

$$= \log[(N^*-n+1) /(N^*+1)]$$

Thus the inequality (3.2.10) reduce to,

$$\log[(N^*-n+1)/(N^*+1)] + n \log[((N^*+1)T-k)/(N^*T-k)] \geq 0 \qquad (3.2.11)$$

Similarly, from the condition $l(N^*) \geq l(N^*-1)$, we can have,

$$\log[N^*/(N^*-n)] + n \log[((N^*-1)T-k)/(N^*T-k)] \geq 0 \tag{3.2.12}$$

The value of $\hat{N}$ is obtained by numerically solving the above two equations. In order to assure that $\hat{N}$ is unique, we will have to establish that the projection of $l(N^*)$ is unimodal. Assuming that $l(N^*)$ is unimodal, $\hat{N}$ is unique.

Note that in order to solve (3.2.11) and (3.2.12) it is enough to know n and k/T only. However in section (3.3) we show that, not all k/T values gives finite estimate of N. We prove a necessary and sufficient condition for the MLE $\hat{N}$, to be finite. In the table (3.1) we give some numerical computation for the MLE $\hat{N}$; for various values of n and k/T.

§ 3.3 Conditions for finite MLE

For the likelihood function given in (3.2.1) we have

$$L(N,\Lambda|\underline{t}) = [\prod_{i=1}^{n}(N-i+1)]\ \Lambda^n \exp\{-\sum_{i=1}^{n} (N-i+1)\Lambda t_i\} \tag{3.3.1}$$

Let, $\mathbb{L}(N,\Lambda|\underline{t}) = \log L(N,\Lambda|\underline{t})$

Thus,

$$\mathbb{L}(N,\Lambda|\underline{t}) = \log \left\{ [\prod_{i=1}^{n}(N-i+1)\ ]\ \Lambda^n \exp\{-\sum_{i=1}^{n} (N-i+1)\Lambda t_i\} \right\}$$

$$= \sum_{i=1}^{n} \log(N-i+1) + n \log\Lambda - \sum_{i=1}^{n} (N-i+1)\Lambda t_i\} \tag{3.3.2}$$

Theorem (3.3.1) : The likelihood function given in (3.3.2) will have a unique maximum at finite N and non-zero Λ if and only if,

$$\left[\sum_{i=1}^{n} (i-1)t_i / \sum_{i=1}^{n} (i-1) \right] > \left[\sum_{i=1}^{n} t_i/n \right]. \qquad (3.3.3)$$

Proof : Define $1/\Lambda = \chi$ ($\chi \geq 0$).

Note that $N\Lambda = \phi$, implies that $N = \phi\, \chi$.

Thus from equation (3.3.2) we have,

$$\mathbb{L}(N,\chi|\underline{t}) = \sum_{i=1}^{n} \log[(N-i+1)\Lambda] - \sum_{i=1}^{n} (N-i+1)\Lambda t_i$$

$$= \sum_{i=1}^{n} \log[(N-i+1)/\chi\ ] - \sum_{i=1}^{n} (N-i+1)t_i/\chi \qquad (3.3.4)$$

$$= \sum_{i=1}^{n} \log[(\phi\chi-i+1)/\chi\ ] - \phi \sum_{i=1}^{n} t_i + \sum_{i=1}^{n} (i-1)t_i/\chi$$

hence,

$$\mathbb{L}(N,\chi|\underline{t}) = \sum_{i=1}^{n} \log(\phi\chi-i+1) - n\log(\chi) - \phi \sum_{i=1}^{n} t_i + \sum_{i=1}^{n} (i-1)t_i/\chi \qquad (3.3.5)$$

Differentiating equation (3.3.5) w.r.t. χ gives,

$$\frac{\partial}{\partial\chi} \mathbb{L}(N,\chi|\underline{t}) = \sum_{i=1}^{n} \phi(\phi\chi-i+1)^{-1} - \frac{n}{\chi} - \sum_{i=1}^{n} (i-1)t_i/\chi^{-2}$$

$$= \sum_{i=1}^{n} \phi(\phi\chi-(i-1))^{-1} - \frac{n}{\chi} - \sum_{i=1}^{n} (i-1)t_i/\chi^{-2}$$

$$= \sum_{i=1}^{n} (1-(i-1)/\phi\chi)^{-1}/\chi - \frac{n}{\chi} - \sum_{i=1}^{n} (i-1)t_i/\chi^{-2}$$

Since $N > n > i$ we have, $0 < (i-1)/\phi\chi < 1$.

Hence,

$$\frac{\partial}{\partial\chi}\mathbb{L}(N,\chi|\underline{t}) = \sum_{i=1}^{n}\frac{1}{\chi}\left\{1+\frac{(i-1)}{\phi\chi}+\frac{(i-1)^2}{(\phi\chi)^2}+\dots\right\} - \frac{n}{\chi} - \sum_{i=1}^{n}\frac{(i-1)t_i}{\chi^2}$$

$$\frac{\partial}{\partial\chi}\mathbb{L}(N,\chi|\underline{t}) = \sum_{i=1}^{n}\frac{1}{\chi}\left\{\frac{(i-1)}{\phi\chi}+\frac{(i-1)^2}{(\phi\chi)^2}+\dots\right\} - \sum_{i=1}^{n}\frac{(i-1)t_i}{\chi^2}$$

$$= \frac{1}{\chi^2}\left\{\sum_{i=1}^{n}\frac{(i-1)}{\phi} - \sum_{i=1}^{n}(i-1)t_i\right\}$$

$$+ \sum_{i=1}^{n}\frac{1}{\chi}\left\{\frac{(i-1)^2}{(\phi\chi)^2}+\dots\right\} \qquad (3.3.6)$$

Clearly, $\partial\,\mathbb{L}(N,\chi|\underline{t})\,/\partial\chi \longrightarrow 0$ as $\chi \longrightarrow \infty$.

$\partial\,\mathbb{L}(N,\chi|\underline{t})\,/\partial\chi$ approaches to zero from above if,

$$\left\{\sum_{i=1}^{n}\frac{(i-1)}{\phi} - \sum_{i=1}^{n}(i-1)t_i\right\} > 0$$

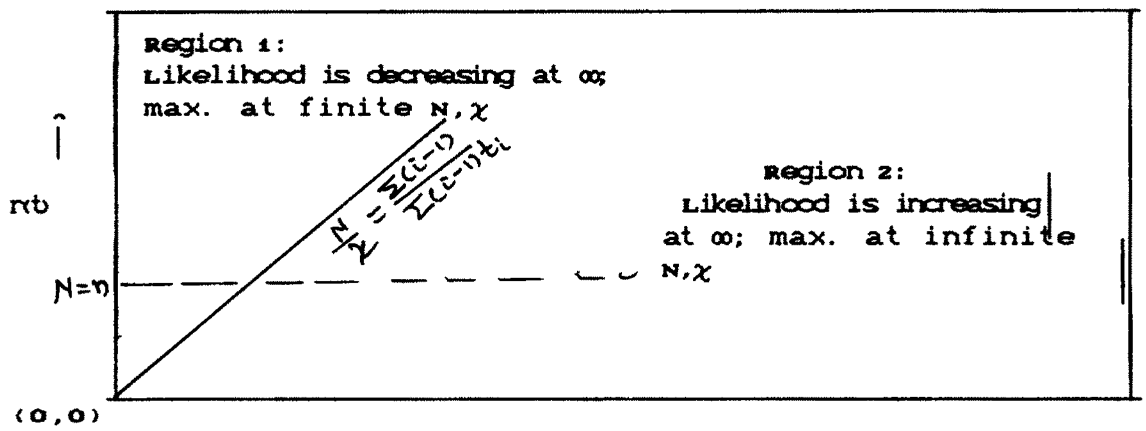


Fig.(3.1): Nature of Likelihood function

This implies that,

$$\left\{\sum_{i=1}^{n} (i-1) \Big/ \sum_{i=1}^{n} (i-1)t_i \right\} > \phi \qquad (3.3.7)$$

and it approaches to from zero below if,

$$\left\{\sum_{i=1}^{n} (i-1) \Big/ \sum_{i=1}^{n} (i-1)t_i \right\} < \phi \qquad (3.3.8)$$

This situation is shown in the figure (3.1). The likelihood curve can be divided into two regions. In region (1) the likelihood has its largest value at finite N, χ. In region (2) there will be a maximum at finite N, χ. Consider the log-likelihood function at $\chi \longrightarrow \infty$, that is,

$$
\begin{aligned}
l(\phi) &= \lim_{\chi\longrightarrow\infty} \mathbb{L}(N,\chi|\underline{t}) \\
&= \lim_{\chi\longrightarrow\infty} \left\{ \sum_{i=1}^{n} \log[(\phi\chi-i+1)/\chi\ ] - \phi \sum_{i=1}^{n} t_i + \sum_{i=1}^{n} (i-1)t_i/\chi \right\} \\
&= \sum_{i=1}^{n} \log\phi - \phi \sum_{i=1}^{n} t_i \\
&= n \log\phi - \phi \sum_{i=1}^{n} t_i \qquad (3.3.9)
\end{aligned}
$$

This implies that,

$$
\begin{aligned}
L(\phi) &= \text{antilog } [\ l(\phi)\ ] \\
&= \text{antilog } [n \log\phi - \phi \sum_{i=1}^{n} t_i] \\
&= \phi^{n} \exp \{- \phi \sum_{i=1}^{n} t_i\} \qquad (3.3.10)
\end{aligned}
$$

This likelihood is maximum at $\phi = n \,/\, \sum_{i=1}^{n} t_i$.

The global maximum must occur at finite N, χ if

$$\left\{ n/\sum_{i=1}^{n} t_i \right\} > \left\{ \sum_{i=1}^{n} (i-1) \,/ \sum_{i=1}^{n} (i-1)t_i \right\}$$

which leads to,

$$\left[\sum_{i=1}^{n} (i-1)t_i / \sum_{i=1}^{n} (i-1) \right] > \left[\sum_{i=1}^{n} t_i/n \right] \tag{3.3.3}$$

The converse can be proved as follow. The maximum value of the likelihood on the arc at infinity, putting $\hat{\phi} = \sum_{i=1}^{n} t_i/n$ in (3.3.10), is

$$L(\hat{\phi}) = \{n/\sum_{i=1}^{n} t_i\}^{n} \exp\{-n\} \tag{3.3.11}$$

It will be sufficient to show that, this exceeds the likelihood at all finite (N,χ) points on $\partial\, \mathbb{L}(N,\chi|\underline{t}) \,/\, \partial\chi = 0$.

Now $\partial\, \mathbb{L}(N,\chi|\underline{t}) \,/\, \partial\chi = 0$ implies that,

$$-\frac{n}{\chi} + \sum_{i=1}^{n} (N-i+1)t_i/\chi^2 = 0 \qquad \{\text{from eq. (3.3.4)}\}$$

This gives,

$$\Rightarrow \chi = \sum_{i=1}^{n} (N-i+1)t_i \,/\, n \tag{3.3.12}$$

And on this the likelihood function (3.2.1) is,

$$L(N|\underline{t}) = \left\{ \prod_{i=1}^{n} (N-i+1) \right\} \left\{ n/\sum_{i=1}^{n} (N-i+1)t_i \right\}^{n} \exp\{-n\} \tag{3.3.13}$$

Here we have to show that,

$$\{n/\sum_{i=1}^{n} t_i\}^n \exp\{-n\} > \left\{ \prod_{i=1}^{n}(N-i+1) \right\} \left\{ n/\sum_{i=1}^{n} (N-i+1)t_i \right\}^n \exp\{-n\}$$

This gives,

$$n/\sum_{i=1}^{n} t_i > \left\{ \prod_{i=1}^{n}(N-i+1) \right\}^{1/n} \left\{ n/\sum_{i=1}^{n} (N-i+1)t_i \right\}$$

Implies that,

$$\sum_{i=1}^{n} (N-i+1)t_i > \{\sum_{i=1}^{n} t_i\} \left\{ \prod_{i=1}^{n}(N-i+1) \right\}^{1/n} \tag{3.3.14}$$

Since, $[\sum_{i=1}^{n} (i-1)t_i/\sum_{i=1}^{n} (i-1)] < [\sum_{i=1}^{n} t_i/n]$,

$$n\sum_{i=1}^{n} (i-1)t_i < [\sum_{i=1}^{n} (i-1)] [\sum_{i=1}^{n} t_i]$$

$$n N [\sum_{i=1}^{n} t_i] - n\sum_{i=1}^{n}(i-1)t_i > n N [\sum_{i=1}^{n} t_i] - [\sum_{i=1}^{n}(i-1)] [\sum_{i=1}^{n} t_i]$$

$$n [\sum_{i=1}^{n}(N-i+1)t_i] > [\sum_{i=1}^{n} t_i][\sum_{i=1}^{n}(N-i+1)]$$

$$[\sum_{i=1}^{n}(N-i+1)t_i] > [\sum_{i=1}^{n} t_i][\sum_{i=1}^{n}(N-i+1)]/n$$

But $$[\sum_{i=1}^{n}(N-i+1)]/n > [\prod_{i=1}^{n}(N-i+1)]^{1/n}$$

This follows from the fact that the arithmetic mean of n non-negative number is always greater than the geometric mean of the same numbers. We have,

$$\sum_{i=1}^{n} (N-i+1)t_i > \{\sum_{i=1}^{n} t_i\} \left\{ \prod_{i=1}^{n}(N-i+1) \right\}^{1/n} \tag{3.3.15}$$

It is remained to prove the uniqueness. It is trivial to show that when the global maximum is at infinity, it is the only maximum at infinity and occurs at:

$$\hat{\phi} = n/\sum_{i=1}^{n} t_i = N\Lambda = N / \chi \tag{3.3.16}$$

The first term of equation (3.3.14) is a polynomial in N with no root in the parameter space ($N \geq n$). The second and third term of the same equation together form a decreasing function in N. The equation (3.3.14) has at most one turning point. In fact the maximum of (3.3.14) could occur at $N = n$.

This proves that, there is only one maximum of likelihood for finite N and χ ($\chi>0$), since $\hat{\chi}$ is uniquely determined by $\hat{N}$.

Corollary (3.3.1) : MLE is finite if $k/T > (n-1)/2$, where

$k = \sum_{i=1}^{n}(i-1)t_i$ and $T = \sum_{i=1}^{n} t_i$.

Proof. Condition (3.3.3) is equivalent to,

$$[\sum_{i=1}^{n} (i-1)t_i] - [\sum_{i=1}^{n} (i-1)] [\sum_{i=1}^{n} t_i/n] > 0 \tag{3.3.17}$$

Using the notations given in equation (3.3.4), we write the above inequality as,

$$k - [n(n+1)/2 - n](T/n) > 0$$

This gives us,

$k > [(n+1)/2 - 1]T$

which equivalent to,

$$k/T > (n-1)/2 \tag{3.3.18}$$

Hence the proof.

§ 3.4 Asymptotic distribution of $\hat{N}$ and $\hat{\Lambda}$.

In this section we obtain asymptotic distribution of $\hat{N}$ and $\hat{\Lambda}$. Using the same, asymptotic confidence interval for N and Λ have been proposed.

Theorem (3.4.1) : Asymptotic distributions of $\hat{N}$ and $\hat{\Lambda}$ are given by

$$\sqrt{n}\,(\hat{N} - N) \xrightarrow{d} AN\,(0, \sigma_{\hat{N}})$$

and

$$\sqrt{n}\,(\hat{\underline{\Lambda}} - \underline{\Lambda}) \xrightarrow{d} AN\,(0, \sigma_{\hat{\Lambda}}) \quad \text{as } n \longrightarrow \infty$$

where,

$$\sigma_{\hat{N}} = = n \Big/ \left\{ n \sum_{i=1}^{n} [1/(N-i+1)]^2 - \{ \sum_{i=1}^{n} 1/(N-i+1) \}^2 \right\}$$

and

$$\sigma_{\hat{\Lambda}} = \Lambda^2 \sum_{i=1}^{n} [1/(N-i+1)]^2 \Big/ \left\{ n \sum_{i=1}^{n} [1/(N-i+1)]^2 - \{ \sum_{i=1}^{n} 1/(N-i+1) \}^2 \right\}$$

Proof : Based upon the n observed interoccurence time $t_1, t_2, \ldots, t_n$ we have the likelihood function as,

$$L(N,\Lambda|\underline{t}) = \begin{cases} \prod_{i=1}^{n} (N-i+1)\ \Lambda \ \exp(-(N-i+1)\Lambda t_i) & \text{for } N \geq n \\ o & \text{otherwise} \end{cases}$$

Thus for $N \geq n$ and $t_i \geq 0$ $(i=1,2,\ldots,n)$, we have,

$$L(N,\Lambda|\underline{t}) = [\prod_{i=1}^{n}(N-i+1)\ ]\ \Lambda^n \exp\{-\sum_{i=1}^{n} (N-i+1)\Lambda t_i\}$$

Let,

$$\mathbb{L}(N,\Lambda|\underline{t}) = \log L(N,\Lambda|\underline{t})$$

$$= \log \left\{ [\prod_{i=1}^{n}(N-i+1)\ ]\ \Lambda^n \exp\{-\sum_{i=1}^{n} (N-i+1)\Lambda t_i\} \right\}$$

$$= \sum_{i=1}^{n} \log(N-i+1) + n \log\Lambda - \sum_{i=1}^{n}(N-i+1)\Lambda t_i\} \qquad (3.4.1)$$

Here the parameter of interest is $\underline{\theta} = (N,\Lambda)'$. Let $\hat{\underline{\theta}} = (\hat{N},\hat{\Lambda})'$ be MLE for $\underline{\theta} = (N,\Lambda)'$. Since Likelihood function satisfies regularity conditions, we have from Zacks (1973)

$$\sqrt{n}\,(\hat{\underline{\theta}} - \underline{\theta}) \xrightarrow{d} N_2\ (\underline{0}\ ,\ I^{-1}(\underline{\theta})) \quad \text{as } n \longrightarrow \infty$$

where $I(\underline{\theta})$ is a positive definite matrix of order 2 and is given

$$I(\underline{\theta}) = \begin{bmatrix} -E\left[\frac{\partial^2}{\partial N^2}\mathbb{L}(N,\Lambda)\right] & -E\left[\frac{\partial}{\partial N}\frac{\partial}{\partial \Lambda}\mathbb{L}(N,\Lambda)\right] \\ -E\left[\frac{\partial}{\partial N}\frac{\partial}{\partial \Lambda}\mathbb{L}(N,\Lambda)\right] & -E\left[\frac{\partial^2}{\partial \Lambda^2}\mathbb{L}(N,\Lambda)\right] \end{bmatrix}$$

From equation (3.4.1) we get,

$$\frac{\partial}{\partial N}\mathbb{L}(N,\Lambda|\underline{t}) = \sum_{i=1}^{n}[1/(N-i+1)] - \Lambda\sum_{i=1}^{n} t_i$$

$$\Rightarrow \frac{\partial^2}{\partial N^2}\mathbb{L}(N,\Lambda|\underline{t}) = -\sum_{i=1}^{n}[1/(N-i+1)]^2$$

$$\therefore -E\left[\frac{\partial^2}{\partial N^2}\mathbb{L}(N,\Lambda|\underline{t})\right] = \sum_{i=1}^{n}[1/(N-i+1)]^2 \qquad (3.4.2)$$

$$\frac{\partial}{\partial \Lambda}\mathbb{L}(N,\Lambda|\underline{t}) = \frac{n}{\Lambda} - \sum_{i=1}^{n}(N-i+1)\ t_i$$

$$\Rightarrow \frac{\partial^2}{\partial \Lambda^2}\mathbb{L}(N,\Lambda|\underline{t}) = -\frac{n}{\Lambda^2}$$

$$E\left[\frac{\partial^2}{\partial \Lambda^2}\mathbb{L}(N,\Lambda|\underline{t})\right] = \frac{n}{\Lambda^2} \qquad (3.4.3)$$

$$\text{and } \frac{\partial}{\partial N}\frac{\partial}{\partial \Lambda}\mathbb{L}(N,\Lambda|\underline{t}) = -\sum_{i=1}^{n} t_i$$

$$\Rightarrow E\left[\frac{\partial}{\partial N}\frac{\partial}{\partial \Lambda}\mathbb{L}(N,\Lambda|\underline{t})\right] = -\sum_{i=1}^{n} t_i \qquad (3.4.4)$$

Therefore using equation (3.4.2), (3.4.3) and (3.4.4), we have,

$$I(\underline{\Theta}) = \begin{bmatrix} \sum_{i=1}^{n} 1/(N-i+1)^2 & \sum_{i=1}^{n} 1/[(N-i+1)\Lambda] \\ \sum_{i=1}^{n} 1/[(N-i+1)\Lambda] & n/\Lambda^2 \end{bmatrix}$$

Hence, the inverse matrix of $I(\underline{\Theta})$, $I^{-1}(\underline{\Theta})$ is,

$$I^{-1}(\underline{\Theta}) = \begin{bmatrix} n/(\Lambda^2 D) & \left[\sum_{i=1}^{n}[1/(N-i+1)\Lambda]\right]/D \\ \left[\sum_{i=1}^{n}[1/(N-i+1)\Lambda]\right]/D & \left[\sum_{i=1}^{n} 1/(N-i+1)^2\ D\right] \end{bmatrix}$$

where $D = |I(\underline{\theta})|$

$$= \frac{1}{\Lambda^2} \left\{ n \sum_{i=1}^{n} [1/(N-i+1)]^2 - \{ \sum_{i=1}^{n} 1/(N-i+1) \}^2 \right\}$$

Therefore we can write,

$$\sqrt{n}\ (\hat{N} - N) \xrightarrow{d} AN\ (0, \sigma_{\hat{N}})$$

and

$$\sqrt{n}\ (\underline{\hat{\Lambda}} - \underline{\Lambda}) \xrightarrow{d} AN\ (0, \sigma_{\hat{\Lambda}}) \quad \text{as } n \longrightarrow \infty \tag{3.4.5}$$

where,

$$\sigma_{\hat{N}} = I_{11}^{-1}(\underline{\theta})$$

$$= n \Big/ \left\{ n \sum_{i=1}^{n} [1/(N-i+1)]^2 - \{ \sum_{i=1}^{n} 1/(N-i+1) \}^2 \right\} \tag{3.4.6}$$

and

$$\sigma_{\hat{\Lambda}} = I_{22}^{-1}(\underline{\theta})$$

$$= \Lambda^2 \sum_{i=1}^{n} [1/(N-i+1)]^2$$

$$\Big/ \left\{ n \sum_{i=1}^{n} [1/(N-i+1)]^2 - \{ \sum_{i=1}^{n} 1/(N-i+1) \}^2 \right\} \tag{3.4.7}$$

Hence the proof.

Remark : In practice, $\sigma_{\hat{N}}$ and $\sigma_{\hat{\Lambda}}$ are unknown, the estimate of $\sigma_{\hat{N}}$ and $\sigma_{\hat{\Lambda}}$ are obtained by replacing N and Λ by $\hat{N}$ and $\hat{\Lambda}$ respectively in (3.4.6) and (3.4.7). Thus,

$$\hat{\sigma}_{\hat{N}} = \text{Est}[\ \sigma_{\hat{N}}\ ]$$
$$= \left[\ \sigma_{\hat{N}}\ \right]_{N=\hat{N},\Lambda=\hat{\Lambda}} \qquad \text{and}$$
$$\sigma_{\hat{\Lambda}} = \text{Est}[\ \sigma_{\hat{\Lambda}}\ ]$$
$$= \left[\ \sigma_{\hat{\Lambda}}\ \right]_{N=\hat{N},\Lambda=\hat{\Lambda}}$$

3.4.1 Asymptotic Confidence Interval for $\hat{N}$ and $\hat{\Lambda}$

Since,

$$\sqrt{n}\ (\hat{N} - N\ ) \xrightarrow{d} AN\ (0,\ \sigma_{\hat{N}}) \quad \text{and}$$
$$\sqrt{n}\ (\hat{\Lambda} - \Lambda) \xrightarrow{d} AN\ (0,\ \sigma_{\hat{\Lambda}}) \quad \text{as } n \longrightarrow \infty$$

where,

$$\sigma_{\hat{N}} = = n \Big/ \left\{ n \sum_{i=1}^{n} [1/(N-i+1)]^2 - \{ \sum_{i=1}^{n} 1/(N-i+1)\ \}^2 \right\}$$

and

$$\sigma_{\hat{\Lambda}} = \Lambda^2 \sum_{i=1}^{n} [1/(N-i+1)]^2$$
$$\Big/ \left\{ n \sum_{i=1}^{n} [1/(N-i+1)]^2 - \{ \sum_{i=1}^{n} 1/(N-i+1)\ \}^2 \right\}$$

Asymptotic Confidence Interval for $\hat{N}$ and $\hat{\Lambda}$, are

$$\left[\ \hat{N} - \hat{\sigma}_{\hat{N}}\ Z_{\alpha/2}\ /\sqrt{n}\ \}\ ,\ \hat{N} + \hat{\sigma}_{\hat{N}}\ Z_{\alpha/2}\ /\sqrt{n}\ \}\right] \tag{3.4.8}$$

and

$$\left[\ \hat{\Lambda} - \hat{\sigma}_{\hat{\Lambda}}\ Z_{\alpha/2}\ /\sqrt{n}\ \}\ ,\ \hat{\Lambda} + \hat{\sigma}_{\hat{\Lambda}}\ Z_{\alpha/2}\ /\sqrt{n}\ \}\right] \tag{3.4.9}$$

§ 3.5 Empirical Study

In table 3.1 we present computations for MLE for some values of n as given in Forman and Singpurwalla (1977).

Table 3.1

Maximum Likelihood Estimate for N

n = 2		n = 10	
k/T	$\hat{N}$	k/T	$\hat{N}$
1.9	2.0	9.0	10.0
0.8	2.0	7.0	10.0
0.7	2.0	6.0	11.17
0.55	3.06	5.5	13.52
0.54	5.44	5.0	21.30
0.53	8.81	4.8	32.30
0.52	12.93	4.7	45.90
0.51	25.60	4.65	59.90
0.50	10550.00		

From the table we observe that,

i) for some values of k/T, the MLE turns out to be finite. But for other values of k/T, it is vary large.

ii) as n increases range over finiteness of MLE is observed also increases.

iii) on the boundary, where $k/T \cong (n-1)/2$ MLE is relatively large.

Below we give some graphs of empirical distribution of k/T, wherein we take $N = 50$ and $\Lambda = 1$. For different values of n, say 10, 20, 30 and 40; samples have generated and empirical distribution of k/T is plotted. Further from the empirical distribution we estimate probability that the MLE is finite. That is, $P[\ k/T > (n-1)/2\ ]$ for different values of n (see table 3.2). We observe that, as n increases to N; distribution of k/T goes away from origin (it moves towards right along X-axis) and $P[k/T>(n-1)/2]$ increases as desired.

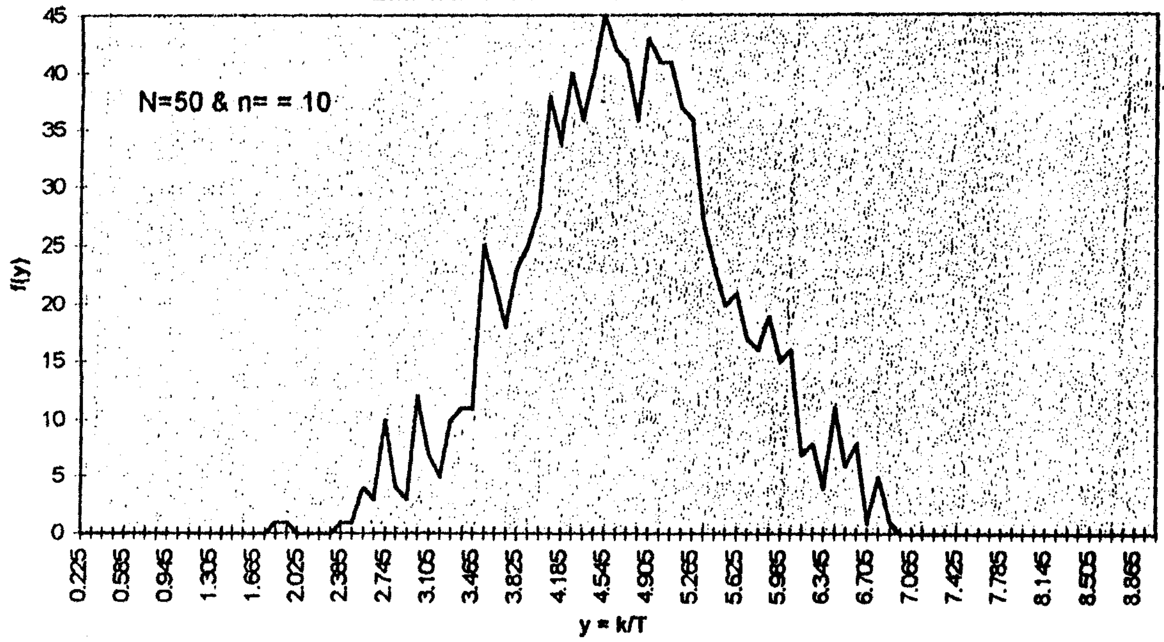
Table 3.2

Estimate of $P[\ k/T > (n-1)/2]$

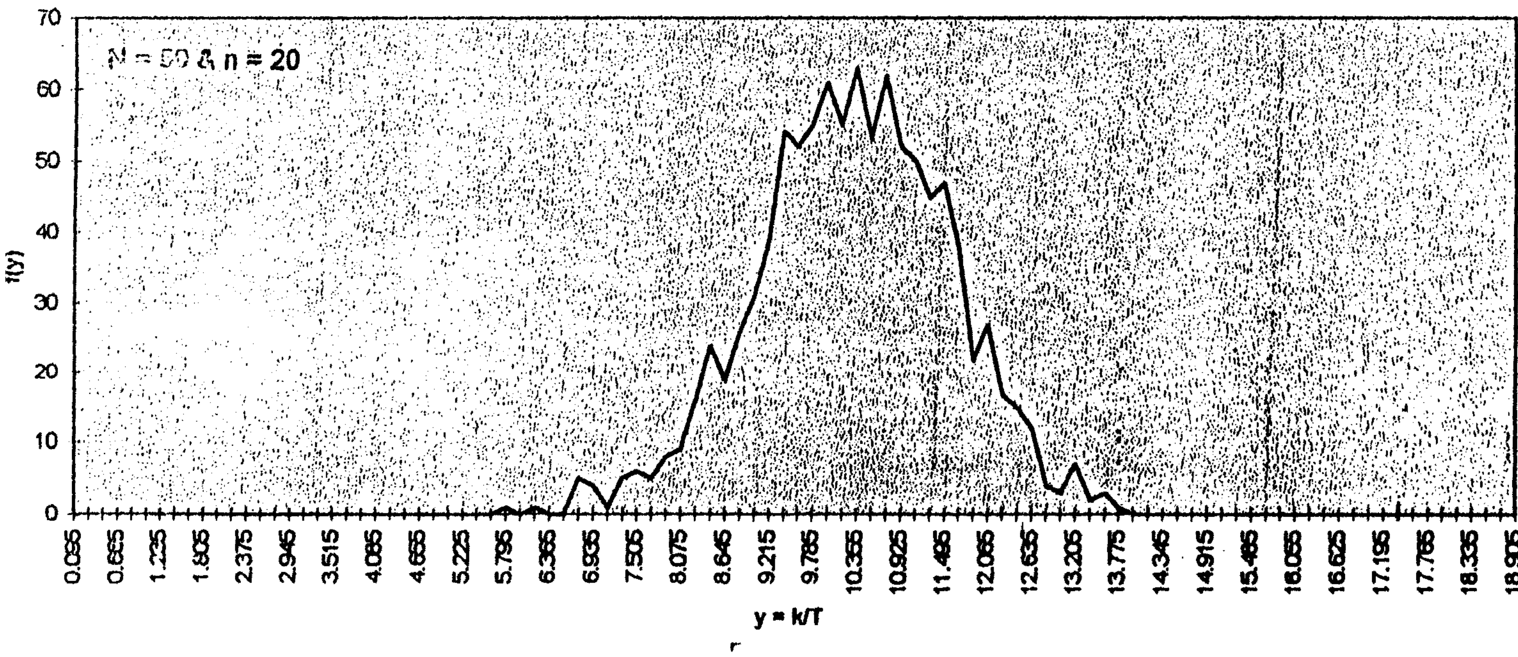
Sr. No.	n	Est{ $P[\ k/T > (n-1)/2\ ]$ }
1	10	0.5870
2	20	0.7217
3	30	0.8891
4	40	0.9879

[The values are obtained for $N = 50$ and $\Lambda = 1$].

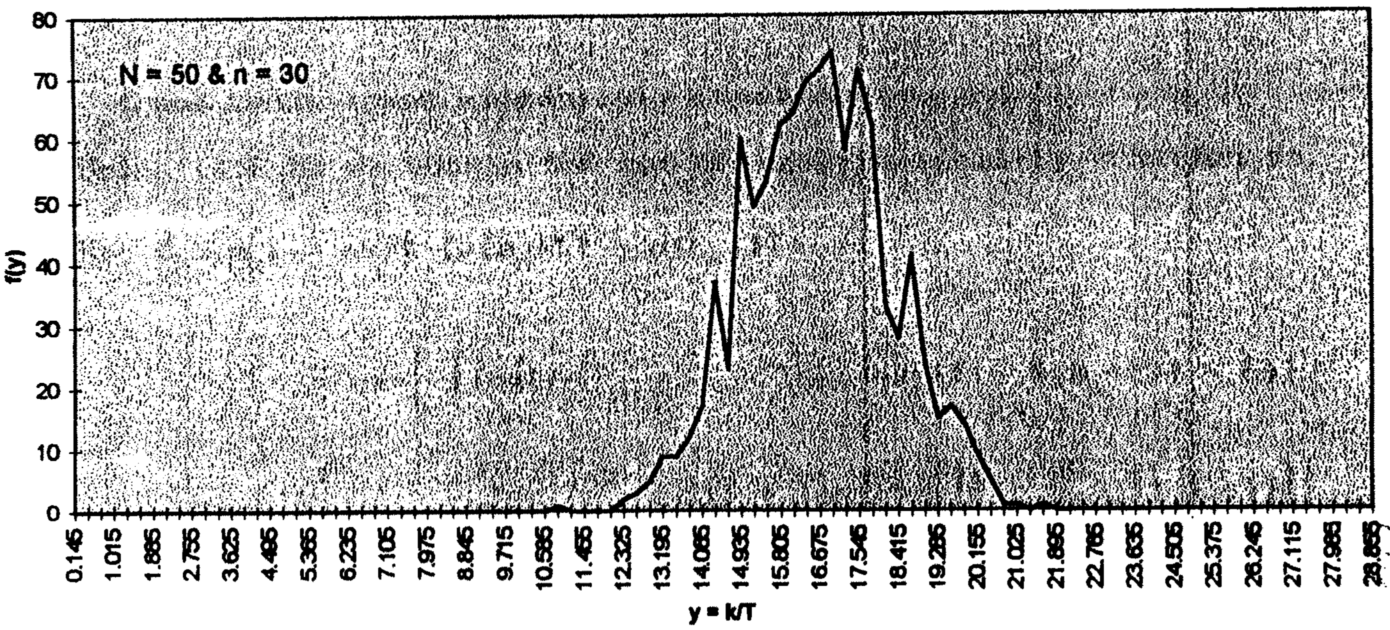
EMPIRICAL DISTRIBUTION OF k/T



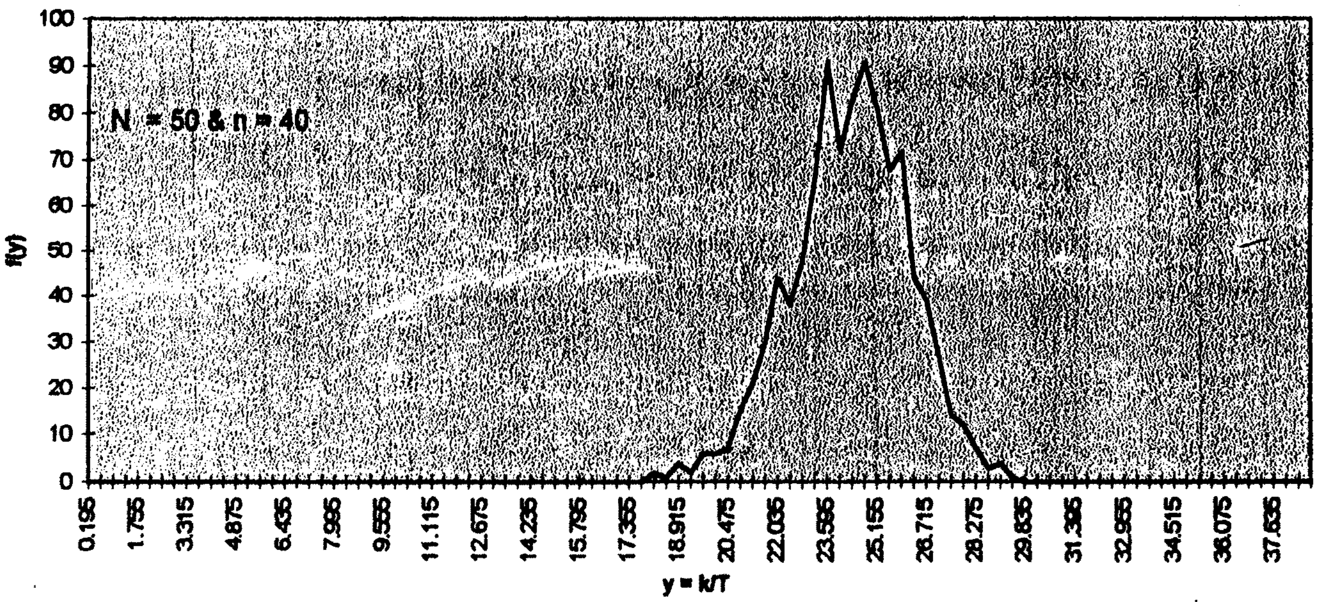
EMPIRICAL DISTRIBUTION OF k/T



EMPIRICAL DISTRIBUTION OF k/T



EMPIRICAL dISTRIBUTION OF k/T



§ 3.6 MLE for parameters of De-eutrophication model by Moranda

Consider the De-dutrophication model given in section (2.5) of Chapter - II. If t_1, t_2, ... , t_n are the observed successive time between failures of a piece of software, then by using equation (2.5.2), we can write the likelihood function as,

$$L(\, D,\, k\,) = \prod_{i=1}^{n} Dk^{i-1} \exp\{-\, Dk^{i-1}\, t_i\} \quad ;\ t_i \geq 0,\ 0 < k < 1$$

$$= D^{n}\, k^{\Sigma(i-1)} \prod_{i=1}^{n} \exp\{-\, Dk^{i-1}\, t_i\} \quad ;\ t_i \geq 0,\ 0<k<1$$

The summation is taken over i = 1, 2, ... , n. Thus,

$$\mathbb{L}(D,k) = \log_e L(D,k)$$

$$= n \log(D) + \sum_{i=1}^{n} (i-1) \log(k) - D \sum_{i=1}^{n} k^{i-1}\, t_i \qquad (3.6.1)$$

Then the MLE of k and D are the solutions of the equations, $\partial\, \mathbb{L}\,(D,k)\, /\partial k = 0$ and $\partial\, \mathbb{L}(D,k)\, /\, \partial D = 0$. Here,

$$\frac{\partial\, \mathbb{L}(D,k)}{\partial k} = \sum_{i=1}^{n} (i-1)/\, k - D \sum_{i=1}^{n} (i-1)\, k^{i-2}\, t_i \quad \text{and}$$

$$\frac{\partial\, \mathbb{L}(D,k)}{\partial D} = (n/D) - \sum_{i=1}^{n} k^{i-1}\, t_i\, .$$

Thus $\partial\, \mathbb{L}\,(D,k)\, /\partial k = 0$ and $\partial\, \mathbb{L}\,(D,k)\, /\, \partial D = 0$ gives,

$$\sum_{i=1}^{n} (i-1)/\, k - D \sum_{i=1}^{n} (i-1)\, k^{i-2}\, t_i = 0 \qquad \text{and} \qquad (3.6.2)$$

$$D = n\, /\, \left\{ \sum_{i=1}^{n} k^{i-1}\, t_i \right\} \qquad (3.6.3)$$

From equation (3.6.2) and (3.6.3), we get,

$$\sum_{i=1}^{n} (i-1)/ k - \left\{ \sum_{i=1}^{n} (i-1) k^{i-2} t_i \right\} n/ \left\{ \sum_{i=1}^{n} k^{i-1} t_i \right\} = 0$$

Since, $0 < k < 1$, $\sum_{i=1}^{n} k^{i-1} t_i \neq 0$.

$$\Rightarrow \sum_{i=1}^{n} k^{i-1} t_i \left\{ \sum_{i=1}^{n} (i-1)/ k \right\} - \left\{ n \sum_{i=1}^{n} (i-1)k^{i-2} t_i \right\} = 0$$

$$\frac{n}{k}\left\{ \frac{n+1}{2} -1 \right\} \sum_{i=1}^{n} k^{i-1} t_i - \left\{ n \sum_{i=1}^{n} (i-1)k^{i-2} t_i \right\} = 0$$

$$n \sum_{i=1}^{n} k^{i-2} t_i \left\{ \frac{n+1}{2} -1 -i + 1\right\} = 0$$

$$(n/k^2) \sum_{i=1}^{n} k^{i} t_i \left\{ \frac{n+1}{2} -i\right\} = 0$$

Since $0 <k < 1$, we have $(n/k^2) \neq o$ and hence,

$$\sum_{i=1}^{n} k^{i} t_i \left\{ \frac{n+1}{2} -i \right\} = 0 \qquad (3.6.4)$$

Hence the MLE of k, $\hat{k}$, is the solution to the polynomial equation (3.6.4). Having obtained the MLE of k, the MLE for D, $\hat{D}$ is

$$\hat{D} = n / \left\{ \sum_{i=1}^{n} \hat{k}^{i-1} t_i \right\} \qquad (3.6.5)$$

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